Expected Value of Asymmetric Coordinated Search Technique for Detecting a Randomly Located Target on the Plane

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Abstract

This paper proposes Asymmetric coordination strategy, we study coordinated search technique for a lost located object on the plan, where there are two searchers beginning their search from the same initial point (0, 0), but both sides do not have the same area. Here we introduce a model of search plan and investigate the expected value of detecting the lost target in both sides to avoid wasting time. We present an algorithm to facilitate the search technique. An illustrative application from real life has been introduced to demonstrate the applicability of this search technique. The effectiveness of this model has been illustrated by numerical results

Keywords: Expected value; coordinated search; minimizing the expected time; AST, probability theory 2-dimentional algorithm; lost submarine

1. Introduction

Every person is exposed to the possibility of losing anything on a regular basis, but teenagers and the elderly are more prone to do so, given that most of their owners are distracted, hurried, fearful, and impulsive. If the target is a missing person or a ship, how can we know? Or a search for explosive devices in the desert? The search could also be related to the detection of diseases or cancerous cells in the human body. As a result, in emergency situations, we can hire many searchers to quickly discover the lost objective and, in some cases, save money.

Search theory has a wide range of applications. The antisubmarine warfare operations research department of the US Navy was founded during WWII. The primary purpose was to gain a better understanding of the German submarine threat in the Atlantic.

To begin, one of the most important models of search algorithms is the three-dimensional space. El-Hadidy and El-Bagoury [1,2] introduced and evaluated a current three-dimensional search model for one or two searchers to find a three-dimensional randomly located object. T. Caraballo et al. [3] suggested a new search theory model for discovering a randomly placed three-dimensional item. El-Bagoury et al.[4] have investigated a more advanced three-dimensional search model to locate a 3-D randomly positioned object. Using a monitoring system, A. Teamah et al.[5] established an optimal discrete search for randomly delivering COVID-19 across many cells in an area of the human body. in discrete search issues, using a monitoring system [6]. A. A. Mohamed, Fergani, and El-Hadidy [7] outlined the method of coordinated search in the plane, while Thomas [8] depicted the coordinated search on a circle with a known radius, where the target is equally likely to be anywhere on its circumference. AL-Aziz, S.N, El-Bagoury, etc. The study by A.H. and W. Afifi comprehensively characterizes the search for a randomly situated object on the plane using a coordinated search technique. When searchers try for a missing

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object, the main goal is to prevent wasting time and lowering the enormous financial costs. The estimated time to detect the target is calculated using a symmetric distribution for the target's position. [9] . This study will discuss a novel coordinated strategy, one of which is the elimination of time spent seeking for the objective. Two searchers will go on this journey from the origin point of a sea surface circle zone. The first seeks the target on the right side of the known zone, whereas the second seeks it on the left. Both searchers will not return to their starting place. We'll use modern communication tools, including sign language, to save time and effort on our excursion.

2. Problem Formulations

The fear of submarine disappearance has grown in any army. Finding submerged submarines in the depths of seas or oceans is difficult. The ARA San Juan submarine, according to an Argentine Navy official, went missing on November 15, 2017. The search for the ARA San Juan submarine and any survivors drew nearly 24 countries from around the world.

They also stated that they had received a call from the missing submarine requesting its location. The navy sent ships to look for the missing ship. We will demonstrate a coordinated technique to address this type of problem in order to save time, when the user knows the probability distribution function of the missing submarine position. in real-life searches. Using a symmetric search approach and a bivariate standard normal distribution, the main goal is to determine the optimum search strategy that minimizes the expected value of the time it takes to find the missing submarine. What will we do if the two zones are not equal, and one of the coral reef rocks has a more intricate nature than the other? Another approach (AST), asymmetric search technique, will be used to make the search operation easier. Figure 1 shows how two planes partition a two-dimensional zone. The planes cross at (0,0), and the target is randomly placed inside the search region. Two searchers start together from the starting point to the goal, where distance equals time. Submarine disappearances have become a common concern for any army. Finding the best means to discover sunken submarines in the depths of seas or oceans is difficult. This research looks at the problem of finding a missing submarine in a 2-dimensional known region. Two searchers are striving to discover the target in the quickest period feasible. The target's location is unknown.

2.1. The Searching Framework

[1] **The search area** is a circular two-dimensional area.

The target: The target is a lost submarine that was randomly located on the sea surface.

Methods of search: Two searchers on the circular portion were looking for the lost objective. The two searchers begin their search for the objective at (0,0) (the center of the circular known area), where the region is divided into two portions that intersect in the center of the area, one of which is right and the other is left. As shown in Fig. 1, we shall divide the region into multiple circles.

