



Application of Integral Operator Generated by Touchard Polynomials to Certain Subclasses of Harmonic Functions

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Abstract

Let $\mathcal{S}_{\mathcal{H}}$ denote the class of functions $f = h + \bar{g}$ which are harmonic univalent and sense-preserving in the unite disk $\mathbb{U} = \{z : |z| < 1\}$ where $h(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $g(z) = \sum_{k=1}^{\infty} b_k z^k$ ($|b_1| < 1$). In this paper we establish connections between various subclasses of harmonic univalent functions by applying certain integral operator involving the Touchard Polynomials.

Keywords: Harmonic univalent; Touchard Polynomials; Integral Operator

1 Introduction and preliminary results

A continuous functions $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $\mathcal{D} \subset \mathbb{C}$ we can write $f(z) = h + \bar{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f . A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$. See Clunie and Sheil-Small (see⁶).

Denote by $\mathcal{S}_{\mathcal{H}}$ the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the unit disk $\mathbb{U} = \{z : |z| < 1\}$ for which $f(0) = h(0) = f_z(0) - 1 = 0$. For $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k \quad |b_1| < 1. \quad (1)$$

Note that $\mathcal{S}_{\mathcal{H}}$ reduce to class of $\mathcal{S}$ of normalized analytic univalent functions if the co-analytic part of its member is zero. Let the subclass $\mathcal{S}_{\mathcal{H}}^0$ of $\mathcal{S}_{\mathcal{H}}$ defined by

$$\mathcal{S}_{\mathcal{H}}^0 = \{f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}} : g'(0) = b_1 = 0\}.$$

Analogous to well-known subclasses of the family $\mathcal{S}$, one can define various subclasses of the family $\mathcal{S}_{\mathcal{H}}$. A sense-preserving harmonic mapping $f \in \mathcal{S}_{\mathcal{H}}$ is in the class $\mathcal{S}_{\mathcal{H}}^*$ if the range $f(\mathbb{U})$ is starlike with respect to the

origin. A function $f \in \mathcal{S}_{\mathcal{H}}^*(\alpha)$ is called a harmonic starlike of order α , ($0 \leq \alpha < 1$) mapping in $\mathbb{U}$. Likewise a function f defined in $\mathbb{U}$ belongs to the class $\mathcal{C}_{\mathcal{H}}$ if $f \in \mathcal{S}_{\mathcal{H}}$ and if $f(\mathbb{U})$ is a convex domain. A function $f \in \mathcal{C}_{\mathcal{H}}(\alpha)$ is called harmonic convex of order α , ($0 \leq \alpha < 1$) $\mathbb{U}$. Analytically, we have

$$f \in \mathcal{S}_{\mathcal{H}}^*(\alpha) \Leftrightarrow \operatorname{Re} \left\{ \frac{zh'(z) - \overline{zg'(z)}}{h(z) + g(z)} \right\} > \alpha, \quad z \in \mathbb{U}.$$

$$f \in \mathcal{C}_{\mathcal{H}}(\alpha) \Leftrightarrow \operatorname{Re} \left\{ \frac{zh''(z) + h'(z) - \overline{zg''(z) + zg'(z)}}{h'(z) - \overline{g'(z)}} \right\} > \alpha, \quad z \in \mathbb{U}.$$

These classes have been extensively studied by Jahangiri⁷ and.²

For $\alpha = 0$, these classes $\mathcal{S}_{\mathcal{H}}^*(\alpha)$ and $\mathcal{C}_{\mathcal{H}}(\alpha)$ were denoted by $\mathcal{S}_{\mathcal{H}}^*$ and $\mathcal{C}_{\mathcal{H}}$ respectively were studied by Avci and Zlotkiewicz,⁸ Silverman⁴ and Silvia.⁵ Further, we let $\mathcal{S}_{\mathcal{H}}^{*,0}$, $\mathcal{C}_{\mathcal{H}}^0$ and $\mathcal{K}_{\mathcal{H}}^0$ denote the subclasses of $\mathcal{S}_{\mathcal{H}}^0$ of harmonic function which are, respectively, starlike, convex and close-to-convex.

