Bipolar Hypersoft Homeomorphism Maps and Bipolar Hypersoft Compact Spaces

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Abstract

Herein, we further contribute and promote topological structures via bipolar hypersoft (BHS) setting by introducing new types of maps called BHS continuous, BHS open, BHS closed, and BHS homeomorphism maps. We investigate their characterizations and establish their main properties. By providing a thorough picture of the proposed maps, we investigate the concept of BHS compact space and obtain several results relating to this concept. We point out that BH compactness preserved under BH continuous map. The relationships among these concepts with their counterparts in hypersoft (HS) structures are discussed.

Keywords: BHS continuous map; BHS open map; BHS closed map; BHS homeomorphism map; BHS compact space

1 Introduction

Molodtsov[1] introduced soft sets, a new mathematical method for dealing with vagueness, in 1999. Many scholars and researchers have researched soft set applications in various fields such as decision-making,[2] forecasting,[3] computer science,[4] data mining,[5] and medical diagnosis.[6]

In a variety of everyday problems, parametric values are further separated into sets of disjoint attribute-values. Existing soft set theory is inadequate for dealing with attributive-valued sets. To tackle real-life scenarios, HS set theory,[7,8] defined by Smarandache in 2018, was specifically designed to make soft set theory in keeping with attributive-valued sets. Certain properties, operations, laws, relations, mappings, and applications of HS set theory and its extensions were discussed in.[9–21] The utilization of HS set theory in defining hypersoft topological space (HSTS),[22] HS connected spaces,[23] HS separation axioms,[24] HS continuity and HS compact spaces[25] were also explored.

Recently, Musa and Asaad[26] proposed the concept of BHS set. It consists of two HS sets, one that provides positive information and the other that provides negative information. Under BHS set environment, they defined set-theoretic operations and discussed some of their properties. This concept was used by Musa and Asaad to define bipolar hypersoft topological space (BHSTS) in.[27] Many subsequent research, such as BHS connected spaces[28] and BHS separation axioms,[29,30] have looked at topological notions in bipolar hypersoft topologies (BHSTs). The authors,[31] defined mappings between BHS families. In order to contributes to this concept we define, in this article, continuity and compactness in the context of BHS sets.

This article is presented as follows: BH set theory and BHST are discussed first. Section 3 introduces and investigates BHS continuous, BHS open, BHS closed, and BHS homeomorphism maps, which are all new types of BHS maps. In section 4, we study BHS compact space and discuss some of its characteristics. Finally, in section 5, we conclude the paper and make recommendations for further research.

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2 Preliminary Concepts

2.1 Bipolar Hypersoft Sets

Throughout this work, $\Re$ denotes the universal set; $2^{\Re}$ denotes the set of all subsets of $\Re$, and Σ denotes the universal parameter set where $\Sigma = \sigma_1 \times \sigma_2 \times \ldots \times \sigma_n$ with $\sigma_i \cap \sigma_j = \varphi$ and $i \neq j$. Also, $\Lambda, \Delta \subseteq \Sigma$.

Definition 2.1. [26] A triple $(g, \widehat{g}, \Lambda)$ is called a BHS set over $\Re$, where g and $\widehat{g}$ are mappings given by $g : \Lambda \to 2^{\Re}$ and $\widehat{g} : \neg\Lambda \to 2^{\Re}$ such that $g(\ell) \cap \widehat{g}(\neg\ell) = \varphi$ for all $\ell \in \Lambda$.

We write $(g, \widehat{g}, \Lambda) = \{(\ell, g(\ell), \widehat{g}(\neg\ell)) : \ell \in \Lambda \text{ and } g(\ell) \cap \widehat{g}(\neg\ell) = \varphi\}$.

The collection of all BHS sets on $\Re$ with the set of parameters Σ is denoted by $\mho_{(\Re,\Sigma)}$.

Definition 2.2. [26] Let $(g_1, \widehat{g_1}, \Lambda)$, $(g_2, \widehat{g_2}, \Delta) \in \mho_{(\Re,\Sigma)}$. Then

i. $(g_1, \widehat{g_1}, \Lambda)$ is a BHS subset of $(g_2, \widehat{g_2}, \Delta)$, denoted by $(g_1, \widehat{g_1}, \Lambda) \,\widetilde{\widetilde{\sqsubseteq}}\, (g_2, \widehat{g_2}, \Delta)$, if $\Lambda \subseteq \Delta$ and $g_1(\ell) \subseteq g_2(\ell)$, $\widehat{g_2}(\neg\ell) \subseteq \widehat{g_1}(\neg\ell)$ for all $\ell \in \Lambda$.

ii. $(g_1, \widehat{g_1}, \Lambda)$ and $(g_2, \widehat{g_2}, \Delta)$ are BHS equal, if $(g_1, \widehat{g_1}, \Lambda) \,\widetilde{\widetilde{\sqsubseteq}}\, (g_2, \widehat{g_2}, \Delta)$ and $(g_2, \widehat{g_2}, \Delta) \,\widetilde{\widetilde{\sqsubseteq}}\, (g_1, \widehat{g_1}, \Lambda)$.

iii. If $g_1(\ell) = \varphi$ and $\widehat{g_1}(\neg\ell) = \Re$ for all $\ell \in \Lambda$, then $(g_1, \widehat{g_1}, \Lambda)$ is called a relative null BHS set and denoted by $(\Phi, \widetilde{\Re}, \Lambda)$.

iv. If $g_1(\ell) = \Re$ and $\widehat{g_1}(\neg\ell) = \varphi$ for all $\ell \in \Lambda$, then $(g_1, \widehat{g_1}, \Lambda)$ is called a relative whole BHS set and denoted by $(\widetilde{\Re}, \Phi, \Lambda)$.

v. The complement of $(g_1, \widehat{g_1}, \Lambda)$ is a BHS set $(g_1, \widehat{g_1}, \Lambda)^c = (g_1^c, \widehat{g_1}^c, \Lambda)$ where $g_1^c(\ell) = \widehat{g_1}(\neg\ell)$ and $\widehat{g_1}^c(\neg\ell) = g_1(\ell)$ for all $\ell \in \Lambda$.

vi. The union of $(g_1, \widehat{g_1}, \Lambda)$ and $(g_2, \widehat{g_2}, \Delta)$, denoted by $(g_1, \widehat{g_1}, \Lambda) \,\widetilde{\widetilde{\sqcup}}\, (g_2, \widehat{g_2}, \Delta)$, is a BHS set $(g, \widehat{g}, C)$, where $C = \Lambda \cap \Delta$ and for all $\ell \in C$: $g(\ell) = g_1(\ell) \cup g_2(\ell)$ and $\widehat{g}(\neg\ell) = \widehat{g_1}(\neg\ell) \cap \widehat{g_2}(\neg\ell)$.

vii. The intersection of $(g_1, \widehat{g_1}, \Lambda)$ and $(g_2, \widehat{g_2}, \Delta)$, denoted by $(g_1, \widehat{g_1}, \Lambda) \,\widetilde{\widetilde{\sqcap}}\, (g_2, \widehat{g_2}, \Delta)$, is a BHS set $(g, \widehat{g}, C)$, where $C = \Lambda \cap \Delta$ and for all $\ell \in C$: $g(\ell) = g_1(\ell) \cap g_2(\ell)$ and $\widehat{g}(\neg\ell) = \widehat{g_1}(\neg\ell) \cup \widehat{g_2}(\neg\ell)$.

Definition 2.3. [27] Let $(g, \widehat{g}, \Sigma)$ be a BHS set over $\Re$ and Υ be a non-empty subset of $\Re$. Then the sub BHS set of $(g, \widehat{g}, \Sigma)$ over Υ denoted by $(g_\Upsilon, \widehat{g}_\Upsilon, \Sigma)$, is defined as follows:

$$g_\Upsilon(\ell) = \Upsilon \cap g(\ell) \text{ and } \widehat{g}_\Upsilon(\neg\ell) = \Upsilon \cap \widehat{g}(\neg\ell), \text{ for each } \ell \in \Sigma.$$

Definition 2.4. [31] Let $\gamma : \Re \to \aleph$ be an injective map. Let $\delta : \Sigma \to \acute{\Sigma}$ and $\lambda : \neg\Sigma \to \neg\acute{\Sigma}$ be two maps such that $\lambda(\neg\ell) = \neg\delta(\ell)$ for all $\neg\ell \in \neg\Sigma$ and $\Psi_{\gamma\delta\lambda} : \mho_{(\Re,\Sigma)} \to \mho_{(\aleph,\acute{\Sigma})}$ be a BHS map. Then:

1. The image of $(g, \widehat{g}, \Lambda)$, $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Lambda)) = (\Psi_{\gamma\delta\lambda}(g), \Psi_{\gamma\delta\lambda}(\widehat{g}), \acute{\Sigma})$ is a BHS set in $\mho_{(\aleph,\acute{\Sigma})}$ given as, for all $\acute{\ell} \in \acute{\Sigma}$

$$\Psi_{\gamma\delta\lambda}(g)(\acute{\ell}) = \begin{cases} \gamma\left(\bigcup_{\ell\in\delta^{-1}(\acute{\ell})\cap\Lambda} g(\ell)\right), & \text{if } \delta^{-1}(\acute{\ell}) \cap \Lambda \neq \varphi \\ \varphi, & \text{otherwise} \end{cases}$$

$$\Psi_{\gamma\delta\lambda}(\widehat{g})(\neg\acute{\ell}) = \begin{cases} \gamma\left(\bigcap_{\neg\ell\in\lambda^{-1}(\neg\acute{\ell})\cap\neg\Lambda} \widehat{g}(\neg\ell)\right), & \text{if } \lambda^{-1}(\neg\acute{\ell}) \cap \neg\Lambda \neq \varphi \\ \aleph, & \text{otherwise} \end{cases}$$

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2. The inverse image of $(f, \widehat{f}, \acute{\Lambda})$, $\Psi^{-1}_{\gamma\delta\acute{\Lambda}}((f, \widehat{f}, \acute{\Lambda})) = (\Psi^{-1}_{\gamma\delta\lambda}(f), \Psi^{-1}_{\gamma\delta\lambda}(\widehat{f}), \Sigma)$ is a BHS set in $\mho_{(\Re,\Sigma)}$ given as, for all $\ell \in \Sigma$

$$\Psi^{-1}_{\gamma\delta\lambda}(f)(\ell) = \begin{cases} \gamma^{-1}\left(f(\delta(\ell))\right), & \text{if } \delta(\ell) \in \acute{\Lambda} \\ \varphi, & \text{if } \delta(\ell) \notin \acute{\Lambda} \end{cases}$$

$$\Psi^{-1}_{\gamma\delta\lambda}(\widehat{f})(\neg\ell) = \begin{cases} \gamma^{-1}(\widehat{f}(\lambda(\neg\ell))), & \text{if } \lambda(\neg\ell) \in \neg\acute{\Lambda} \\ \Re, & \text{if } \lambda(\neg\ell) \notin \neg\acute{\Lambda} \end{cases}$$

Definition 2.5. [31] We call a BHS map $\Psi_{\gamma\delta\lambda}$ BHS surjective (resp., BHS injective, BHS bijective) if the maps γ and δ are surjective (resp., injective, bijective).