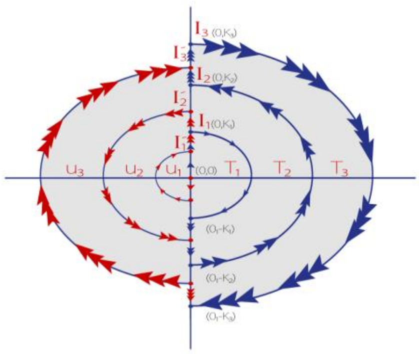

Figure 1: The search Path

2.2. The Searching Technique

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In this concept, the search for the missing submarine is conducted via a synchronized movement between the two searchers, with both searchers sending radio telex signals to a marine submarine signal reception center after each move. Two types of detections exist:

1- Perfect Detection(Discovery): This indicates that one of the searchers finds the lost ship in the search area and gives a positive signal to the marine. Both searchers send signals to a marine submarine signal reception center through radio telex.
2- False Detection: When the two searchers fail to find the missing person, they send radio telex signals to a marine submarine signal reception center in the section designated for the missing person..
Let (X, Y) be two independent random variables which represent the position of the located target in the region with a cumulative distribution function (CDF) $F(x, y)$, and its corresponding probability density function (PDF) is $f(x, y)$.

The two searchers p_1 and p_2 start searching together for the lost both searchers send signals to a marine submarine signal's reception center by radio telex (target) from the origin point (0, 0) with equal speeds $v_1 = v_2 = 1$ respectively, and they search with regular speed on the sectors and its tracks, The searcher s_1 searches on the right hand-side and the searcher p_2 searches on the left hand-side. The searcher p_1 goes through a positive part y axis with a distance k_1 (radius of the first circle), and then it starts searching on the sector T_1, after completing the search on the T_1 reach to y-axis again but does not return to the origin point, and move towards the second sector T_2 with distance $k_2 - k_1$, when it completes the searching on the sector T_2 and doesn't find the target it continues searching on the sector T_3 and so on. Another searcher p_2 is searching the opposite part, on the left hand-side, goes through a negative part y axis with distance k_1, and starts searching on the sector U_1 after completing the searching on this sector and it didn't find the target it moves to the second sector U_1 with distance $k'_2 - k'_1$. When it completes searching on the sector U_2 and didn't find the target it continues searching on the sector U_3 and so on. The searchers p_1 and p_2 search the sector T_1 and U_1 at the same time. If one of them detects the lost target it will send a positive sign to the marine submarine signal's reception center and the search will end. But, if the two searchers send a negative sign to a marine submarine signal's reception center the two searchers complete their search and move to the next sectors T_2 and U_2, and so on until one of the searchers detects the lost submarine. The main reason why the searchers don't not to return to the origin point again is to save search time and reduce the expected length of the time for detecting the target.

Any track has a width $k_i - k_{i-1}$, the searcher moves on the y axis (Positive, Negative) until he detects the target. we aim to calculate the expected length of the time for detecting the target and wish to find the optimal search plan to detect the target.
The searchers p_1 and p_2 follow a coordinated search path to find the target. Let G_i be the search path for the first searcher p_1, i=1,2,…, and H_i the search path of the second searcher p_2, i=1,2,……Also, let T_i and U_i,i=1,2,….. be the sectors which the searchers p_1 and p_2 search respectively. Let q_1 be the time which the first searcher p_1 takes to search each sector in the path G_i, and q_2 is the time which the second searcher s_2 takes to search each sector in the path H_i.

The search process will be done according to the following procedures:

Step 1: To find the target, the two searchers proceed from (0,0). The first searcher takes the following search path k'_1 : If the target is not found, the searcher p1 moves to $(0, k'_1)$ with distance $|k'_1|$ through a positive part in y axis and completes searching on the first sector T1, which it tracks until the point $(0, -k'_1)$, while the searcher p2 moves to $(0, -k'_1)$ with distance $|k'_1|$ through a negative part in x axis and completes searching on the first sector U1, which it tracks until until you get to $(0, k'_1)$. At this point, the two searchers send their signals to a marine ship signal reception center; if one of them provides a positive signal, the other sends a negative signal, and will move to the following step.

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Step 2: the searcher p_1 finishes looking for the missing target and goes to (0, k_2) with a distance $|k_2 - k_1|$, the first searcher takes the following search path k_2 , If the target is not located ,the searcher p_1 moves to (0, k_2) with distance $|k_2 - k_1|$ through a positive part in y axis, if the target is not found, it moves to the second sector T2, and continues tracking until it reaches (0, $-k_2$).). If the target is not located, the searcher p2 moves to (0,$-k'_2$) with distance $|k'_2 - k'_1|$ through a negative part in x axis. If the target is not found, it moves to the second sector U_2, and continues tracking until it reaches (0, k'_2). At this point, the two searchers send their signals to a marine ship signal reception center; if one of them produces a positive signal, the search is completed; if both searchers send negative signals, the search continues.

Step 3: the searcher p_1 finishes looking for the missing target and advances to (0, k_3) with a distance $|k_3 - k_2|$, The first searcher takes the following search path k_3 as follows: the searcher p_1 moves to (0, k_3) with distance $|k_3 - k_2|$ and a positive portion in y axis. If the target is not discovered, the search moves on to the second sector T2 and continues until the point (0, k_2). At the same time, the searcher p2 travels to (0, k'_3) along the negative part of the $x -$ axis with the distance $|k'_3 - k'_2|$. is not discovered, the search continues on the second sector U3 until it reaches point (0, k'_3). During this time, the two searchers send signals to a marine ship signal reception center; if one of them sends a positive signal, the search is completed; if the two searchers send a negative signal, the search is continued to the next sector, and so on, until one of the two searchers detects the lost submarine.