Recently, the author³ introduce a series with Touchard polynomials coefficients after the second force as following:

$$F_n(z, m) = z + \sum_{k=2}^{\infty} \frac{m^{k-1}(k-1)^n}{(k-1)!} e^{-m} z^k. \quad (2)$$

It can be easily by ratio test showed that the above series is convergent and the radius of convergence is infinity.

For harmonic functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}$ and $\Omega(z) = z + \sum_{k=2}^{\infty} \psi_k z^k + \overline{\sum_{k=1}^{\infty} \varphi_k z^k}$ the convolution of f and Ω is given by

$$(f * \Omega)(z) = f(z) * \Omega(z) = z + \sum_{k=2}^{\infty} a_k \psi_k z^k + \overline{\sum_{k=1}^{\infty} b_k \varphi_k z^k}. \quad (3)$$

Now, we introduce the integral operator $I : \mathcal{S}_{\mathcal{H}} \rightarrow \mathcal{S}_{\mathcal{H}}$ as following:

$$I_n(f) \equiv I_n(m_1, m_2)f(z) = H(z) + \overline{G(z)},$$

where

$$H(z) = h(z) * \int_0^z \frac{F_n(t, m_1)}{t} dt, \quad G(z) = g(z) * \int_0^z \frac{F_n(t, m_2)}{t} dt,$$

or equivalently

$$H(z) = z + \sum_{k=2}^{\infty} \frac{m_1^{k-1}(k-1)^n}{k!} e^{-m_1} a_k z^k, \quad G(z) = b_1 z + \sum_{k=2}^{\infty} \frac{m_2^{k-1}(k-1)^n}{k!} e^{-m_2} b_k z^k. \quad (4)$$

In this paper we will apply the integral operator I for the various subclasses of harmonic univalent function.

2 Preliminary Lemmas

To prove our results, we need the following Lemmas:

Lemma 2.1. ⁽¹⁾ If $f = h + \bar{g} \in \mathcal{C}_{\mathcal{H}}^0$ where h and g are given by (1.1) with $b_1 = 0$, then

$$|a_k| \leq \frac{k+1}{2}, \quad |b_k| \leq \frac{k-1}{2},$$

Lemma 2.2. ⁽²⁾ If $f = h + \bar{g}$ where h and g are given by (1.1). If for some α ($0 \leq \alpha < 1$) and the inequality

$$\sum_{k=2}^{\infty} (k-\alpha)a_k + \sum_{k=1}^{\infty} (k+\alpha)b_k \leq 1-\alpha, \quad (5)$$

is satisfied, then f is harmonic, sense-preserving, univalent function in $\mathbb{U}$ and $f \in \mathcal{S}_{\mathcal{H}}^*(\alpha)$.

Define $\mathcal{TS}_{\mathcal{H}}^*(\alpha) = \mathcal{S}_{\mathcal{H}}^*(\alpha) \cap \mathcal{T}$, where $\mathcal{T}$ consists of the function $f = h + \bar{g}$ in $\mathcal{S}_{\mathcal{H}}$ so that h and g are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \quad g(z) = \sum_{k=1}^{\infty} |b_k| z^k \quad |b_1| < 1. \quad (6)$$

Lemma 2.3. In ⁽²⁾, it is also show that $f = h + \bar{g}$ where h and g are given by (2.2) is in the class $\mathcal{TS}_{\mathcal{H}}(\alpha)$, if and only if the coefficient conation (2.3) holds. Moreover, if $f \in \mathcal{TS}_{\mathcal{H}}^*(\alpha)$, then

$$|a_n| \leq \frac{1-\alpha}{k-\alpha}, \quad k \geq 2 \quad |b_k| \leq \frac{1-\alpha}{k+\alpha}, \quad k \geq 1.$$

Lemma 2.4. ⁽²⁾ If $f = h + \bar{g}$ where h and g are given by (1.1). If for some α ($0 \leq \alpha < 1$) and the inequality

$$\sum_{k=2}^{\infty} k(k-\alpha)a_k + \sum_{k=1}^{\infty} k(k+\alpha)b_k \leq 1-\alpha, \quad (7)$$

is satisfied, then f is harmonic, sense-preserving, univalent function in $\mathbb{U}$ and $f \in \mathcal{C}_{\mathcal{H}}(\alpha)$.