Proposition 2.6. *If $\Psi_{\gamma\delta\lambda} : \mho_{(\Re,\Sigma)} \to \mho_{(\aleph,\acute{\Sigma})}$ is a BHS bijective map, then $\Psi^{-1}_{\gamma\delta\lambda} : \mho_{(\aleph,\acute{\Sigma})} \to \mho_{(\Re,\Sigma)}$ is also a BHS bijective map.*

Proof. Let $(f_1, \widehat{f_1}, \acute{\Sigma}) \neq (f_2, \widehat{f_2}, \acute{\Sigma}) \in \mho_{(\aleph,\acute{\Sigma})}$. Since $\Psi_{\gamma\delta\lambda}$ is a BHS bijective map, then there exist $(g_1, \widehat{g_1}, \Sigma) \neq (g_2, \widehat{g_2}, \Sigma) \in \mho_{(\Re,\Sigma)}$ such that $\Psi_{\gamma\delta\lambda}(g_1, \widehat{g_1}, \Sigma) = (f_1, \widehat{f_1}, \acute{\Sigma})$ and $\Psi_{\gamma\delta\lambda}(g_2, \widehat{g_2}, \Sigma) = (f_2, \widehat{f_2}, \acute{\Sigma})$. Also, we have $(g_1, \widehat{g_1}, \Sigma) = \Psi^{-1}_{\gamma\delta\lambda}(f_1, \widehat{f_1}, \acute{\Sigma})$ and $(g_2, \widehat{g_2}, \Sigma) = \Psi^{-1}_{\gamma\delta\lambda}(f_2, \widehat{f_2}, \acute{\Sigma})$. So, $\Psi^{-1}_{\gamma\delta\lambda}(f_1, \widehat{f_1}, \acute{\Sigma}) \neq \Psi^{-1}_{\gamma\delta\lambda}(f_2, \widehat{f_2}, \acute{\Sigma})$ and $\Psi^{-1}_{\gamma\delta\lambda}$ is BHS injective map. Now, let $(g, \widehat{g}, \Sigma) \in \mho_{(\Re,\Sigma)}$. Then, there exists $(f, \widehat{f}, \acute{\Sigma}) \in \mho_{(\aleph,\acute{\Sigma})}$ such that $\Psi_{\gamma\delta\lambda}(g, \widehat{g}, \Sigma) = (f, \widehat{f}, \acute{\Sigma})$. Since $\Psi_{\gamma\delta\lambda}$ is BHS injective map, then $(g, \widehat{g}, \Sigma) = \Psi^{-1}_{\gamma\delta\lambda}(f, \widehat{f}, \acute{\Sigma})$. Hence, $\Psi^{-1}_{\gamma\delta\lambda}$ is BHS surjective map. Consequently, $\Psi^{-1}_{\gamma\delta\lambda}$ is BHS bijective map. ☐

Definition 2.7. Let $\Psi_{\gamma\delta\lambda} : \mho_{(\Re,\Sigma)} \to \mho_{(\aleph,\acute{\Sigma})}$ and $\Theta_{\gamma\delta\lambda} : \mho_{(\aleph,\acute{\Sigma})} \to \mho_{(M,\acute{\acute{\Sigma}})}$ be two BHS maps. Then the BHS composite map $\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda} : \mho_{(\Re,\Sigma)} \to \mho_{(M,\acute{\acute{\Sigma}})}$ is defined by $(\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda})((g, \widehat{g}, \Sigma)) = \Theta_{\gamma\delta\lambda}(\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)))$ for $(g, \widehat{g}, \Sigma) \in \mho_{(\Re,\Sigma)}$.

Proposition 2.8. *Let $\Psi_{\gamma\delta\lambda} : \mho_{(\Re,\Sigma)} \to \mho_{(\aleph,\acute{\Sigma})}$ and $\Theta_{\gamma\delta\lambda} : \mho_{(\aleph,\acute{\Sigma})} \to \mho_{(M,\acute{\acute{\Sigma}})}$ be BHS bijective maps. Then $\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda} : \mho_{(\Re,\Sigma)} \to \mho_{(M,\acute{\acute{\Sigma}})}$ is also a BHS bijective map and $(\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda})^{-1} = \Psi^{-1}_{\gamma\delta\lambda} \circ \Theta^{-1}_{\gamma\delta\lambda}$.*

Proof. Let $(g_1, \widehat{g_1}, \Sigma) \neq (g_2, \widehat{g_2}, \Sigma) \in \mho_{(\Re,\Sigma)}$. Since $\Psi_{\gamma\delta\lambda}$ is BHS injective map, then $\Psi_{\gamma\delta\lambda}((g_1, \widehat{g_1}, \Sigma)) \neq \Psi_{\gamma\delta\lambda}((g_2, \widehat{g_2}, \Sigma))$. Again, since $\Psi_{\gamma\delta\lambda}$ is BHS surjective map, then there exists $(g, \widehat{g}, \Sigma) \in \mho_{(\Re,\Sigma)}$ such that $\Psi_{\gamma\delta\lambda}(g, \widehat{g}, \Sigma) = (f, \widehat{f}, \acute{\Sigma})$. Then, $\Theta_{\gamma\delta\lambda}(\Psi_{\gamma\delta\lambda}(g, \widehat{g}, \Sigma) = (h, \widehat{h}, \acute{\acute{\Sigma}})$ and hence $\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda}$ is a BHS surjective map. Therefore, $\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda}$ is BHS bijective map. Next, let $(g, \widehat{g}, \Sigma) \in \mho_{(\Re,\Sigma)}$, $(f, \widehat{f}, \acute{\Sigma}) \in \mho_{(\aleph,\acute{\Sigma})}$, and $(h, \widehat{h}, \acute{\acute{\Sigma}}) \in \mho_{(M,\acute{\acute{\Sigma}})}$ such that $\Psi_{\gamma\delta\lambda}(g, \widehat{g}, \Sigma) = (f, \widehat{f}, \acute{\Sigma})$ and $\Theta_{\gamma\delta\lambda}(f, \widehat{f}, \acute{\Sigma}) = (h, \widehat{h}, \acute{\acute{\Sigma}})$. Since, $\Psi_{\gamma\delta\lambda}$ and $\Theta_{\gamma\delta\lambda}$ are BHS injective maps, then $(g, \widehat{g}, \Sigma) = \Psi^{-1}_{\gamma\delta\lambda}(f, \widehat{f}, \acute{\Sigma})$ and $(f, \widehat{f}, \acute{\Sigma}) = \Theta^{-1}_{\gamma\delta\lambda}(h, \widehat{h}, \acute{\acute{\Sigma}})$. Now, $(\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda})((g, \widehat{g}, \Sigma)) = \Theta_{\gamma\delta\lambda}(\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma))) = \Theta_{\gamma\delta\lambda}(f, \widehat{f}, \acute{\Sigma}) = (h, \widehat{h}, \acute{\acute{\Sigma}})$. Since, $\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda}$ is BHS injective map, then $(\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda})^{-1}((h, \widehat{h}, \acute{\acute{\Sigma}})) = (g, \widehat{g}, \Sigma)$. Also, $(\Psi^{-1}_{\gamma\delta\lambda} \circ \Theta^{-1}_{\gamma\delta\lambda})((h, \widehat{h}, \acute{\acute{\Sigma}})) = \Psi^{-1}_{\gamma\delta\lambda}(\Theta^{-1}_{\gamma\delta\lambda}((h, \widehat{h}, \acute{\acute{\Sigma}}))) = \Psi^{-1}_{\gamma\delta\lambda}((f, \widehat{f}, \acute{\Sigma})) = (g, \widehat{g}, \Sigma)$. Hence, $(\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda})^{-1} = \Psi^{-1}_{\gamma\delta\lambda} \circ \Theta^{-1}_{\gamma\delta\lambda}$. ☐

2.2 Bipolar Hypersoft Topological Spaces

Definition 2.9. [27] Let $\mathcal{T}_{\Re}$ be the collection of BHS sets over $\Re$, then $\mathcal{T}_{\Re}$ is said to be a BHST on $\Re$ if

1. $(\Phi, \widetilde{\Re}, \Sigma)$, $(\widetilde{\Re}, \Phi, \Sigma)$ belong to $\mathcal{T}_{\Re}$;
2. the intersection of any two BHS sets in $\mathcal{T}_{\Re}$ belongs to $\mathcal{T}_{\Re}$;

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3. the union of any number of BHS sets in $\mathcal{T}_{\Re}$ belongs to $\mathcal{T}_{\Re}$.

Then $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$ is called a BHSTS over $\Re$.

Definition 2.10. [27] Let $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$ be a BHSTS over $\Re$ and Υ be a non-empty subset of $\Re$. Then

$$\mathcal{T}_{\Re_{\Upsilon}} = \{(g_{\Upsilon}, \widehat{g}_{\Upsilon}, \Sigma) \mid (g, \widehat{g}, \Sigma) \,\widetilde{\widetilde{\in}}\, \mathcal{T}_{\Re}\}$$

is said to be the relative BHST and $(\Upsilon, \mathcal{T}_{\Re_{\Upsilon}}, \Sigma, \neg\Sigma)$ is called a BHS subspace of $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$

Definition 2.11. [27] Let $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$ be a BHSTS and $(g, \widehat{g}, \Sigma)$ be a BHS set over $\Re$. Then:

1. the intersection of all BHS closed supersets of $(g, \widehat{g}, \Sigma)$ is called the BHS closure of $(g, \widehat{g}, \Sigma)$ and is denoted by $\overline{(g, \widehat{g}, \Sigma)}$.
2. the union of all BHS open subsets of $(g, \widehat{g}, \Sigma)$ is called the BHS interior of $(g, \widehat{g}, \Sigma)$ and is denoted by $(g, \widehat{g}, \Sigma)^o$.