The search path is followed by the searchers p1 and p2. Let G_i the first searcher's path where $i \geq 0$, q_1 be the first searcher's time, L_i is the sector, which p_1 searches i=1, 2, …. , and H_i be the second searcher's path, where $i \geq 0$, T_i the sector, that p_2 looks for i=1,2,….. the two searchers proceed from (0,0) to discover the target. The first search path k_1 of p_1 as follows: p_1 gets to (0, k_1) with distance $|k_1|$ via a positive part in y axis if the target not discovered , the search is completed on the first sector T_1, and the tracking continues until the point (0,$-k_1$). So in this step the time is taken.

in this step $q_1 = |k_1| + \frac{\pi}{w_1}$, Then it advances to point (0, k_2) with distance $|k_2 - k_1|$, taking time $q_1 = |k_2 - k_1| + \frac{\pi}{w_2}$, the target is not found, it moves to sector T2 and continues tracking. If the target is not discovered, it advances to (0, k_3) on sector T3 and continues until the point (0,$-k_3$) with distance $|k_3 - k_2|$, is reached, which takes time $q_1 = |k_3 - k_2| + \frac{\pi}{w_3}$ and so on. The number of circles i, i=1, 2,…. These circles are divided into two sectors T_i, U_i.

The searcher p_2 proceed to (0,$-k'_1$) with distance $|k'_1|$ through a negative part in x axis if the target is not found complete searching on the first sector U_1, and it's tracking. until the point (0, k'_1) with time $q_2 = |k'_1| + \frac{\pi}{w_1}$, then travel to point (0, k_2) with distance $|k'_2 - k'_1|$ it takes time $q_2 = |k'_2 - k'_1| + \frac{\pi}{w_2}$, if not discovered, it moves to sector U_2, until point (0, k'_2). If the target is not discovered it goes to (0,-k'_3) the search on the sector U_3 and it keeps tracking until point (0,k'_3) with distance $|k'_3 - k'_2|$ and so on.

3. Time Value Expected to Detect the Target

Theorem 1: The time value expected for detecting the target is given by:

$$E(t_\psi) = \Bigg[\left[\left[\left(|k_1| + \frac{\pi}{w_1}\right)\int_0^{\frac{\pi}{2}}\int_0^{k_1} g_1(k,\theta)kdkd\theta\right] + \left[\left(|k_2 - k_1| + \frac{\pi}{w_2}\right)\int_0^{\frac{\pi}{2}}\int_{k_1}^{k_2} g_2(k,\theta)kdkd\theta\right]\right.$$

$$+\left[\left(|k_3 - k_2| + \frac{\pi}{w_3}\right)\int_0^{\frac{\pi}{2}}\int_{k_2}^{k_3} g_3(k,\theta)kdkd\theta\right] + \dots\dots\dots\dots + \left[\left(|k_n - k_{n-1}| + \frac{\pi}{w_n}\right)\int_0^{\frac{\pi}{2}}\int_{k_{n-1}}^{k_n} g_n(k,\theta)kdkd\theta\right] + \Bigg]$$

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$$\left[\left(\left|k_1'\right|+\frac{\pi}{w_1}\right)\int_0^{\frac{\pi}{2}}\int_0^{k_1'} g_1(k,\theta)kdkd\theta\right]+\left[\left(\left|k_2'-k_1'\right|+\frac{\pi}{w_2}\right)\int_0^{\frac{\pi}{2}}\int_{k_1'}^{k_2'} g_2(k,\theta)kdkd\theta\right]$$

$$\left[\left(\left|k_3'-k_2'\right|+\frac{\pi}{w_3}\right)\int_0^{\frac{\pi}{2}}\int_{k_2'}^{k_3'} g_3(k,\theta)kdkd\theta\right]+\ldots\ldots\ldots\ldots+\left[\left(\left|k_n'-k_{n-1}'\right|+\frac{\pi}{w_n}\right)\int_0^{\frac{\pi}{2}}\int_{k_{n-1}'}^{k_n'} g_n(k,\theta)kdkd\theta\right]$$

$$=\left[\left(\left|k_i'-k_{i-1}'\right|+\frac{\pi}{w_i}\right)\int_0^{\frac{\pi}{2}}\int_{k_{i-1}'}^{k_i'} g_i(k,\theta)kdkd\theta\right]+\left[\left(\left|k_i'-k_{i-1}'\right|+\frac{\pi}{w_i}\right)\int_0^{\frac{\pi}{2}}\int_{k_{i-1}'}^{k_i'} g_i(k,\theta)kdkd\theta\right]$$

(1)

Where, i=1, 2, 3,…, n, k_0 =0

Proof

1)For the first searcher p_1

If the target is located in any place in the first sector T1, then $q_1=|k_1|+\frac{\pi}{w_1}$.

If the target is located in any place in the second sector T2, then $q_1=|k_2-k_1|+\frac{\pi}{w_2}$.

If the target is located in any place in the third sector T3, then $q_1=|k_3-k_2|+\frac{\pi}{w_3}$. And so on.