Lemma 2.5. In ⁽²⁾, it is also show that $f = h + \bar{g}$ where h and g are given by (2.2) is in the class $\mathcal{TC}_{\mathcal{H}}(\alpha)$, if and only if the coefficient conation (2.3) holds. Moreover, if $f \in \mathcal{TC}_{\mathcal{H}}(\alpha)$, then

$$|a_k| \leq \frac{1-\alpha}{k(k-\alpha)}, \quad k \geq 2 \quad |b_k| \leq \frac{1-\alpha}{k(k+\alpha)}, \quad k \geq 1.$$

Lemma 2.6. ⁽¹⁾ Let $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}^{*,0}$ or $\mathcal{C}_{\mathcal{H}}^0$, where h and g are given by (1.1), then

$$|a_k| \leq \frac{(2k+1)(k+1)}{6}, \quad |b_k| \leq \frac{(2k-1)(k-1)}{6}, \quad k \geq 2.$$

3 Main Results

In our first result, we determine conditions which guarantee that the integral operator I is a harmonic starlike in $\mathbb{U}$.

Theorem 3.1. If $0 \leq \alpha < 1$, $m_1, m_2 > 0$. Also, suppose $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}$ is given by (1.1). If the inequalities

$$(i) \sum_{k=2}^{\infty} |a_k| + \sum_{k=1}^{\infty} |a_k| \leq 1, \quad |b_1| < 1,$$

$$(ii) \sum_{i=0}^n \binom{n}{k} k^{n-i} (2 - e^{-m_1} - e^{-m_2}) \leq 1 - |b_1|$$

are satisfied, then the integral operator I is sense-preserving, harmonic univalent function and maps $\mathcal{S}_{\mathcal{H}}$ in to $\mathcal{S}_{\mathcal{H}}^*$.

Proof. Note that

$$I(m_1, m_2)f(z) = H(z) + \overline{G(z)},$$

where $H(z)$ and $G(z)$ are given by (1.4). In order to show that $I(f)$ is locally univalent and sense-preserving it suffices to show that $|H'(z)| - |G'(z)| > 0$ in $\mathbb{U}$. Using the condition (i), we have

$$\begin{aligned} & |H'(z)| - |G'(z)| \\ & > 1 - \sum_{k=2}^{\infty} k \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{k!} - \sum_{k=2}^{\infty} k \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{k!} - |b_1| \\ & = 1 - \sum_{k=0}^{\infty} \frac{e^{-m_1} m_1^{k+1} (k+1)^n}{(k+1)!} - \sum_{k=2}^{\infty} \frac{e^{-m_2} m_2^{k+1} (k+1)^n}{(k+1)!} - |b_1| \\ & = 1 - \sum_{i=0}^n \binom{n}{k} k^{n-i} \sum_{k=0}^{\infty} e^{-m_1} \frac{m_1^{k+1}}{(k+1)!} - |b_1| - \sum_{i=0}^n \binom{n}{k} k^{n-i} \sum_{k=0}^{\infty} e^{-m_2} \frac{m_2^{k+1}}{(k+1)!} \\ & = 1 - \sum_{i=0}^n \binom{n}{k} k^{n-i} (1 - e^{-m_1}) - |b_1| - \sum_{i=0}^n \binom{n}{k} k^{n-i} (1 - e^{-m_2}) \\ & = 1 - |b_1| - \sum_{i=0}^n \binom{n}{k} k^{n-i} (2 - e^{-m_1} - e^{-m_2}) \\ & \geq 0, \text{ from (ii).} \end{aligned}$$

To show that $I(f)$ is univalent in $\mathbb{U}$, we follow the method of Theorem 1 in.² That is, for $z_1 \neq z_2$ in $\mathbb{U}$, it suffices to prove that

$$\Re \frac{f(z_2) - f(z_1)}{z_2 - z_1} > \int_0^1 (\Re H'(z(t)) - |G'(z(t))|) dt. \quad (8)$$

Since from the given condition (i), we have

$$\Re H'(z(t)) - |G'(z(t))| > 1 - \sum_{k=2}^{\infty} k \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{k!} - \sum_{k=2}^{\infty} k \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{k!} - |b_1|,$$

it follows from the given hypothesis that the last inequality is positive.