Proposition 2.12. [27] *Let $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$ be a BHSTS over $\Re$. Then the collection $\mathcal{T}^h_{\Re} = \{(g, \Sigma) \mid (g, \widehat{g}, \Sigma) \,\widetilde{\widetilde{\in}}\, \mathcal{T}_{\Re}\}$ defines hypersoft topology (HST) on $\Re$.*

Proposition 2.13. [27] *Suppose that $(\Re, \mathcal{T}^h_{\Re}, \Sigma)$ is a HSTS over $\Re$. Then $\mathcal{T}_{\Re}$ consisting of BHS sets $(g, \widehat{g}, \Sigma)$ such that $(g, \Sigma) \,\widetilde{\in}\, \mathcal{T}^h_{\Re}$ and $\widehat{g}(\neg\ell) = \Re \setminus g(\ell)$ for all $\neg\ell \in \neg\Sigma$, defines a BHST over $\Re$.*

Definition 2.14. [25] A HS map $\Psi_{\gamma\delta} : (\Re, \mathcal{T}^h_{\Re}, \Sigma) \to (\aleph, \mathcal{T}^h_{\aleph}, \acute{\Sigma})$ is said to be HS continuous if $\Psi^{-1}_{\gamma\delta}((f, \acute{\Sigma})) \in \mathcal{T}^h_{\Re}$ for every $(f, \acute{\Sigma}) \in \mathcal{T}^h_{\aleph}$.

Definition 2.15. [25] A HS map $\Psi_{\gamma\delta} : (\Re, \mathcal{T}^h_{\Re}, \Sigma) \to (\aleph, \mathcal{T}^h_{\aleph}, \acute{\Sigma})$ is said to be:

1. HS open if the HS image of every HS open set is HS open.
2. HS closed if the HS image of every HS closed set is HS closed.

Definition 2.16. [25] A HSTS $(\Re, \mathcal{T}^h_{\Re}, \Sigma)$ is said to be HS compact space if each HS open cover of $(\widetilde{\Re}, \Sigma)$ has finite HS subcover.

3 Bipolar Hypersoft Homeomorphism Maps

BH continuous, BH open, BH closed, and BH homeomorphism maps, which are novel types of BHS maps, are discussed in this section. We look into their descriptions and look for key characteristics.

Definition 3.1. A BHS map $\Psi_{\gamma\delta\lambda} : (\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$ is said to be BHS continuous if $\Psi^{-1}_{\gamma\delta\lambda}((f, \widehat{f}, \acute{\Sigma})) \in \mathcal{T}_{\Re}$ for every $(f, \widehat{f}, \acute{\Sigma}) \in \mathcal{T}_{\aleph}$.

Example 3.2. Let $\Re = \{r_1, r_2, r_3\}$ and $\aleph = \{\eta_1, \eta_2, \eta_3, \eta_4\}$ be two sets, $\sigma_1 = \{\ell_1, \ell_2, \ell_3, \ell_4\}, \sigma_2 = \{\ell_5\}, \sigma_3 = \{\ell_6\}$, and $\acute{\sigma_1} = \{\acute{\ell_1}, \acute{\ell_2}, \acute{\ell_3}, \acute{\ell_4}\}, \acute{\sigma_2} = \{\acute{\ell_5}\}, \acute{\sigma_3} = \{\acute{\ell_6}\}$ be sets of parameters, $\gamma : \Re \to \aleph$ be a mapping defined as $\gamma(r_i) = \eta_i$ for $i = 1, 2, 3$, the mapping $\delta : \Sigma \to \acute{\Sigma}$ be defined as $\delta((\ell_1, \ell_5, \ell_6)) = \delta((\ell_2, \ell_5, \ell_6)) = (\acute{\ell_1}, \acute{\ell_5}, \acute{\ell_6})$, $\delta((\ell_3, \ell_5, \ell_6)) = (\acute{\ell_3}, \acute{\ell_5}, \acute{\ell_6})$, $\delta((\ell_4, \ell_5, \ell_6)) = (\acute{\ell_4}, \acute{\ell_5}, \acute{\ell_6})$, the mapping $\lambda : \neg\Sigma \to \neg\acute{\Sigma}$ be defined as $\lambda(\neg\ell_i) = \neg\delta(\ell_i)$ for $i = 1, 2, 3$. Let $\mathcal{T}_{\Re} = \{(\Phi, \widetilde{\Re}, \Sigma), (\widetilde{\Re}, \Phi, \Sigma), (g, \widehat{g}, \Sigma)\}$ and $\mathcal{T}_{\aleph} = \{(\Phi, \widetilde{\aleph}, \acute{\Sigma}), (\widetilde{\aleph}, \Phi, \acute{\Sigma}), (f, \widehat{f}, \acute{\Sigma})\}$ be two BHSTs defined respectively on $\Re$ and $\aleph$ where $(g, \widehat{g}, \Sigma)$ and $(f, \widehat{f}, \acute{\Sigma})$ are BHS sets defined as follows

$(g, \widehat{g}, \Sigma) = \{((\ell_1, \ell_5, \ell_6), \varphi, \{r_3\}), ((\ell_2, \ell_5, \ell_6), \varphi, \{r_3\}), ((\ell_3, \ell_5, \ell_6), \{r_1\}, \{r_3\}), ((\ell_4, \ell_5, \ell_6), \varphi, \Re)\}$, and
$(f, \widehat{f}, \acute{\Sigma}) = \{((\acute{\ell_1}, \acute{\ell_5}, \acute{\ell_6}), \{\eta_4\}, \{\eta_3\}), ((\acute{\ell_2}, \acute{\ell_5}, \acute{\ell_6}), \{\eta_1\}, \{\eta_2\}), ((\acute{\ell_3}, \acute{\ell_5}, \acute{\ell_6}), \{\eta_1\}, \{\eta_3, \eta_4\}), ((\acute{\ell_4}, \acute{\ell_5}, \acute{\ell_6}), \varphi, \aleph)\}$.

Then, it is easy to see that $\Psi_{\gamma\delta\lambda} : (\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$ is BHS continuous map.

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Proposition 3.3. *Let $\Psi_{\gamma\delta\lambda} : (\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma) \rightarrow (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$ be a BHS map, then the following statements are equivalent:*

i. $\Psi_{\gamma\delta\lambda}$ *is a BHS continuous map.*

ii. *The BHS inverse image of each BHS closed set is BHS closed set.*

iii. $\overline{\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma}))} \,\tilde{\tilde{\sqsubseteq}}\, \Psi^{-1}_{\gamma\delta\lambda}(\overline{(f,\widehat{f},\acute{\Sigma})})$ *for all* $(f,\widehat{f},\acute{\Sigma}) \,\tilde{\tilde{\sqsubseteq}}\, (\widetilde{\aleph}, \Phi, \acute{\Sigma})$.

iv. $\Psi_{\gamma\delta\lambda}(\overline{(g,\widehat{g},\Sigma)}) \,\tilde{\tilde{\sqsubseteq}}\, \overline{\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma))}$ *for all* $(g,\widehat{g},\Sigma) \,\tilde{\tilde{\sqsubseteq}}\, (\widetilde{\Re}, \Phi, \Sigma)$.

v. $\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})^o) \,\tilde{\tilde{\sqsubseteq}}\, (\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})))^o$ *for all* $(f,\widehat{f},\acute{\Sigma}) \,\tilde{\tilde{\sqsubseteq}}\, (\widetilde{\aleph}, \Phi, \acute{\Sigma})$.

Proof. (i.) $\Rightarrow$ (ii.): Let $(f,\widehat{f},\acute{\Sigma})$ be a BHS closed set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$, then $\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma}))^c$ is a BHS open subset of $(\widetilde{\Re}, \Phi, \Sigma)$. But, $\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma}))^c = (\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})))^c$, then $\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma}))$ is a BHS closed subset of $(\widetilde{\Re}, \Phi, \Sigma)$.

(ii.) $\Rightarrow$ (iii.): Obviously, $\overline{(f,\widehat{f},\acute{\Sigma})}$ is a BHS closed subset of $(\widetilde{\aleph}, \Phi, \acute{\Sigma})$. From (ii.), $\Psi^{-1}_{\gamma\delta\lambda}(\overline{(f,\widehat{f},\acute{\Sigma})})$ is a BHS closed subset of $(\widetilde{\Re}, \Phi, \Sigma)$ and hence $\overline{\Psi^{-1}_{\gamma\delta\lambda}(\overline{(f,\widehat{f},\acute{\Sigma})})} = \Psi^{-1}_{\gamma\delta\lambda}(\overline{(f,\widehat{f},\acute{\Sigma})})$. Obviously, $(f,\widehat{f},\acute{\Sigma}) \,\tilde{\tilde{\sqsubseteq}}\, \overline{(f,\widehat{f},\acute{\Sigma})}$ and $\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})) \,\tilde{\tilde{\sqsubseteq}}\, \Psi^{-1}_{\gamma\delta\lambda}(\overline{(f,\widehat{f},\acute{\Sigma})})$. Then, $\overline{\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma}))} \,\tilde{\tilde{\sqsubseteq}}\, \overline{\Psi^{-1}_{\gamma\delta\lambda}(\overline{(f,\widehat{f},\acute{\Sigma})})} = \Psi^{-1}_{\gamma\delta\lambda}(\overline{(f,\widehat{f},\acute{\Sigma})}) = \Psi^{-1}_{\gamma\delta\lambda}(\overline{(f,\widehat{f},\acute{\Sigma})})$.

(iii.) $\Rightarrow$ (iv.): Let $(g,\widehat{g},\Sigma) \,\tilde{\tilde{\sqsubseteq}}\, (\widetilde{\Re}, \Phi, \Sigma)$, then $\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma)) \,\tilde{\tilde{\sqsubseteq}}\, (\widetilde{\aleph}, \Phi, \acute{\Sigma})$. From (iii.), $\overline{\Psi^{-1}_{\gamma\delta\lambda}(\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma)))} \,\tilde{\tilde{\sqsubseteq}}\, \Psi^{-1}_{\gamma\delta\lambda}(\overline{\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma))})$. Then, $\overline{(g,\widehat{g},\Sigma)} \,\tilde{\tilde{\sqsubseteq}}\, \overline{\Psi^{-1}_{\gamma\delta\lambda}(\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma)))} \,\tilde{\tilde{\sqsubseteq}}\, \Psi^{-1}_{\gamma\delta\lambda}(\overline{\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma))})$. It follows that, $\Psi_{\gamma\delta\lambda}(\overline{(g,\widehat{g},\Sigma)}) \,\tilde{\tilde{\sqsubseteq}}\, \Psi_{\gamma\delta\lambda}(\Psi^{-1}_{\gamma\delta\lambda}(\overline{\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma))})) \,\tilde{\tilde{\sqsubseteq}}\, \overline{\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma))}$.