2) For the second searcher p_2

If the target is located in any place in the first sector U1, then $q_2=|\acute{k}_1|+\frac{\pi}{w_1}$..

If the target is located in any place in the second sector U2, then $q_2=|\acute{k}_2-\acute{k}_1|+\frac{\pi}{w_2}$.

If the target is located in any place in the third sector U3, then $q_2=|\acute{k}_3-\acute{k}_2|+\frac{\pi}{w_3}$..

And so on, then

$$E(t_\psi)=\left[\left(\left|k_1\right|+\frac{\pi}{w_1}\right)\int_0^{\frac{\pi}{2}}\int_0^{k_1} g_1(k,\theta)kdkd\theta\right]+\left[\left(\left|k_2-k_1\right|+\frac{\pi}{w_2}\right)\int_0^{\frac{\pi}{2}}\int_{k_1}^{k_2} g_2(k,\theta)kdkd\theta\right]+$$

$$\left[\left(\left|k_3-k_2\right|+\frac{\pi}{w_3}\right)\int_0^{\frac{\pi}{2}}\int_{k_2}^{k_3} g_3(k,\theta)kdkd\theta\right]$$

$$+\ldots\ldots\ldots\ldots\ldots+\left[\left(\left|k_n-k_{n-1}\right|+\frac{\pi}{w_n}\right)\int_0^{\frac{\pi}{2}}\int_{k_{n-1}}^{k_n} g_n(k,\theta)kdkd\theta\right]+$$

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$$\left[\left(|k_1'|+\frac{\pi}{w_1}\right)\int_0^{\frac{\pi}{2}}\int_0^{k_1'} g_1(k,\theta)kdkd\theta\right]+\left[\left(|k_2'-k_1'|+\frac{\pi}{w_2}\right)\int_0^{\frac{\pi}{2}}\int_{k_1'}^{k_2'} g_2(k,\theta)kdkd\theta\right]+$$

$$\left[\left(|k_3'-k_2'|+\frac{\pi}{w_3}\right)\int_0^{\frac{\pi}{2}}\int_{k_2'}^{k_3'} g_3(k,\theta)kdkd\theta\right]+...............+\left[\left(|k_n'-k_{n-1}'|+\frac{\pi}{w_n}\right)\int_0^{\frac{\pi}{2}}\int_{k_{n-1}'}^{k_n'} g_n(k,\theta)kdkd\theta\right]$$

.

$$=\left[\left[\left(|k_i'-k_{i-1}'|+\frac{\pi}{w_i}\right)\int_0^{\frac{\pi}{2}}\int_{k_{i-1}'}^{k_i'} g_i(k,\theta)kdkd\theta\right]+\left[\left(|k_i'-k_{i-1}'|+\frac{\pi}{w_i}\right)\int_0^{\frac{\pi}{2}}\int_{k_{i-1}'}^{k_i'} g_i(k,\theta)kdkd\theta\right]\right].$$

4. Algorithm and Flowchart

To calculate the expected value of time for detecting the lost target, we use the following algorithm

Main algorithm

- initialize the Searcher $\mathbf{p_1}$ and Searcher $\mathbf{p_2}$ to his start point (0,0) ;
- Velocities $\mathbf{V_1}$ = Velocities $\mathbf{V_2}$ = 1 ;
- i = 1 ;
- $\mathbf{k_{i-1} = 0}$ **;**

- for (i to m)
- distance $\mathcal{D}$= | $k_i - k_{i-1}$ | ;

- if (i = odd number) then
  - moves p_1 towards point (0, k_i) with distance $\mathcal{D}$ && moves p2 towards point (0,-k_i) with distance $\mathcal{D}$;
  - p1 complete searching in the sector **Ti** with same distance $\mathcal{D}$ and tracks until the point (0,-k'_i) && p2 complete searching in the sector **Ui** with same distance $\mathcal{D}$ and tracks until the point (0,k_i)

- else
  - moves p1 towards the point (0,-k_i) with distance $\mathcal{D}$ && moves p2 towards point (0, k'_i) with distance $\mathcal{D}$;
  - p1 complete searching in the sector **Ti** with same distance $\mathcal{D}$ and tracks until point (0, k'_i) && p2 complete searching in the sector **Ui** with same distance $\mathcal{D}$ and tracks until point (0,-k_i)

- end if
- calculate time for the step $\mathbf{Q_i}$ = distance $\mathcal{D}$ + (3.14 / w_i)

- if (target not found in p1 || p2) then
  - print p1 + "not found target in " + T_i)
  - print p2 + "not found target in " + U_i)

- else
  - if (target found in P_1) then
    - print green signal+**p1+ " found the target in " +** $\mathbf{Q_i}$ **+ "time:" +** $\mathbf{Q_i}$
  - else

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- print green signal+**p2+ " found the target in " + U_i +"time:" + Q_i**

  - exit a for loop
  - end if

- end if
- End for
- End Main

[2]
[3]

5. Application:

P1 and P2 are two searchers tasked with finding a lost submarine on the seabed. The position of the missing submarine follows the standard bivariate normal distribution for two independent random.

variables x, y are defined as $f(x,y)=\frac{1}{2\pi}e^{\frac{-x^2+y^2}{2}}$. By changing to polar coordinates $x=k\cos\theta$, $y=k\sin\theta$.