Therefore, from the inequality (3.1) we have

$$\Re \frac{f(z_2) - f(z_1)}{z_2 - z_1} > 0.$$

This proves the univalence of $I(f)$.

In order to prove that $I(f) \in \mathcal{S}_{\mathcal{H}}^* \equiv \mathcal{S}_{\mathcal{H}}^*(0)$, by using Lemma 2.2 it suffices to show that

$$\sum_{k=2}^{\infty} k \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{k!} |a_k| + |b_1| + \sum_{k=2}^{\infty} k \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{k!} |b_k| \leq 1.$$

Since $|a_k| \leq 1$, $|b_k| \leq 1$, $\forall k \geq 2$, because of given condition (i), we obtain that

$$\begin{aligned} & \sum_{k=2}^{\infty} k \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{k!} |a_k| + |b_1| + \sum_{k=2}^{\infty} k \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{k!} |b_k| \\ & \leq \sum_{k=2}^{\infty} \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{(k-1)!} + |b_1| + \sum_{k=2}^{\infty} \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{(k-1)!} \\ & = \sum_{i=0}^n \binom{n}{k} k^{n-i} (1 - e^{-m_1}) + |b_1| + \sum_{i=0}^n \binom{n}{k} k^{n-i} (1 - e^{-m_2}) \\ & \leq 1, \text{ from (ii).} \end{aligned}$$

This completes the proof of Theorem 3.1.

We next find a sufficient condition for which the integral operator I maps $\mathcal{C}_{\mathcal{H}}^0$ into $\mathcal{S}_{\mathcal{H}}^*(\alpha)$.

Theorem 3.2. *If $0 \leq \alpha < 1$, $m_1, m_2 > 0$ and the inequality*

$$\sum_{i=0}^n \binom{n}{k} k^{n-i} \left[m_1 + m_2 + (2 - \alpha)(1 - e^{-m_1}) + \alpha(1 - e^{-m_2}) - \frac{\alpha}{m_1}(1 - e^{-m_1} - m_1 e^{-m_1}) \right. \\ \left. - \frac{\alpha}{m_2}(1 - e^{-m_2} - m_2 e^{-m_2}) \right] \leq 2(1 - \alpha)$$

is satisfied, then $I(\mathcal{C}_{\mathcal{H}}^0) \subset \mathcal{S}_{\mathcal{H}}^(\alpha)$.*

Proof. Let $f = h + \bar{g} \in \mathcal{C}_{\mathcal{H}}^0$ where h and g are given (1.1) with $b_1 = 0$. We need to prove that $I(f) = H + \bar{G} \in \mathcal{S}_{\mathcal{H}}^*(\alpha)$ where h and G are given by (1.4) with $b_1 = 0$ are analytic function in $\mathbb{U}$. In view of Lemma 2.2, it is enough to show that

$$\sum_{k=2}^{\infty} (k - \alpha) \left| \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{k!} a_k \right| + \sum_{k=2}^{\infty} (k + \alpha) \left| \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{k!} b_k \right| \leq 1 - \alpha. \quad (9)$$

Applying Lemma 2.1, we have

$$\begin{aligned} & \sum_{k=2}^{\infty} (k - \alpha) \left| \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{k!} a_k \right| + \sum_{k=2}^{\infty} (k + \alpha) \left| \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{k!} b_k \right| \\ & \leq \frac{1}{2} \left[\sum_{k=2}^{\infty} (k - \alpha)(k + 1) \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{k!} + \sum_{k=2}^{\infty} (k + \alpha)(k - 1) \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{k!} \right] \\ & = \frac{1}{2} \left[\sum_{k=2}^{\infty} \{k(k-1) + k(2 - \alpha) - \alpha\} \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{k!} \right. \\ & \quad \left. + \sum_{k=2}^{\infty} \{k(k-1) + k\alpha - \alpha\} \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{k!} \right] \\ & = \frac{1}{2} \left[\sum_{k=2}^{\infty} \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{(k-2)!} + (2 - \alpha) \sum_{k=2}^{\infty} \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{(k-1)!} - \alpha \sum_{k=2}^{\infty} \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{k!} \right. \\ & \quad \left. + \sum_{k=2}^{\infty} \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{(k-2)!} + \alpha \sum_{k=2}^{\infty} \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{(k-1)!} - \alpha \sum_{k=2}^{\infty} \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{k!} \right] \\ & = \frac{1}{2} \sum_{i=0}^n \binom{n}{k} k^{n-i} \left[m_1 + m_2 + (2 - \alpha)(1 - e^{-m_1}) + \alpha(1 - e^{-m_2}) \right. \\ & \quad \left. - \frac{\alpha}{m_1}(1 - e^{-m_1} - m_1 e^{-m_1}) - \frac{\alpha}{m_2}(1 - e^{-m_2} - m_2 e^{-m_2}) \right]. \end{aligned}$$