(iv.) $\Rightarrow$ (v.): Let $(f,\widehat{f},\acute{\Sigma}) \,\tilde{\tilde{\sqsubseteq}}\, (\widetilde{\aleph}, \Phi, \acute{\Sigma})$, then apply (iv.) to $(f,\widehat{f},\acute{\Sigma})^c$, we obtain $\Psi_{\gamma\delta\lambda}(\overline{\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})^c)}) \,\tilde{\tilde{\sqsubseteq}}\, \overline{\Psi_{\gamma\delta\lambda}(\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})^c))}$. Hence, $\Psi_{\gamma\delta\lambda}(((\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})))^o)^c) \,\tilde{\tilde{\sqsubseteq}}\, \overline{(f,\widehat{f},\acute{\Sigma})^c} = ((f,\widehat{f},\acute{\Sigma})^o)^c$. Thus, $((\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})))^o)^c \,\tilde{\tilde{\sqsubseteq}}\, \Psi^{-1}_{\gamma\delta\lambda}(((f,\widehat{f},\acute{\Sigma})^o)^c)$. Therefore, $\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})^o) \,\tilde{\tilde{\sqsubseteq}}\, (\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})))^o$.

(v.) $\Rightarrow$ (i.): Let $(f,\widehat{f},\acute{\Sigma})$ be a BHS open subset of $(\widetilde{\aleph}, \Phi, \acute{\Sigma})$. From (v.), $\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})^o) = \Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})) \,\tilde{\tilde{\sqsubseteq}}\, (\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})))^o$. But, $(\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})))^o \,\tilde{\tilde{\sqsubseteq}}\, \Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma}))$. So, $(\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})))^o = \Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma}))$ and $\Psi^{-1}_{\gamma\delta\lambda}(f,\widehat{f},\acute{\Sigma})$ is a BHS open subset of $(\widetilde{\Re}, \Phi, \Sigma)$. Therefore, $\Psi_{\gamma\delta\lambda}$ is a BHS continuous map. □

Proposition 3.4. *A BHS bijective map $\Psi_{\gamma\delta\lambda} : (\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma) \rightarrow (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$ is a BHS continuous if and only if $(\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma)))^o \,\tilde{\tilde{\sqsubseteq}}\, \Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma)^o)$ for all $(g,\widehat{g},\Sigma) \,\tilde{\tilde{\sqsubseteq}}\, (\widetilde{\Re}, \Phi, \Sigma)$.*

Proof. Let $\Psi_{\gamma\delta\lambda}$ be a BHS continuous map. Let $(g,\widehat{g},\Sigma) \,\tilde{\tilde{\sqsubseteq}}\, (\widetilde{\Re}, \Phi, \Sigma)$, then $\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma)) \,\tilde{\tilde{\sqsubseteq}}\, (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$. Obviously, $(\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma)))^o$ is a BHS open in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$. As, $\Psi_{\gamma\delta\lambda}$ is a BHS continuous map, then $\Psi^{-1}_{\gamma\delta\lambda}((\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma)))^o)$ is a BHS open in $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$ and $(\Psi^{-1}_{\gamma\delta\lambda}((\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma)))^o))^o = \Psi^{-1}_{\gamma\delta\lambda}((\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma)))^o)$. Clearly, $(\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma)))^o \,\tilde{\tilde{\sqsubseteq}}\, \Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma))$. As, $\Psi_{\gamma\delta\lambda}$ is a BHS injective map, then $\Psi^{-1}_{\gamma\delta\lambda}((\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma)))^o) \,\tilde{\tilde{\sqsubseteq}}\, (g,\widehat{g},\Sigma)$ and $(\Psi^{-1}_{\gamma\delta\lambda}((\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma)))^o))^o = \Psi^{-1}_{\gamma\delta\lambda}((\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma)))^o) \,\tilde{\tilde{\sqsubseteq}}\, (g,\widehat{g},\Sigma)^o$. Again, as $\Psi_{\gamma\delta\lambda}$ is a BHS surjective map, then $(\Psi_{\gamma\delta\lambda}(g,\widehat{g},\Sigma))^o \,\tilde{\tilde{\sqsubseteq}}\, \Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma)^o)$. Conversely, let $(f,\widehat{f},\acute{\Sigma})$ be any BHS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$, then $\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})) \,\tilde{\tilde{\sqsubseteq}}\, (\widetilde{\Re}, \Phi, \Sigma)$. By hypothesis, $(\Psi_{\gamma\delta\lambda}(\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma}))))^o \,\tilde{\tilde{\sqsubseteq}}\, \Psi_{\gamma\delta\lambda}((\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})))^o)$. Since, $\Psi_{\gamma\delta\lambda}$ is a BHS surjective map, then $(f,\widehat{f},\acute{\Sigma})^o = (f,\widehat{f},\acute{\Sigma}) \,\tilde{\tilde{\sqsubseteq}}\, \Psi_{\gamma\delta\lambda}((\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})))^o)$. Again, since $\Psi_{\gamma\delta\lambda}$ is a BHS injective map, then

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$\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})) \,\widetilde{\widetilde{\sqsubseteq}}\, (\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})))^o$. But, $(\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})))^o \,\widetilde{\widetilde{\sqsubseteq}}\, \Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma}))$ and so $(\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})))^o$ $= \Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma}))$. Therefore, $\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma}))$ is a BHS open in $(\Re,\mathcal{T}_\Re,\Sigma,\neg\Sigma)$ and consequently $\Psi_{\gamma\delta\lambda}$ is a BHS continuous map. □

The next two results are straightforward, so we cancel their proofs.

Proposition 3.5. *A BHS map $\Psi_{\gamma\delta\lambda} : (\Re,\mathcal{T}_\Re,\Sigma,\neg\Sigma) \to (\aleph,\mathcal{T}_\aleph,\acute{\Sigma},\neg\acute{\Sigma})$ is a BHS continuous map if:*

i. $(\aleph,\mathcal{T}_\aleph,\acute{\Sigma},\neg\acute{\Sigma})$ *is a BHS indiscerte space.*

ii. $(\Re,\mathcal{T}_\Re,\Sigma,\neg\Sigma)$ *is a BHS discerte space.*

Proposition 3.6. *Let $\Psi_{\gamma\delta\lambda} : (\Re,\mathcal{T}_\Re,\Sigma,\neg\Sigma) \to (\aleph,\mathcal{T}_\aleph,\acute{\Sigma},\neg\acute{\Sigma})$ be a BHS continuous map:*

i. If $\mathcal{T}^*_\Re$ is a BHS finer than $\mathcal{T}_\Re$, then $\Psi_{\gamma\delta\lambda} : (\Re,\mathcal{T}^*_\Re,\Sigma,\neg\Sigma) \to (\aleph,\mathcal{T}_\aleph,\acute{\Sigma},\neg\acute{\Sigma})$ is a BHS continuous map.

ii. If $\mathcal{T}^*_\aleph$ is a BHS coarser than $\mathcal{T}_\aleph$, then $\Psi_{\gamma\delta\lambda} : (\Re,\mathcal{T}_\Re,\Sigma,\neg\Sigma) \to (\aleph,\mathcal{T}^*_\aleph,\acute{\Sigma},\neg\acute{\Sigma})$ is a BHS continuous map.

Definition 3.7. Let $(\Re,\mathcal{T}_\Re,\Sigma,\neg\Sigma)$ be a BHSTS. A subcollection β of $\mathcal{T}_\Re$ is called a BHS base for $\mathcal{T}_\Re$ if every element of $\mathcal{T}_\Re$ can be expressed as the union of members of β. Each element of β is called BHS basis element.

Proposition 3.8. *A BHS map $\Psi_{\gamma\delta\lambda} : (\Re,\mathcal{T}_\Re,\Sigma,\neg\Sigma) \to (\aleph,\mathcal{T}_\aleph,\acute{\Sigma},\neg\acute{\Sigma})$ is a BHS continuous if and only if the BHS inverse image of every member of a BHS base β for $(\aleph,\mathcal{T}_\aleph,\acute{\Sigma},\neg\acute{\Sigma})$ is a BHS open in $(\Re,\mathcal{T}_\Re,\Sigma,\neg\Sigma)$.*

Proof. Let $\Psi_{\gamma\delta\lambda}$ be a BHS continuous and $(\text{ɓ},\widehat{\text{ɓ}},\acute{\Sigma})$ be any BHS basis element for $(\aleph,\mathcal{T}_\aleph,\acute{\Sigma},\neg\acute{\Sigma})$. Since $(\text{ɓ},\widehat{\text{ɓ}},\acute{\Sigma})$ is a BHS open set in $(\aleph,\mathcal{T}_\aleph,\acute{\Sigma},\neg\acute{\Sigma})$ and $\Psi_{\gamma\delta\lambda}$ is a BHS continuous map, then $\Psi^{-1}_{\gamma\delta\lambda}((\text{ɓ},\widehat{\text{ɓ}},\acute{\Sigma}))$ is a BHS open set in $(\Re,\mathcal{T}_\Re,\Sigma,\neg\Sigma)$. Conversely, let $\Psi^{-1}_{\gamma\delta\lambda}((\text{ɓ},\widehat{\text{ɓ}},\acute{\Sigma}))$ be a BHS open set in $(\Re,\mathcal{T}_\Re,\Sigma,\neg\Sigma)$ for every $(\text{ɓ},\widehat{\text{ɓ}},\acute{\Sigma}) \in \beta$, and let $(f,\widehat{f},\acute{\Sigma})$ be any BHS open set in $(\aleph,\mathcal{T}_\aleph,\acute{\Sigma},\neg\acute{\Sigma})$. Then

$$\begin{aligned}
(f,\widehat{f},\acute{\Sigma}) &= \widetilde{\widetilde{\sqcup}}\{(\text{ɓ},\widehat{\text{ɓ}},\acute{\Sigma}) : (\text{ɓ},\widehat{\text{ɓ}},\acute{\Sigma}) \in \beta\}\\
&= \Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma}))\\
&= \Psi^{-1}_{\gamma\delta\lambda}(\widetilde{\widetilde{\sqcup}}\{(\text{ɓ},\widehat{\text{ɓ}},\acute{\Sigma}) : (\text{ɓ},\widehat{\text{ɓ}},\acute{\Sigma}) \in \beta\})\\
&= \widetilde{\widetilde{\sqcup}}\{\Psi^{-1}_{\gamma\delta\lambda}(\{(\text{ɓ},\widehat{\text{ɓ}},\acute{\Sigma}) : (\text{ɓ},\widehat{\text{ɓ}},\acute{\Sigma}) \in \beta\})\}
\end{aligned}$$