The Jacobin matrix defined as:

$$J(u,v)=\det\begin{pmatrix}\frac{\partial x}{\partial k} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial k} & \frac{\partial y}{\partial\theta}\end{pmatrix}=J(u,v)=\det\begin{pmatrix}\cos\theta & -k\sin\theta\\ \sin\theta & k\cos\theta\end{pmatrix}$$

$= k\cos^2\theta + k\sin^2\theta = k(\cos^2\theta+\sin^2\theta) = k$

So, $g(k,\theta)=\frac{1}{2\pi}e^{\frac{-k^2}{2}}$ substituting in (1), so the expected value will be

$$E(t_\psi)=(|k_{i-}k_{i-1}|+\frac{\pi}{w_i})\int_0^{\frac{\pi}{2}}\int_{k_{i-1}}^{k_i}\frac{1}{2\pi}ke^{\frac{-k^2}{2}}kdkd\theta+(|k'_{i-}k'_{i-1}|+\frac{\pi}{w_i})\int_0^{\frac{\pi}{2}}\int_{k'_{i-1}}^{k'_i}\frac{1}{2\pi}ke^{\frac{-k^2}{2}}kdkd\theta$$

$$=\ 0.5\int_0^{\frac{\pi}{2}}\int_{k_{i-1}}^{k_i}k^2e^{-\frac{k^2}{2}}dkd\theta+0.5\int_0^{\frac{\pi}{2}}\int_{k'_{i-1}}^{k'_i}k^2e^{-\frac{k^2}{2}}dkd\theta=\frac{\pi}{4}\int_{k_{i-1}}^{k_i}k^2e^{-\frac{k^2}{2}}dk+\frac{\pi}{4}\int_{k'_{i-1}}^{k'_i}k^2e^{-\frac{k^2}{2}}dk$$

The first search if k: 0 $\longrightarrow$ k_1, then

$$E(t_\psi)=\frac{\pi}{4}\left[\frac{e^{-\frac{k_1^2}{2}}\left(\sqrt{\pi}e^{\frac{k_1^2}{2}}ekf\,\frac{k_1}{\sqrt{2}}-\sqrt{2}k_1\right)}{\sqrt{2}}+\frac{e^{-\frac{k_1'^2}{2}}\left(\sqrt{\pi}e^{\frac{k_1'^2}{2}}ekf\,\frac{k'_1}{\sqrt{2}}-\sqrt{2}k'_1\right)}{\sqrt{2}}\right]$$

The Second search if k: $k_1 \longrightarrow k_2$, then the expected value will be as a following:

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$$=\frac{\pi}{4}\left[\frac{e^{-\frac{k_2^2}{2}}\left(\sqrt{\pi}e^{\frac{k_2^2}{2}}erf\,\frac{k_2}{\sqrt{2}}-\sqrt{2}k_2\right)}{\sqrt{2}}-\frac{e^{-\frac{k_1^2}{2}}\left(\sqrt{\pi}e^{\frac{k_1^2}{2}}erf\,\frac{k_1}{\sqrt{2}}-\sqrt{2}k_1\right)}{\sqrt{2}}\right]+$$

$$\frac{\pi}{4}\left[\frac{e^{-\frac{k_2'^2}{2}}\left(\sqrt{\pi}e^{\frac{k_2'^2}{2}}erf\,\frac{k_2'}{\sqrt{2}}-\sqrt{2}k_2'\right)}{\sqrt{2}}-\frac{e^{-\frac{k_1'^2}{2}}\left(\sqrt{\pi}e^{\frac{k_1'^2}{2}}erf\,\frac{k_1'}{\sqrt{2}}-\sqrt{2}k_1'\right)}{\sqrt{2}}\right]$$

$$=\frac{\pi}{4}\left[\frac{e^{-\frac{k_i^2}{2}}\left(\sqrt{\pi}e^{\frac{k_i^2}{2}}erf\,\frac{k_i}{\sqrt{2}}-\sqrt{2}k_i\right)-e^{-\frac{k_{i-1}^2}{2}}\left(\sqrt{\pi}e^{\frac{k_{i-1}^2}{2}}erf\,\frac{k_{i-1}}{\sqrt{2}}-\sqrt{2}k_{i-1}\right)}{\sqrt{2}}\right]+$$

$$\frac{\pi}{4}\left[\frac{e^{-\frac{k_i'^2}{2}}\left(\sqrt{\pi}e^{\frac{k_i'^2}{2}}erf\,\frac{k_i'}{\sqrt{2}}-\sqrt{2}k_i'\right)-e^{-\frac{k_{i-1}'^2}{2}}\left(\sqrt{\pi}e^{\frac{k_{i-1}'^2}{2}}erf\,\frac{k_{i-1}'}{\sqrt{2}}-\sqrt{2}k_{i-1}'\right)}{\sqrt{2}}\right]$$

