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On Some Results about Finite Non Simple Groups

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Abstract

This paper is dedicated to study some properties of finite groups, where we present the following results: 1) If all centralizers of a group G are cyclic, then G is not simple. 2) If the center of the subgroup A is not trivial and B is cyclic, then G=AB is not simple. On the other hand, we define the m-solvable groups, and we examine some of their elementary properties.

Keywords: m-power closed group; m-solvable group; super solvable group

1.Introduction

The theory of groups is a rich material for mathematical research, where groups are studied and used in many other algebraic structures and games [15-18,26-27].

Many finite groups were studied in [1-3,7-14], their structures as products of subgroups were studied and handled.

Also, novel generalization of abelian groups and solvable groups were released recently [15], such as m-abelian solvable groups. These groups play an important role since they are solvable for prime numbers in the finite case.

In this work, we are motivated to present a novel generalization of solvable groups by using m-power closed groups. Also, we present many results about finite groups.

For definitions and properties of m-power closed groups and m-abelian solvable groups check [15,25].

Main Results:

The Janko simple group defined in [8] with order 175560 is not a product of any two proper subgroups.

Proof.

Every maximal subgroup of Janko group has one of the following orders:

$$2^3 \times 3 \times 7 \,, 2^2 \times 3 \times 5 \,, 2 \times 3 \times 7 \,, 2 \times 5 \times 11 \,, 2 \times 3 \times 19 \,, 2^3 \times 3 \times 5 \,, 2^3 \times 3 \times 5 \times 11$$

We assume that Janko group $G = AB$ where A, B are two subgroups.

Let $|A| = 2 \times 3 \times 19$, then $2^2 \times 5 \times 7 \times 11 \setminus |B|$which is a contradiction.

Theorem:

The Suzuki simple group $S_Z(2^{2k+1}) \cong A$ is not a product of any two proper subgroups.

proof.

Let $q = 2^{2k+1}$and $G \cong S_Z(q)$, then $|G| = q^2(q-1)(q^2+1)$. Each maximal subgroup of G is isomorphic to:

1). $M_1$with order $q^2(q-1)$.

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2). $M_2$with order $4(q+\sqrt{2q}+1)$.
3). $M_3$with order $4(q-\sqrt{2q}+1)$.
4). $M_4$with order $2(q-1)$.
5). $M_5$with order ${q_0}^2(q_0-1)({q_0}^2+1)$, where $q_0 = 2^t$ and $t$is a prime divisor of $2k+1$ [12].
We assume that $G = AB$, where A, B are two proper subgroups.
Suppose that $A = M_1$, then $q^2+1 \setminus |B|$.

$$q^2+1 > q+\sqrt{2q}+1 > q-1 > q-\sqrt{2q}+1$$

Hence $q^2+1 \nmid |M_j|$; $2 \le j \le 4$.
If $B = |M_s|$, then $q_0 - 1 \nmid q-1$ and $q^2+1 \setminus {q_0}^2+1$ which is not possible, so that $A \neq M_1$.

$$|M_2M_3| \le 2^4[(q+1)^2-2q] = 2^4(q^2+1)^2 < |G|$$

$$|M_2M_4| \le 2^4(q+\sqrt{2q}+1)(q-1) < |G|$$

And $|M_3M_4| < |M_2M_4| < |G|$
The only possible cases are $A = M_i$; $i = 2,3,4$ with $B = M_5$.
The 2-Sylow subgroup of Ghas order less or equal 2^{2t+2}.
We have $2t+2 < 2(2k+1)$ and the order of 2-sylow subgroup in G is equal to $q^2, q^{2(2k+1)}$, which is a contradiction

Theorem:

Let G be a pi-group and X is a proper subgroup with order divisible by 8, then one of the following properties is true:

1) $O(X) \setminus 6(q+1)$.
2) X is isomorphic to a subgroup of $N_G(S)$.
3) X is isomorphic to a subgroup of $C_G(i) = \langle i \rangle \times PSL(2, 3^{2n+1})$
4) $X \cong PSL(2,8)$ or $X \cong Aut(PSL(2,8))$.
5) X is isomorphic to a simple pi-group.

Proof.

Let $\alpha = 3(2n+1)$, we assume that$M = O(X) \neq 1$, $|N_G(M)| = 6(q+1)$, thus $|X| \setminus 6(q+1)$.
Let $O(X) = 1$, then if X is solvable, it has 2-sylow subgroup (invariant subgroup), hence is isomorphic to a subgroup of$N_G(s)$.
Now, we suppose that $X$is not solvable.
Let $K = O_2(X) = 1, X \setminus K$ is not solvable, thus $4 \setminus |X \setminus K|$which means that $|K| = 2$and$K \le Z(X)$, thus $X$is compatible with a subgroup of$\langle i \rangle \times PSL(2, 3^{2n+1})$.
In the last case, the 3-sylow subgroup has order 3^{2n+1}, then $|X_3| \le 3^{2n+1} \le 3^{\alpha-2(2n+1)} \le 3^{\alpha-6}$.
Now, let $O_2(X) = 1$, we denote by N to the minimal invariant subgroup of X.
The subgroup N is simple with $C_X(N) \cap N = 1$, thus $X \cong Aut(N)$

Theorem:

If the centralizer A is not trivial and $B$is B-Dedkind, then $G = AB$ is not simple.