Hence $\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma}))$ is a BHS open set in $(\Re,\mathcal{T}_\Re,\Sigma,\neg\Sigma)$ since each $\Psi^{-1}_{\gamma\delta\lambda}((\text{ɓ},\widehat{\text{ɓ}},\acute{\Sigma}))$ is a BHS open set in $(\Re,\mathcal{T}_\Re,\Sigma,\neg\Sigma)$ by hypothesis. Therefore, $\Psi_{\gamma\delta\lambda}$ is a BHS continuous map. □

Proposition 3.9. *Let $\Psi_{\gamma\delta\lambda} : (\Re,\mathcal{T}_\Re,\Sigma,\neg\Sigma) \to (\aleph,\mathcal{T}_\aleph,\acute{\Sigma},\neg\acute{\Sigma})$ and $\Theta_{\gamma\delta\lambda} : (\aleph,\mathcal{T}_\aleph,\acute{\Sigma},\neg\acute{\Sigma}) \to (M,\mathcal{T}_M,\acute{\acute{\Sigma}},\neg\acute{\acute{\Sigma}})$ be BHS continuous maps, then $\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda} : (\Re,\mathcal{T}_\Re,\Sigma,\neg\Sigma) \to (M,\mathcal{T}_M,\acute{\acute{\Sigma}},\neg\acute{\acute{\Sigma}})$ is also a BHS continuous map.*

Proof. Let $(\hbar,\widehat{\hbar},\acute{\acute{\Sigma}})$ be any BHS open set in $(M,\mathcal{T}_M,\acute{\acute{\Sigma}},\neg\acute{\acute{\Sigma}})$. Since $\Theta_{\gamma\delta\lambda}$ is BHS continuous map, then $\Theta^{-1}_{\gamma\delta\lambda}((\hbar,\widehat{\hbar},\acute{\acute{\Sigma}}))$ is BHS open in $(\aleph,\mathcal{T}_\aleph,\acute{\Sigma},\neg\acute{\Sigma})$. Again, since $\Psi_{\gamma\delta\lambda}$ is BHS continuous, then $\Psi^{-1}_{\gamma\delta\lambda}(\Theta^{-1}_{\gamma\delta\lambda}(\hbar,\widehat{\hbar},\acute{\acute{\Sigma}}))$ is BHS open in $(\Re,\mathcal{T}_\Re,\Sigma,\neg\Sigma)$. But

$$\Psi^{-1}_{\gamma\delta\lambda}(\Theta^{-1}_{\gamma\delta\lambda}(\hbar,\widehat{\hbar},\acute{\acute{\Sigma}})) = (\Psi^{-1}_{\gamma\delta\lambda} \circ \Theta^{-1}_{\gamma\delta\lambda})(\hbar,\widehat{\hbar},\acute{\acute{\Sigma}}) = (\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda})^{-1}(\hbar,\widehat{\hbar},\acute{\acute{\Sigma}})$$

Thus, the BHS inverse image under $\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda}$ of every BHS open set in $(M,\mathcal{T}_M,\acute{\acute{\Sigma}},\neg\acute{\acute{\Sigma}})$ is BHS open set in $(\Re,\mathcal{T}_\Re,\Sigma,\neg\Sigma)$ and therefore $\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda}$ is a BHS continuous map. □

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Proposition 3.10. *If a BHS map $\Psi_{\gamma\delta\lambda} : (\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$ is BHS continuous, then the HS map $\Psi_{\gamma\delta} : (\Re, \mathcal{T}^h_{\Re}, \Sigma) \to (\aleph, \mathcal{T}^h_{\aleph}, \acute{\Sigma})$ is HS continuous.*

Proof. Straightforward. □

Proposition 3.11. *Let the condition of constructing a BHST from HST as in Proposition 2.13 hold. If a HS map $\Psi_{\gamma\delta} : (\Re, \mathcal{T}^h_{\Re}, \Sigma) \to (\aleph, \mathcal{T}^h_{\aleph}, \acute{\Sigma})$ is HS continuous, then the BHS map $\Psi_{\gamma\delta\lambda} : (\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$ is BHS continuous.*

Proof. Let $\Psi_{\gamma\delta}$ be a HS continuous map and let $(f, \widehat{f}, \acute{\Sigma})$ be a BHS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$. Then $(f, \acute{\Sigma})$ is a HS open set in $(\aleph, \mathcal{T}^h_{\aleph}, \acute{\Sigma})$. Since $\Psi_{\gamma\delta}$ is a HS continuous map, then $\Psi^{-1}_{\gamma\delta}((f, \acute{\Sigma}))$ is a HS open set in $(\Re, \mathcal{T}^h_{\Re}, \Sigma)$. Hence, $\Psi^{-1}_{\gamma\delta\lambda}((f, \widehat{f}, \acute{\Sigma}))$ is a BHS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$. Thus, $\Psi_{\gamma\delta\lambda}$ is a BHS continuous map. □

Definition 3.12. A BHS map $\Psi_{\gamma\delta\lambda} : (\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$ is said to be:

1. BHS open if the BHS image of every BHS open set is BHS open.
2. BHS closed if the BHS image of every BHS closed set is BHS closed.

Proposition 3.13. *A BHS map $\Psi_{\gamma\delta\lambda} : (\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$ is a BHS open if and only if $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)^o) \mathrel{\tilde{\tilde{\sqsubseteq}}} (\Psi_{\gamma\delta\lambda}(g, \widehat{g}, \Sigma))^o$ for every $(g, \widehat{g}, \Sigma) \mathrel{\tilde{\tilde{\sqsubseteq}}} (\widetilde{\Re}, \Phi, \Sigma)$.*

Proof. Let $\Psi_{\gamma\delta\lambda}$ be a BHS open map and let $(g, \widehat{g}, \Sigma) \mathrel{\tilde{\tilde{\sqsubseteq}}} (\widetilde{\Re}, \Phi, \Sigma)$. We know that, $(g, \widehat{g}, \Sigma)^o$ is a BHS open set. Since $\Psi_{\gamma\delta\lambda}$ is a BHS open map, then $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)^o)$ is a BHS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$ and $(\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)^o))^o = \Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)^o)$. Obviously, $(g, \widehat{g}, \Sigma)^o \mathrel{\tilde{\tilde{\sqsubseteq}}} (g, \widehat{g}, \Sigma)$ and $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)^o) \mathrel{\tilde{\tilde{\sqsubseteq}}} \Psi_{\gamma\delta\lambda}(g, \widehat{g}, \Sigma)$. Therefore, $(\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)^o))^o = \Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)^o) \mathrel{\tilde{\tilde{\sqsubseteq}}} (\Psi_{\gamma\delta\lambda}(g, \widehat{g}, \Sigma))^o$. Conversely, let $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)^o) \mathrel{\tilde{\tilde{\sqsubseteq}}} (\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)))^o$ for every $(g, \widehat{g}, \Sigma) \mathrel{\tilde{\tilde{\sqsubseteq}}} (\widetilde{\Re}, \Phi, \Sigma)$ and let $(g, \widehat{g}, \Sigma)$ be any BHS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$ so that $(g, \widehat{g}, \Sigma)^o = (g, \widehat{g}, \Sigma)$. This implies $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)^o) = \Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)) \mathrel{\tilde{\tilde{\sqsubseteq}}} (\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)))^o$. But $(\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)))^o \mathrel{\tilde{\tilde{\sqsubseteq}}} \Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma))$ and so $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)) = (\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)))^o$. Therefore, $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma))$ is BHS open set and hence $\Psi_{\gamma\delta\lambda}$ is BHS open map. □

Proposition 3.14. *A BHS map $\Psi_{\gamma\delta\lambda} : (\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$ is a BHS closed if and only if $\overline{\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma))} \mathrel{\tilde{\tilde{\sqsubseteq}}} \Psi_{\gamma\delta\lambda}(\overline{(g, \widehat{g}, \Sigma)})$ for every $(g, \widehat{g}, \Sigma) \mathrel{\tilde{\tilde{\sqsubseteq}}} (\widetilde{\Re}, \Phi, \Sigma)$.*

Proof. Let $\Psi_{\gamma\delta\lambda}$ be a BHS closed map and let $(g, \widehat{g}, \Sigma) \mathrel{\tilde{\tilde{\sqsubseteq}}} (\widetilde{\Re}, \Phi, \Sigma)$. We know that, $\overline{(g, \widehat{g}, \Sigma)}$ is a BHS closed set. Since $\Psi_{\gamma\delta\lambda}$ is a BHS closed map, then $\Psi_{\gamma\delta\lambda}(\overline{(g, \widehat{g}, \Sigma)})$ is a BHS closed set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$ and $\overline{\Psi_{\gamma\delta\lambda}(\overline{(g, \widehat{g}, \Sigma)})} = \Psi_{\gamma\delta\lambda}(\overline{(g, \widehat{g}, \Sigma)})$. Obviously, $(g, \widehat{g}, \Sigma) \mathrel{\tilde{\tilde{\sqsubseteq}}} \overline{(g, \widehat{g}, \Sigma)}$ and $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)) \mathrel{\tilde{\tilde{\sqsubseteq}}} \Psi_{\gamma\delta\lambda}(\overline{(g, \widehat{g}, \Sigma)})$. Therefore, $\overline{\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma))} \mathrel{\tilde{\tilde{\sqsubseteq}}} \overline{\Psi_{\gamma\delta\lambda}(\overline{(g, \widehat{g}, \Sigma)})} = \Psi_{\gamma\delta\lambda}(\overline{(g, \widehat{g}, \Sigma)})$. Conversely, suppose that $\overline{\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma))} \mathrel{\tilde{\tilde{\sqsubseteq}}} \Psi_{\gamma\delta\lambda}(\overline{(g, \widehat{g}, \Sigma)})$ for every $(g, \widehat{g}, \Sigma) \mathrel{\tilde{\tilde{\sqsubseteq}}} (\widetilde{\Re}, \Phi, \Sigma)$ and let $(g, \widehat{g}, \Sigma)$ be any BHS closed set in $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$ so that $\overline{(g, \widehat{g}, \Sigma)} = (g, \widehat{g}, \Sigma)$. This implies $\overline{\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma))} \mathrel{\tilde{\tilde{\sqsubseteq}}} \Psi_{\gamma\delta\lambda}(\overline{(g, \widehat{g}, \Sigma)}) = \Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma))$. But $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)) \mathrel{\tilde{\tilde{\sqsubseteq}}} \overline{\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma))}$ and hence $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)) = \overline{\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma))}$. Therefore, $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma))$ is BHS closed set and hence $\Psi_{\gamma\delta\lambda}$ is BHS closed map. □

Proposition 3.15. *Let $\Psi_{\gamma\delta\lambda} : (\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$ and $\Theta_{\gamma\delta\lambda} : (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma}) \to (M, \mathcal{T}_M, \acute{\acute{\Sigma}}, \neg\acute{\acute{\Sigma}})$ be two BHS maps, then:*

i. If $\Psi_{\gamma\delta\lambda}$ and $\Theta_{\gamma\delta\lambda}$ are BHS open maps, then $\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda}$ is also a BHS open map.