$$E(t_\psi)=\frac{1}{2\pi}\cdot\frac{\pi}{2\sqrt{2}}\left[\left(\left|k_i-k_{i-1}\right|+\frac{\pi}{w_i}\right)\left(-k_ie^{-\frac{k^2}{2}}+\sqrt{\frac{\pi}{2}}erf\,\frac{k_i}{\sqrt{2}}+k_{i-1}e^{-\frac{k_i^2}{2}}-\sqrt{\frac{\pi}{2}}erf\,\frac{k_{i-1}}{\sqrt{2}}\right)\right]+$$

$$\frac{1}{2\pi}\cdot\frac{\pi}{2\sqrt{2}}\left[\left(\left|k_i'-k_{i-1}'\right|+\frac{\pi}{w_i}\right)\left(-k_i'e^{-\frac{k^2}{2}}+\sqrt{\frac{\pi}{2}}erf\,\frac{k_i'}{\sqrt{2}}+k_{i-1}'e^{-\frac{k_i'^2}{2}}-\sqrt{\frac{\pi}{2}}erf\,\frac{k_{i-1}'}{\sqrt{2}}\right)\right]+$$

$$E(t_\psi)=\frac{1}{4\sqrt{2}}\left[\left|k_i-k_{i-1}\right|+\frac{\pi}{w_i}\right]\left[-k_ie^{-\frac{k_i^2}{2}}+k_{i-1}e^{-\frac{k_{i-1}^2}{2}}+\sqrt{\frac{\pi}{2}}(erf\,\frac{k_i}{\sqrt{2}}-erf\,\frac{k_{i-1}}{\sqrt{2}}\right]_+$$

$$\frac{1}{4\sqrt{2}}\left[\left|k_i'-k_{i-1}'\right|+\frac{\pi}{w_i}\right]\left[-k_i'e^{-\frac{k_i'^2}{2}}+k_{i-1}'e^{-\frac{k_{i-1}'^2}{2}}+\sqrt{\frac{\pi}{2}}(erf\,\frac{k_i'}{\sqrt{2}}-erf\,\frac{k_{i-1}'}{\sqrt{2}}\right] \quad (2)$$

Special cases:

If $k_1-k_0=k\rightarrow(3)$ where k_0=0, then $k_2-k_1=k\rightarrow(4)$, and $k_2=2k$ .Also, $k_3-k_2=k\Rightarrow k_3=3k$.

K=1,2,3,4,…………, k' =1,2,3,4,……………………

Hence,

$$E(t_\psi)=\frac{1}{4\sqrt{2}}\left[k+\frac{\pi}{w_i}\right]\left[-k_ie^{-\frac{k_i^2}{2}}+(k_i-k)e^{-\frac{(k_i-k)^2}{2}}+\sqrt{\frac{\pi}{2}}\left(erf\,\frac{k_i}{\sqrt{2}}-erf\,\frac{(k_i-k)}{\sqrt{2}}\right)\right]$$

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$$\frac{1}{4\sqrt{2}}\left[k'+\frac{\pi}{w_i}\right]\left[-k_i'e^{-\frac{k_i'^2}{2}}+(k_i'-k')e^{-\frac{(k_i'-k)^2}{2}}+\sqrt{\frac{\pi}{2}}\left(erf\,\frac{k_i'}{\sqrt{2}}-erf\,\frac{(k_i'-k')}{\sqrt{2}}\right)\right]$$

If K=2k'

$$E(t_\psi)=\frac{1}{4\sqrt{2}}\left[k+\frac{\pi}{w_i}\right]\left[-ike^{-\frac{(ik)^2}{2}}+k(i-1)e^{-\frac{(k(i-k))^2}{2}}+\sqrt{\frac{\pi}{2}}\left(erf\,\frac{ik}{\sqrt{2}}-erf\,\frac{k(i-1)}{\sqrt{2}}\right)\right]+$$

$$\frac{1}{4\sqrt{2}}\left[k'+\frac{\pi}{w_i}\right]\left[-ik'e^{-\frac{(ik')^2}{2}}+k'(i-1)e^{-\frac{(k'(i-k'))^2}{2}}+\sqrt{\frac{\pi}{2}}\left(erf\,\frac{ik'}{\sqrt{2}}-erf\,\frac{k'(i-1)}{\sqrt{2}}\right)\right].(5)$$

$$E(t_\psi)=\frac{1}{4\sqrt{2}}\left[2k'+\frac{\pi}{w_i}\right]\left[-i2k'e^{-\frac{(i2k')^2}{2}}+2k'(i-1)e^{-\frac{(2k'(i-2k'))^2}{2}}+\sqrt{\frac{\pi}{2}}\left(erf\,\frac{i2k'}{\sqrt{2}}-erf\,\frac{2k'(i-1)}{\sqrt{2}}\right)\right]$$
$$+\frac{1}{4\sqrt{2}}\left[k'+\frac{\pi}{w_i}\right]\left[-ik'e^{-\frac{(ik')^2}{2}}+k'(i-1)e^{-\frac{(k'(i-k'))^2}{2}}+\sqrt{\frac{\pi}{2}}\left(erf\,\frac{ik'}{\sqrt{2}}-erf\,\frac{k'(i-1)}{\sqrt{2}}\right)\right].(6)$$