Proof.

We assume that the theorem is not true and G is simple with minimal order.
Let Mbe a maximal subgroup of Gwhich contains$A$, $G = MB$, $M = A(M \cap B)$.
If$M \cap B \neq 1$, then$M \cap B \lhd B$ and $(M \cap B)^G = (M \cap B)^A \le M$.
By the maximality of A, we get that $G$is primitive group and B is 2-transitive.
Now, let $x$be an element of prime order P from A, for $P = 2$ we will get a contradiction.
This means that $P > 2$ and $\langle x \rangle = P$ is cyclic normal subgroup of A with order P.
G will has one of the following properties:
1). G has a normal subgroup.
2). $P^G \cong PSL(2,q)$ or $P^G \cong PSU(3,q^2)$.
3). $P^G \cong PSL(2,8)$ and $G \cong R(3)$ with power 28.
By the assumption, G is simple and $P^G = G$.
G is isomorphic to $PSL(2,q)$ or $PSU(3,q^2)$, hence $A$is compatible with the centralizer of the Sylow subgroup$Q$ of G with $q \setminus |Q|$.
This is a contradiction, hence $G \cong PSU(3,q^2)$.
$|Q| = q^3$ and $A \setminus Q$ is cyclic with order $\frac{1}{d}(q^2-1)$, where $d = (3, q-1)$, see [4].
If q is odd, then the 2-sylow ssubgroup of A is not trivial and cyclic, thus G is not imple [11].
Also, $q = 2^n$ and $C_G(Q) = Z(Q)$ which contradicts $x \in C_G(Q)$.

Result:

If the centralizer $A$is not trivial and $B$is syclic, then $G = AB$ is not simple.

m-solvable groups

Definition:
Let G be a group, consider the following subnormal series $H_0 = \{e\} \leq H_1 \leq H_2 \leq \cdots \leq H_n = G$ then G is called m-solvable group for a fixed positive integer m if $H_i \setminus H_{i-1}$ is m-group $1 \leq i \leq n$.

Remark:
1). G is m-solvable group if and only if $H_{i-1} \lhd_m H_i$.
2). Every solvable group is m-solvable group for all $m \in N$.

Theorem:
Let G be m-solvable group and H be a subgroup of G, then H is m-solvable.

Proof.
We consider the m-solvable series $\{e\} = H_0 \leq H_1 \leq H_2 \leq \cdots \leq H_n = G$.
We have the subnormal series $\{e\} = H_0 \cap H \leq H_1 \cap H \leq \cdots \leq H_n \cap H = H$.

$$H_i \cap H \setminus H_{i-1} \cap H = H_i \cap H \setminus (H_i \cap H_{i-1}) \cap H \cong H_{i-1}(H_i \cap H) \setminus H_{i-1}$$

Now, we must prove that, $H_{i-1} \lhd_m H_{i-1}(H_i \cap H)$.
Let $x, y \in H_{i-1}(H_i \cap H)$.

Theorem:
Let G be m-solvable group, then the homomorphic image is m-solvable.

Proof.
Let $f: G \to K$ be a group homomorphism. Consider the following m-solvable series.

$$\{e\} = H_0 \leq H_1 \leq H_2 \leq \cdots \leq H_n = G$$

Hence, $\{e\} = f(H_0) \leq f(H_1) \leq \cdots \leq f(G)$. Now, we must the factor $f(H_i) \setminus f(H_{i-1})$.
$\forall\, x_1, y_1 \in f(H_i)$ that $x_1 = f(x)\,, y_1 = f(y)$.
There exists $z \in H_i$ such that $z^m x^m y^m \in H_{i-1}$.
Hence, $[f(z)]^m . [f(x)]^m . [f(y)]^m \in f(H_{i-1})$.
This implies that $f(H_{i-1}) \lhd_m f(H_i)$.

Theorem:
Let G be a group and H be normal in G, if $H, G \setminus H$ are m-solvable, then G is m-solvable.

Proof.
Consider the following two m-solvable series.

$$\{e\} = H_0 \leq H_1 \leq H_2 \leq \cdots \leq H_k = H$$
$$\{e\} = K_1 \setminus H \leq K_2 \setminus H \leq \cdots \leq K_n \setminus H = G \setminus H$$

We have: $K_i \setminus H \setminus K_{i-1} \setminus H \cong K_i \setminus K_{i-1}$ is m-group.
Thus:

$$\{e\} = H_0 \leq H_1 \leq H_2 \leq \cdots \leq H \leq K_2 \leq \cdots \leq K_n = G$$

Then, G is m-solvable.

Example:
Consider the group $G = S_5$, we have $\{e\} = A_5 \leq S_5$, $|S_5 \setminus A_5| = 2\,, |A_5| = 60$. Hence S_5 is 60-abelian solvable.

Conclusion

In this paper, we have studied some of elementary properties of finite groups, where we proved the following results:

1). If all centralizers of a group G are cyclic, then $G$is not simple.
2). If the centre of the subgroup A is not trivial and B is cyclic, then $G = AB$ is not simple.
Also, we have generalized the concept of m-abelian solvable groups by m-power closed groups and presented some properties of this new generalization in terms of theorems.

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