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ii. *If* $\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda}$ *is a BHS open map and* $\Psi_{\gamma\delta\lambda}$ *is a BHS surjective BHS continuous map, then* $\Theta_{\gamma\delta\lambda}$ *is a BHS open map.*

iii. *If* $\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda}$ *is a BHS open map and* $\Theta_{\gamma\delta\lambda}$ *is a BHS injective BHS continuous map, then* $\Psi_{\gamma\delta\lambda}$ *is a BHS open map.*

Proof. i. Let $(g, \widehat{g}, \Sigma)$ be a BHS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$. Since $\Psi_{\gamma\delta\lambda}$ is a BHS open map, then $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma))$ is a BHS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$. Again, since $\Theta_{\gamma\delta\lambda}$ is a BHS open map, then $\Theta_{\gamma\delta\lambda}(\Psi_{\gamma\delta\lambda}(g, \widehat{g}, \Sigma)) = (\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda})((g, \widehat{g}, \Sigma))$ is a BHS open set in $(M, \mathcal{T}_M, \acute{\acute{\Sigma}}, \neg\acute{\acute{\Sigma}})$. Hence, $\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda}$ is a BHS open map.

ii. Let $(f, \widehat{f}, \acute{\Sigma})$ be a BHS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$. Since $\Psi_{\gamma\delta\lambda}$ is a BHS continuous, then $\Psi^{-1}_{\gamma\delta\lambda}((f, \widehat{f}, \acute{\Sigma}))$ is a BHS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$. Again, since $\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda}$ is a BHS open map, then $(\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda})(\Psi^{-1}_{\gamma\delta\lambda}((f, \widehat{f}, \acute{\Sigma})))$ is a BHS open set in $(M, \mathcal{T}_M, \acute{\acute{\Sigma}}, \neg\acute{\acute{\Sigma}})$. As, $\Psi_{\gamma\delta\lambda}$ is a BHS surjective, then $(\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda})(\Psi^{-1}_{\gamma\delta\lambda}((f, \widehat{f}, \acute{\Sigma}))) = \Theta_{\gamma\delta\lambda}(\Psi_{\gamma\delta\lambda}(\Psi^{-1}_{\gamma\delta\lambda}(f, \widehat{f}, \acute{\Sigma}))) = \Theta_{\gamma\delta\lambda}((f, \widehat{f}, \acute{\Sigma}))$ is a BHS open set in $(M, \mathcal{T}_M, \acute{\acute{\Sigma}}, \neg\acute{\acute{\Sigma}})$. Hence, $\Theta_{\gamma\delta\lambda}$ is a BHS open map.

iii. Let $(g, \widehat{g}, \Sigma)$ be a BHS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$. Since $\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda}$ is a BHS open map, then $(\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda})((g, \widehat{g}, \Sigma))$ is a BHS open set in $(M, \mathcal{T}_M, \acute{\acute{\Sigma}}, \neg\acute{\acute{\Sigma}})$. Again, since $\Theta_{\gamma\delta\lambda}$ is a BHS continuous map, then $\Theta^{-1}_{\gamma\delta\lambda}((\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda})((g, \widehat{g}, \Sigma)))$ is a BHS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$. As, $\Theta_{\gamma\delta\lambda}$ is a BHS injective, then $\Theta^{-1}_{\gamma\delta\lambda}(\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma))) = \Theta^{-1}_{\gamma\delta\lambda}(\Theta_{\gamma\delta\lambda}(\Psi_{\gamma\delta\lambda}(g, \widehat{g}, \Sigma))) = \Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma))$ is a BHS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$. Hence, $\Psi_{\gamma\delta\lambda}$ is a BHS open map.

□

Proposition 3.16. *Let* $\Psi_{\gamma\delta\lambda} : (\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$ *and* $\Theta_{\gamma\delta\lambda} : (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma}) \to (M, \mathcal{T}_M, \acute{\acute{\Sigma}}, \neg\acute{\acute{\Sigma}})$ *be two BHS maps, then:*

i. *If* $\Psi_{\gamma\delta\lambda}$ *and* $\Theta_{\gamma\delta\lambda}$ *are BHS closed maps, then* $\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda}$ *is also a BHS closed map.*

ii. *If* $\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda}$ *is a BHS closed map and* $\Psi_{\gamma\delta\lambda}$ *is a BHS surjective BHS continuous map, then* $\Theta_{\gamma\delta\lambda}$ *is a BHS closed map.*

iii. *If* $\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda}$ *is a BHS closed map and* $\Theta_{\gamma\delta\lambda}$ *is a BHS injective BHS continuous map, then* $\Psi_{\gamma\delta\lambda}$ *is a BHS closed map.*

Proof. Similar to the proof of Proposition 3.15. □

Proposition 3.17. *If a BHS map* $\Psi_{\gamma\delta\lambda} : (\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$ *is BHS open (closed), then the HS map* $\Psi_{\gamma\delta} : (\Re, \mathcal{T}^h_{\Re}, \Sigma) \to (\aleph, \mathcal{T}^h_{\aleph}, \acute{\Sigma})$ *is HS open (closed).*

Proof. Straightforward. □

Proposition 3.18. *Let the condition of constructing a BHST from HST as in Proposition 2.13 hold. If a HS map* $\Psi_{\gamma\delta} : (\Re, \mathcal{T}^h_{\Re}, \Sigma) \to (\aleph, \mathcal{T}^h_{\aleph}, \acute{\Sigma})$ *is HS open (closed), then the BHS map* $\Psi_{\gamma\delta\lambda} : (\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$ *is BHS open (closed).*

Proof. Similar to the proof of Proposition 3.11. □

Definition 3.19. A BHS bijective map $\Psi_{\gamma\delta\lambda} : (\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$ is said to be BHS homeomorphism if $\Psi_{\gamma\delta\lambda}$ and $\Psi^{-1}_{\gamma\delta\lambda}$ are BHS continuous maps.

Proposition 3.20. *If* $\Psi_{\gamma\delta\lambda} : (\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$ *is a BHS bijective map, then the following statements are equivalent:*

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i. $\Psi_{\gamma\delta\lambda}$ *is BHS homeomorphism.*

ii. $\Psi_{\gamma\delta\lambda}$ *is BHS continuous and BHS open.*

iii. $\Psi_{\gamma\delta\lambda}$ *is BHS continuous and BHS closed.*

Proof. (i.) $\Rightarrow$ (ii.): Let $\Theta_{\gamma\delta\lambda}$ be the BHS inverse map of $\Psi_{\gamma\delta\lambda}$ so that $\Psi^{-1}_{\gamma\delta\lambda} = \Theta_{\gamma\delta\lambda}$ and $\Theta^{-1}_{\gamma\delta\lambda} = \Psi_{\gamma\delta\lambda}$. Since $\Psi_{\gamma\delta\lambda}$ is BHS bijective map, then $\Theta_{\gamma\delta\lambda}$ is also BHS bijective. Let $(g, \widehat{g}, \Sigma)$ be a BHS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$. Since $\Theta_{\gamma\delta\lambda}$ is BHS continuous, then $\Theta^{-1}_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma))$ is a BHS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$. But $\Theta^{-1}_{\gamma\delta\lambda} = \Psi_{\gamma\delta\lambda}$ so that $\Theta^{-1}_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)) = \Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma))$ is a BHS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$. It follows that $\Psi_{\gamma\delta\lambda}$ is BHS open map. Also, $\Psi_{\gamma\delta\lambda}$ is BHS continuous by hypothesis.

(ii.) $\Rightarrow$ (iii.): Let $(g, \widehat{g}, \Sigma)$ be a BHS closed set in $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$, then $(g, \widehat{g}, \Sigma)^c$ is a BHS open set. Since $\Psi_{\gamma\delta\lambda}$ is BHS open map, then $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)^c)$ is a BHS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$. As $\Psi_{\gamma\delta\lambda}$ is a BHS bijective map, then $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)^c) = (\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)))^c$. Hence, $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma))$ is a BHS closed set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$ and consequently $\Psi_{\gamma\delta\lambda}$ is BHS closed map.