By considering the values of i, k' and in the tables (1-10) using the Mathematica program, and substituting in relation (6) we can get the different values of as a following :

Table 1: The expected values of detecting the lost submarine k'=1

i	k'	Wi	π_i	E(t)
1	1	10	3.14	0.436495
2	1	10	3.14	0.998014
3	1	10	3.14	0.289066
4	1	10	3.14	0.00902054
5	1	10	3.14	0.000325687
6	1	10	3.14	4.47332*10^-6
1	1	20	3.14	0.403892
2	1	20	3.14	0.922187
3	1	20	3.14	0.265963
4	1	20	3.14	0.00798526
5	1	20	3.14	0.000286775
6	1	20	3.14	3.93884*10^-6
1	1	30	3.14	0.393024
2	1	30	3.14	0.896911
3	1	30	3.14	0.258262
4	1	30	3.14	0.00764016

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5	1	30	3.14	0.000273805
6	1	30	3.14	3.76068*10^-6

Table 2: The expected values of detecting the lost submarine k'=٢

i	k'	Wi	π_i	E(t)
1	2	10	3.14	1.33335
2	2	10	3.14	0.840929
3	2	10	3.14	0.223521
4	2	10	3.14	9.1522
5	2	10	3.14	0.00409331
6	2	10	3.14	2.44962*10^-13
1	2	20	3.14	1.27291
2	2	20	3.14	0.783876
3	2	20	3.14	0.20842
4	2	20	3.14	8.8191
5	2	20	3.14	0.00394434
6	2	20	3.14	2.34417*10^-13
1	2	30	3.14	1.25277
2	2	30	3.14	0.764858
3	2	30	3.14	0.203386
4	2	30	3.14	8.70806
5	2	30	3.14	0.00389468
6	2	30	3.14	2.30903*10^-13

Table 3: The expected values of detecting the lost submarine k'=٣

i	k'	Wi	π_i	E(t)
1	3	10	3.14	2.11164
2	3	10	3.14	0.0215065
3	3	10	3.14	3.51503
4	3	10	3.14	0.0585726
5	3	10	3.14	5.15049*10^-7
6	3	10	3.14	33.485
1	3	20	3.14	2.04309
2	3	20	3.14	0.0204876
3	3	20	3.14	3.3485
4	3	20	3.14	0.0557978
5	3	20	3.14	4.99832*10^-7
6	3	20	3.14	32.6524
1	3	30	3.14	2.02024
2	3	30	3.14	0.020148
3	3	30	3.14	3.293
4	3	30	3.14	0.0548728

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5	3	30	3.14	4.9476*10^-7
6	3	30	3.14	32.3749

Table 4: The expected values of detecting the lost submarine k'=٤

i	k'	Wi	π_i	E(t)
1	4	10	3.14	2.79673
2	4	10	3.14	6.05425E-05
3	4	10	3.14	0.00204663
4	4	10	3.14	9.15138
5	4	10	3.14	0.00409326
6	4	10	3.14	1.93158*10^- 13
1	4	20	3.14	2.72721
2	4	20	3.14	5.83391E-05
3	4	20	3.14	0.00197215
4	4	20	3.14	8.81833
5	4	20	3.14	0.00394429
6	4	20	3.14	1.86128*10^- 13
1	4	30	3.14	2.70403
2	4	30	3.14	5.76047E-05
3	4	30	3.14	0.00194732
4	4	30	3.14	8.70731
5	4	30	3.14	0.00389464
6	4	30	3.14	1.83785*10^- 13

Table 5: The expected values of detecting the lost submarine k'=٥

i	k'	Wi	π_i	E(t)
1	5	10	3.14	3.46247
2	5	10	3.14	6.7498*10^-7
3	5	10	3.14	1.81185*10^- 21
4	5	10	3.14	5.25118E-05
5	5	10	3.14	18.7878
6	5	10	3.14	8.75196E-05
1	5	20	3.14	3.3929
2	5	20	3.14	6.55038*10^- 7
3	5	20	3.14	1.75832*10^- 21
4	5	20	3.14	5.09603E-05
5	5	20	3.14	18.2327
6	5	20	3.14	8.49339E-05
1	5	30	3.14	3.36971
2	5	30	3.14	6.48391*10^- 7
3	5	30	3.14	1.74048*10^-

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4	5	30	3.14	5.04432E-05
5	5	30	3.14	18.0477
6	5	30	3.14	0.000084072

Table 6: The expected values of detecting the lost submarine k'=٦

i	k'	Wi	π_i	E(t)
1	6	10	3.14	4.12716
2	6	10	3.14	2.76029*10^-9
3	6	10	3.14	-2.9524E-70
4	6	10	3.14	1.08093*10^-30
5	6	10	3.14	4.07981*10^-7
6	6	10	3.14	33.485
7	6	10	3.14	6.11972*10^-7
1	6	20	3.14	4.05759
2	6	20	3.14	2.69166*10^-9
3	6	20	3.14	-2.87898E-70
4	6	20	3.14	1.05406*10^-30
5	6	20	3.14	3.97837*10^-7
6	6	20	3.14	32.6524
7	6	20	3.14	5.96755*10^-7
1	6	30	3.14	4.0344
2	6	30	3.14	2.66878*10^-9
3	6	30	3.14	-2.85451E-70
4	6	30	3.14	1.0451*10^-30
5	6	30	3.14	3.94455*10^-7
6	6	30	3.14	32.3749
7	6	30	3.14	5.91683*10^-7

Table 7: The expected values of detecting the lost submarine k'=7

i	k'	Wi	π_i	E(t)
1	7	10	3.14	4.79183
2	7	10	3.14	4.14778*10^-12
3	7	10	3.14	-4.69749E-95
4	7	10	3.14	4.69749*10^-95
5	7	10	3.14	9.95128*10^-

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6	7	10	3.14	1.03618*10^-9
7	7	10	3.14	54.3037
1	7	20	3.14	4.72226
2	7	20	3.14	4.05874*10^-12
3	7	20	3.14	-4.59666E-95
4	7	20	3.14	4.59666*10^-95
5	7	20	3.14	9.73767*10^-42
6	7	20	3.14	1.01393*10^-9
7	7	20	3.14	53.138
1	7	30	3.14	4.69907
2	7	30	3.14	4.02906*10^-12
3	7	30	3.14	-4.56305E-95
4	7	30	3.14	4.56305*10^-95
5	7	30	3.14	9.66646*10^-42
6	7	30	3.14	1.00652*10^-9
7	7	30	3.14	52.7495

Table 8: The expected values of detecting the lost submarine k'=8

i	k'	Wi	π_i	E(t)
1	8	10	3.14	5.4565
2	8	10	3.14	2.24956*10^-15
3	8	10	3.14	-2.9555E-124
4	8	10	3.14	-5.147E-222
5	8	10	3.14	3.94071*10^-124
6	8	10	3.14	1.51217*10^-54
7	8	10	3.14	8.93414*10^-13
8	8	10	3.14	82.3044
1	8	20	3.14	5.38693
2	8	20	3.14	2.20708*10^-15
3	8	20	3.14	-2.8997E-124
4	8	20	3.14	-5.0498E-222
5	8	20	3.14	3.86629*10^-124
6	8	20	3.14	1.48362*10^-54
7	8	20	3.14	8.76543*10^-13
8	8	20	3.14	80.7502

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1	8	30	3.14	5.36374
2	8	30	3.14	2.19292*10^-15
3	8	30	3.14	-2.8811E-124
4	8	30	3.14	-5.0174E-222
5	8	30	3.14	3.84149*10^-124
6	8	30	3.14	1.4741*10^-54
7	8	30	3.14	8.70919*10^-13
8	8	30	3.14	80.2321

Table 9: The expected values of detecting the lost submarine k'=9

i	k'	Wi	π_i	E(t)
1	9	10	3.14	6.12117
2	9	10	3.14	-1.30655E-69
3	9	10	3.14	-2.2263E-157
4	9	10	3.14	-2.2389E-280
5	9	10	3.14	2.23892*10^-280
6	9	10	3.14	3.71052*10^-157
7	9	10	3.14	3.91966*10^-69
8	9	10	3.14	2.67285*10^-16
9	9	10	3.14	118.548
1	9	20	3.14	6.0516
2	9	20	3.14	-1.28453E-69
3	9	20	3.14	-2.1888E-157
4	9	20	3.14	-2.2012E-280
5	9	20	3.14	2.20118*10^-280
6	9	20	3.14	3.64798*10^-157
7	9	20	3.14	3.85359*10^-69
8	9	20	3.14	2.6278*10^-16
9	9	20	3.14	116.55
1	9	30	3.14	6.02841
2	9	30	3.14	-1.27719E-69
3	9	30	3.14	-2.1763E-157
4	9	30	3.14	-2.1886E-280
5	9	30	3.14	2.1886*10^-280
6	9	30	3.14	3.62713*10^-157
7	9	30	3.14	3.83157*10^-69
8	9	30	3.14	2.61278*10^-

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				16
9	9	30	3.14	115.883

Table 10: The expected values of detecting the lost submarine k'=10

i	k'	Wi	π_i	E(t)
1	10	10	3.14	6.78584
2	10	10	3.14	-5.04645E-86
3	10	10	3.14	-2.0205E-194
4	10	10	3.14	0
5	10	10	3.14	0
6	10	10	3.14	0
7	10	10	3.14	4.04098*10^-194
8	10	10	3.14	1.76626*10^-85
9	10	10	3.14	2.81331*10^-20
1	10	20	3.14	6.71627
2	10	20	3.14	-4.96963E-86
3	10	20	3.14	-1.9897E-194
4	10	20	3.14	0
5	10	20	3.14	0
6	10	20	3.14	0
7	10	20	3.14	3.97947*10^-194
8	10	20	3.14	1.73937*10^-85
9	10	20	3.14	2.77049*10^-20
1	10	30	3.14	6.69308
2	10	30	3.14	-4.94402E-86
3	10	30	3.14	-1.9795E-194
4	10	30	3.14	0
5	10	30	3.14	0
6	10	30	3.14	0
7	10	30	3.14	3.95896*10^-194
8	10	30	3.14	1.73041*10^-85
9	10	30	3.14	2.75621*10^-20

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