(iii.) $\Rightarrow$ (i.): Let $(g, \widehat{g}, \Sigma)$ be a BHS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$, then $(g, \widehat{g}, \Sigma)^c$ is a BHS closed set. Since $\Psi_{\gamma\delta\lambda}$ is BHS closed map, then $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)^c) = \Theta^{-1}_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)^c) = (\Theta^{-1}_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)))^c$ is a BHS closed set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$, that is, $\Theta^{-1}_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma))$ is a BHS open set. Hence, $\Theta_{\gamma\delta\lambda} = \Psi^{-1}_{\gamma\delta\lambda}$ is BHS continuous map. ☐

Proposition 3.21. *Let* $\Psi_{\gamma\delta\lambda} : (\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$ *and* $\Theta_{\gamma\delta\lambda} : (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma}) \to (M, \mathcal{T}_M, \acute{\acute{\Sigma}}, \neg\acute{\acute{\Sigma}})$ *be BHS homeomorphism maps, then* $\Theta_{\gamma\delta\lambda} \circ \Psi_{\gamma\delta\lambda} : (\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma) \to (M, \mathcal{T}_M, \acute{\acute{\Sigma}}, \neg\acute{\acute{\Sigma}})$ *is also a BHS homeomorphism map.*

Proof. Follows from Proposition 3.9, Proposition 3.15 (i.), and Proposition 3.20. ☐

Proposition 3.22. *If* $\Psi_{\gamma\delta\lambda} : (\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}, \neg\acute{\Sigma})$ *is a BHS homeomorphism map, then the following statements hold for all* $(g, \widehat{g}, \Sigma) \,\widetilde{\widetilde{\sqsubseteq}}\, (\widetilde{\Re}, \Phi, \Sigma)$.

i. $\Psi_{\gamma\delta\lambda}(\overline{(g, \widehat{g}, \Sigma)}) = \overline{\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma))}$.

ii. $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Sigma)^o) = (\Psi_{\gamma\delta\lambda}(g, \widehat{g}, \Sigma))^o$.

Proof. i. Follows from Proposition 3.3, Proposition 3.14, and Proposition 3.20.

ii. Follows from Proposition 3.4, Proposition 3.13, and Proposition 3.20. ☐

4 Bipolar Hypersoft Compact Spaces

In this part, we look into BHS compactness, which is another significant property of BHSTSs. The topic of BHS compact spaces is examined, and certain conclusions are drawn.

Definition 4.1. A collection $\{(g_i, \widehat{g}_i, \Sigma) : i \in I\}$ of BHS sets is called the BHS cover of a BHS set $(g, \widehat{g}, \Sigma)$ if $(g, \widehat{g}, \Sigma) \,\widetilde{\widetilde{\sqsubseteq}}\, \widetilde{\widetilde{\sqcup}} \{(g_i, \widehat{g}_i, \Sigma) : i \in I\}$. If each member of $\{(g_i, \widehat{g}_i, \Sigma) : i \in I\}$ is BHS open set, then it is called the BHS open cover of $(g, \widehat{g}, \Sigma)$. A BHS subcover is a subcollection of $\{(g_i, \widehat{g}_i, \Sigma) : i \in I\}$ which is also a BHS cover.

Definition 4.2. A BHSTS $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$ is said to be BHS compact space if each BHS open cover of $(\widetilde{\Re}, \Phi, \Sigma)$ has finite BHS subcover.

Example 4.3. Let $\Re = \{r_1, r_2, r_3\}$, $\sigma_1 = \{\ell_1, \ell_2\}$, $\sigma_2 = \{\ell_3\}$, and $\sigma_3 = \{\ell_4\}$. Let $\mathcal{T}_{\Re} = \{(\Phi, \widetilde{\Re}, \Sigma)$, $(\widetilde{\Re}, \Phi, \Sigma)$, $(g_1, \widehat{g}_1, \Sigma)$, $(g_2, \widehat{g}_2, \Sigma)$, $(g_3, \widehat{g}_3, \Sigma)$, $(g_4, \widehat{g}_4, \Sigma)$, $(g_5, \widehat{g}_5, \Sigma)$, $(g_6, \widehat{g}_6, \Sigma)$, $(g_7, \widehat{g}_7, \Sigma)\}$ be a BHST defined on $\Re$, where

$(g_1, \widehat{g}_1, \Sigma) = \{((\ell_1, \ell_3, \ell_4), \{r_2\}, \{r_3\}), ((\ell_2, \ell_3, \ell_4), \{r_2\}, \{r_3\})\}$.

$(g_2, \widehat{g}_2, \Sigma) = \{((\ell_1, \ell_3, \ell_4), \{r_1\}, \{r_3\}), ((\ell_2, \ell_3, \ell_4), \{r_1\}, \{r_3\})\}$.

$(g_3, \widehat{g}_3, \Sigma) = \{((\ell_1, \ell_3, \ell_4), \{r_1, r_2\}, \{r_3\}), ((\ell_2, \ell_3, \ell_4), \{r_1, r_2\}, \{r_3\})\}$.

$(g_4, \widehat{g}_4, \Sigma) = \{((\ell_1, \ell_3, \ell_4), \{r_2, r_3\}, \varphi), ((\ell_2, \ell_3, \ell_4), \{r_2, r_3\}, \varphi)\}$.

$(g_5, \widehat{g}_5, \Sigma) = \{((\ell_1, \ell_3, \ell_4), \{r_1, r_3\}, \varphi), ((\ell_2, \ell_3, \ell_4), \{r_1, r_3\}, \varphi)\}$.

$(g_6, \widehat{g}_6, \Sigma) = \{((\ell_1, \ell_3, \ell_4), \varphi, \{r_3\}), ((\ell_2, \ell_3, \ell_4), \varphi, \{r_3\})\}$.

$(g_7, \widehat{g}_7, \Sigma) = \{((\ell_1, \ell_3, \ell_4), \{r_3\}, \varphi), ((\ell_2, \ell_3, \ell_4), \{r_3\}, \varphi)\}$.

It is easy to see that $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$ is a BHS compact space.

Example 4.4. Let $\Re = \{r_1, r_2, r_3, ...\}$, $\sigma_1 = \{\ell_1, \ell_2\}$, $\sigma_2 = \{\ell_3\}$, $\sigma_3 = \{\ell_4\}$, and consider the family of BHS sets $\{(g_n, \widehat{g}_n, \Sigma) : n = 1, 2, 3, ...\}$, where

$(g_n, \widehat{g}_n, \Sigma) = \{((\ell_1, \ell_3, \ell_4), \{r_1, r_2, ..., r_n\}, \varphi), ((\ell_2, \ell_3, \ell_4), \{r_1, r_2, ..., r_n\}, \varphi)\}$.

The family $\mathcal{T}_{\Re} = \{(\Phi, \widetilde{\Re}, \Sigma), (\widetilde{\Re}, \Phi, \Sigma), (g_n, \widehat{g}_n, \Sigma)\}$ is a BHST on $\Re$. However, $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$ is not a BHS compact space.

Proposition 4.5. *If $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$ is a BHS compact space and $\mathcal{T}^*_{\Re}$ is BHS coarser than $\mathcal{T}_{\Re}$, then $(\Re, \mathcal{T}^*_{\Re}, \Sigma, \neg\Sigma)$ is BHS compact.*

Proof. Let the collection $\{(g_i, \widehat{g}_i, \Sigma) : i \in I\}$ be a BHS open cover of $(\widetilde{\Re}, \Phi, \Sigma)$ in $(\Re, \mathcal{T}^*_{\Re}, \Sigma, \neg\Sigma)$. Since $\mathcal{T}_{\Re}$ is BHS finer than $\mathcal{T}^*_{\Re}$, then each member of $\{(g_i, \widehat{g}_i, \Sigma) : i \in I\}$ is also a BHS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$. Hence, $\{(g_i, \widehat{g}_i, \Sigma) : i \in I\}$ is also a BHS open cover of $(\widetilde{\Re}, \Phi, \Sigma)$ in $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$. Since $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$ is a BHS compact space, then $\{(g_i, \widehat{g}_i, \Sigma) : i \in I\}$ has a finite BHS subcover. It follows that $(\Re, \mathcal{T}^*_{\Re}, \Sigma, \neg\Sigma)$ is BHS compact space. □

Proposition 4.6. *Let $(\Upsilon, \mathcal{T}_{\Upsilon}, \Sigma, \neg\Sigma)$ be a BHS subspace of $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$. Then $(\Upsilon, \mathcal{T}_{\Upsilon}, \Sigma, \neg\Sigma)$ is BHS compact space if and only if every BHS cover of $(\widetilde{\Upsilon}, \Phi, \Sigma)$ by BHS open sets in $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$ contains a finite BHS subcover.*

Proof. Let $(\Upsilon, \mathcal{T}_{\Upsilon}, \Sigma, \neg\Sigma)$ be a BHS compact space and $\{(g_i, \widehat{g}_i, \Sigma) : i \in I\}$ be a BHS cover of $(\widetilde{\Upsilon}, \Phi, \Sigma)$ by BHS open sets in $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$. Then the collection $\{(g_{i_{\Upsilon}}, \widehat{g}_{i_{\Upsilon}}, \Sigma) : i \in I\}$ be a BHS cover of $(\widetilde{\Upsilon}, \Phi, \Sigma)$ by BHS open sets in $(\Upsilon, \mathcal{T}_{\Upsilon}, \Sigma, \neg\Sigma)$. Since $(\Upsilon, \mathcal{T}_{\Upsilon}, \Sigma, \neg\Sigma)$ is a BHS compact, then the finite subcollection of $\{(g_{i_{\Upsilon}}, \widehat{g}_{i_{\Upsilon}}, \Sigma) : i \in I\}$ is also a BHS cover of $(\widetilde{\Upsilon}, \Phi, \Sigma)$. Thus, the finite subcollection of $\{(g_i, \widehat{g}_i, \Sigma) : i \in I\}$ is also a BHS cover of $(\widetilde{\Upsilon}, \Phi, \Sigma)$. Conversely, let $\{(g_{i_{\Upsilon}}, \widehat{g}_{i_{\Upsilon}}, \Sigma) : i \in I\}$ be a BHS cover of $(\widetilde{\Upsilon}, \Phi, \Sigma)$. Obviously, $\{(g_i, \widehat{g}_i, \Sigma) : i \in I\}$ be a BHS cover of $(\widetilde{\Upsilon}, \Phi, \Sigma)$ by BHS open sets in $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$. By hypothesis, the finite subcollection of $\{(g_i, \widehat{g}_i, \Sigma) : i \in I\}$ is also a BHS cover of $(\widetilde{\Upsilon}, \Phi, \Sigma)$ and hence $\{(g_{i_{\Upsilon}}, \widehat{g}_{i_{\Upsilon}}, \Sigma) : i = 1, 2, ..., n\}$ is a BHS subcover of $(\widetilde{\Upsilon}, \Phi, \Sigma)$. Therefore, $(\Upsilon, \mathcal{T}_{\Upsilon}, \Sigma, \neg\Sigma)$ is BHS compact space. □

Proposition 4.7. *A BHSTS $(\Re, \mathcal{T}_{\Re}, \Sigma, \neg\Sigma)$ is a BHS compact if there exists a BHS basis β for $\mathcal{T}_{\Re}$ such that every BHS cover of $(\widetilde{\Re}, \Phi, \Sigma)$ by the elements of β has a finite BHS subcover.*

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Proof. Let $(\Re, \mathcal{T}_\Re, \Sigma, \neg\Sigma)$ be a BHS compact space. Obviously, $\mathcal{T}_\Re$ is a BHS basis for $\mathcal{T}_\Re$. Hence, every BHS cover of $(\widetilde{\Re}, \Phi, \Sigma)$ by the elements of $\mathcal{T}_\Re$ has a finite BHS subcover. Conversely, let $\{(g_i, \widehat{g}_i, \Sigma) : i \in I\}$ be a BHS cover of $(\widetilde{\Re}, \Phi, \Sigma)$. Now, $\{(g_i, \widehat{g}_i, \Sigma) : i \in I\}$ can be written as the union of some BHS basis elements $\{(b_i, \widehat{b}_i, \Sigma) : i \in I\}$ of β. These elements form a BHS cover of $(\widetilde{\Re}, \Phi, \Sigma)$. By hypothesis, every BHS cover of $(\widetilde{\Re}, \Phi, \Sigma)$ by the elements of β has a finite BHS subcover. Now, we have $\{(b_i, \widehat{b}_i, \Sigma) : i = 1, 2, ..., n\}$ $\widetilde{\widetilde{\sqsubseteq}}$ $\{(g_i, \widehat{g}_i, \Sigma) : i = 1, 2, ..., n\}$. This implies that $\{(g_i, \widehat{g}_i, \Sigma) : i = 1, 2, ..., n\}$ is a finite BHS subcover of $(\widetilde{\Re}, \Phi, \Sigma)$. Hence, $(\Re, \mathcal{T}_\Re, \Sigma, \neg\Sigma)$ is a BHS compact space. □

Proposition 4.8. *If $\Psi_{\gamma\delta\lambda} : (\Re, \mathcal{T}_\Re, \Sigma, \neg\Sigma) \to (\aleph, \mathcal{T}_\aleph, \acute{\Sigma}, \neg\acute{\Sigma})$ is a BHS continuous map and $(\Re, \mathcal{T}_\Re, \Sigma, \neg\Sigma)$ is a BHS compact space, then $\Psi_{\gamma\delta\lambda}((\Re, \mathcal{T}_\Re, \Sigma, \neg\Sigma))$ is a BHS compact.*

Proof. Let $\{(f_i, \widehat{f}_i, \acute{\Sigma}) : i \in I\}$ be a BHS cover of $\Psi_{\gamma\delta\lambda}((\widetilde{\Re}, \Phi, \Sigma))$. Since $\Psi_{\gamma\delta\lambda}$ is a BHS continuous map, then, for each $i \in I$, $\Psi^{-1}_{\gamma\delta\lambda}((f_i, \widehat{f}_i, \acute{\Sigma}))$ is a BHS open set in $(\Re, \mathcal{T}_\Re, \Sigma, \neg\Sigma)$. Then, the collection $\{\Psi^{-1}_{\gamma\delta\lambda}((f_i, \widehat{f}_i, \acute{\Sigma})) : i \in I\}$ forms a BHS cover of $(\widetilde{\Re}, \Phi, \Sigma)$. Since $(\Re, \mathcal{T}_\Re, \Sigma, \neg\Sigma)$ is a BHS compact, then we have $(\widetilde{\Re}, \Phi, \Sigma) = \widetilde{\widetilde{\sqcup}} \{\Psi^{-1}_{\gamma\delta\lambda}((f_i, \widehat{f}_i, \acute{\Sigma})) : i = 1, 2, ..., n\} = \Psi^{-1}_{\gamma\delta\lambda}(\widetilde{\widetilde{\sqcup}}\{(f_i, \widehat{f}_i, \acute{\Sigma}) : i = 1, 2, ..., n\})$ so that $\Psi_{\gamma\delta\lambda}((\widetilde{\Re}, \Phi, \Sigma)) \widetilde{\widetilde{\sqsubseteq}} \widetilde{\widetilde{\sqcup}}\{(f_i, \widehat{f}_i, \acute{\Sigma}) : i = 1, 2, ..., n\}$. Hence, $\Psi_{\gamma\delta\lambda}((\widetilde{\Re}, \Phi, \Sigma))$ is a BHS compact space. □

Definition 4.9. A collection ξ of BHS sets is said to have the finite intersection property if the BHS intersection of members of each finite subcollection of ξ is non-null BHS set.

Proposition 4.10. *A BHSTS $(\Re, \mathcal{T}_\Re, \Sigma, \neg\Sigma)$ is a BHS compact if and only if every collection of BHS closed subsets of $(\widetilde{\Re}, \Phi, \Sigma)$ with the finite intersection property has a non-null BHS intersection.*

Proof. Let $(\Re, \mathcal{T}_\Re, \Sigma, \neg\Sigma)$ be a BHS compact space and let $\mu = \{(g_i, \widehat{g}_i, \Sigma) : i \in I\}$ be a collection of BHS closed subsets of $(\widetilde{\Re}, \Phi, \Sigma)$ with the finite intersection property and suppose, if possible, $\widetilde{\widetilde{\sqcap}}\{(g_i, \widehat{g}_i, \Sigma) : i \in I\} = (\Phi, \widetilde{\Re}, \Sigma)$. Then, $\widetilde{\widetilde{\sqcup}}\{(g_i, \widehat{g}_i, \Sigma)^c : i \in I\} = (\widetilde{\Re}, \Phi, \Sigma)$. This means that $\{(g_i, \widehat{g}_i, \Sigma)^c : i \in I\}$ is BHS open cover of $(\widetilde{\Re}, \Phi, \Sigma)$. Since $(\Re, \mathcal{T}_\Re, \Sigma, \neg\Sigma)$ is a BHS compact space, we have that $\widetilde{\widetilde{\sqcup}}\{(g_i, \widehat{g}_i, \Sigma)^c : i = 1, 2, ..., n\} = (\widetilde{\Re}, \Phi, \Sigma)$ which implies that $\widetilde{\widetilde{\sqcap}}\{(g_i, \widehat{g}_i, \Sigma)^c : i = 1, 2, ..., n\} = (\Phi, \widetilde{\Re}, \Sigma)$. But this contradicts the finite intersection property of μ. Hence, we must have $\{(g_i, \widehat{g}_i, \Sigma) : i \in I\} \neq (\Phi, \widetilde{\Re}, \Sigma)$. Conversely, let every collection of BHS closed subsets of $(\widetilde{\Re}, \Phi, \Sigma)$ with the finite intersection property has a non-null BHS intersection and let $\{(g_i, \widehat{g}_i, \Sigma) : i \in I\}$ be a BHS open cover of $(\widetilde{\Re}, \Phi, \Sigma)$ so that $\widetilde{\widetilde{\sqcup}}\{(g_i, \widehat{g}_i, \Sigma) : i \in I\} = (\widetilde{\Re}, \Phi, \Sigma)$. Then, $\widetilde{\widetilde{\sqcap}}\{(g_i, \widehat{g}_i, \Sigma)^c : i \in I\} = (\Phi, \widetilde{\Re}, \Sigma)$. Hence, by hypothesis $\widetilde{\widetilde{\sqcap}}\{(g_i, \widehat{g}_i, \Sigma)^c : i = 1, 2, ..., n\} = (\Phi, \widetilde{\Re}, \Sigma)$. Then, $\widetilde{\widetilde{\sqcup}}\{(g_i, \widehat{g}_i, \Sigma) : i = 1, 2, ..., n\} = (\widetilde{\Re}, \Phi, \Sigma)$. Thus, $(\Re, \mathcal{T}_\Re, \Sigma, \neg\Sigma)$ is a BHS compact space. □

Proposition 4.11. *If $(\Re, \mathcal{T}_\Re, \Sigma, \neg\Sigma)$ is a BHS compact space, then $(\Re, \mathcal{T}^h_\Re, \Sigma)$ is a HS compact.*

Proof. Straightforward. □

Proposition 4.12. *Let $(\Re, \mathcal{T}^h_\Re, \Sigma)$ be a HSTS and let $(\Re, \mathcal{T}_\Re, \Sigma, \neg\Sigma)$ be a BHSTS constructed from $(\Re, \mathcal{T}^h_\Re, \Sigma)$ as in Proposition 2.13. If $(\Re, \mathcal{T}^h_\Re, \Sigma)$ is a HS compact space, then $(\Re, \mathcal{T}_\Re, \Sigma, \neg\Sigma)$ is a BHS compact.*

Proof. Let $(\Re, \mathcal{T}^h_\Re, \Sigma)$ be a HS compact space and let the collection $\{(g_i, \widehat{g}_i, \Sigma) : i \in I\}$ be a BHS open cover of $(\widetilde{\Re}, \Phi, \Sigma)$ such that $(\widetilde{\Re}, \Phi, \Sigma) = \widetilde{\widetilde{\sqcup}} \{(g_i, \widehat{g}_i, \Sigma) : i \in I\}$. Then, $\Re = \cup\{g_i(\ell) : i \in I\}$ for all $\ell \in \Sigma$. Since $(\Re, \mathcal{T}^h_\Re, \Sigma)$ is a HS compact space, then $\Re = \cup\{g_i(\ell) : i = 1, 2, ..., n\}$. Since $\widehat{g}(\neg\ell) = \Re \setminus g(\ell)$ for all $\ell \in \Sigma$, then $\Phi = \cap\{\widehat{g}(\neg\ell) : i = 1, 2, ..., n\}$. Hence, $(\widetilde{\Re}, \Phi, \Sigma) = \widetilde{\widetilde{\sqcup}} \{(g_i, \widehat{g}_i, \Sigma) : i = 1, 2, ..., n\}$. Thus, $(\Re, \mathcal{T}_\Re, \Sigma, \neg\Sigma)$ is a BHS compact space. □

5 Conclusions

In this paper, we have made a contribution to the field of BHSTs. In the frame of BHS maps, we have defined new types of maps namely continuous map, open map, closed map, and homeomorphism map. In addition, the concept of BHS compact space have been studied. Also, We have created some examples to validate and illustrate the obtained conclusions and relations. The relationships between these notions and their HS analogues have also been discussed. As a future work, one can extend the proposed work to IndetermSoft Set, IndetermHyperSoft Set and TreeSoft Set and their corresponding Fuzzy, Intuitionistic Fuzzy, Neutrosophic forms and other Fuzzy-extension.

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