The last expression is bounded above by $1 - \alpha$ by the given hypothesis.

Thus the proof of Theorem 3.2 is established.

Theorem 3.3. *If $0 \leq \alpha < 1$, $m_1, m_2 > 0$ and if the inequality*

$$\sum_{i=0}^n \binom{n}{k} k^{n-i} \left[2m_1^2 + m(9 - 2\alpha)m_1 + (6 - 5\alpha)(1 - e^{-m_1}) - \frac{\alpha}{m_1}(1 - e^{-m_1} - m_1 e^{-m_1}) \right. \\ \left. 2m_2^2 + m(2\alpha + 3)m_2 - \alpha(1 - e^{-m_1}) - \frac{\alpha}{m_2}(1 - e^{-m_2} - m_2 e^{-m_2}) \right] \leq 6(1 - \alpha)$$

is satisfied, then $I(\mathcal{S}_{\mathcal{H}}^{,0}) \subset \mathcal{S}_{\mathcal{H}}^*(\alpha)$ and $I(\mathcal{C}_{\mathcal{H}}^0) \subset \mathcal{S}_{\mathcal{H}}^*(\alpha)$.*

Proof. Let $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}^{*,0}$ where h and g are given (1.1) with $b_1 = 0$. We need to prove that $I(f) = H + \bar{G} \in \mathcal{S}_{\mathcal{H}}^*(\alpha)$ where h and G are given by (1.4) with $b_1 = 0$ are analytic function in $\mathbb{U}$. In view of Lemma 2.2, it is enough to show that

$$\sum_{k=2}^{\infty} (k - \alpha) \left| \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{k!} a_k \right| + \sum_{k=2}^{\infty} (k + \alpha) \left| \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{k!} b_k \right| \leq 1 - \alpha. \quad (10)$$

Applying Lemma 2.4, we have

$$\begin{aligned} & \sum_{k=2}^{\infty} (k - \alpha) \left| \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{k!} a_k \right| + \sum_{k=2}^{\infty} (k + \alpha) \left| \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{k!} b_k \right| \\ & \leq \frac{1}{6} \left[\sum_{k=2}^{\infty} (k - \alpha)(2k + 1)(k + 1) \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{k!} + \sum_{k=2}^{\infty} (k + \alpha)(2k - 1)(k - 1) \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{k!} \right] \\ & = \frac{1}{6} \left[\sum_{k=2}^{\infty} \{2k(k-1)(k-2) + (9-2\alpha)k(k-1) + (6-5\alpha)k - \alpha\} \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{k!} \right. \\ & \quad \left. + \sum_{k=2}^{\infty} \{2k(k-1)(k-2) + (2\alpha+3)k(k-1) - k\alpha + \alpha\} \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{k!} \right] \\ & = \frac{1}{6} \sum_{i=0}^n \binom{n}{k} k^{n-i} \left[e^{-m_1} \left\{ 2 \sum_{k=2}^{\infty} \frac{m_1^{k-1}}{(k-3)!} + (9-2\alpha) \sum_{k=2}^{\infty} \frac{m_1^{k-1}}{(k-2)!} + (6-5\alpha) \sum_{k=2}^{\infty} \frac{m_1^{k-1}}{(k-1)!} - \alpha \sum_{k=2}^{\infty} \frac{m_1^{k-1}}{k!} \right\} \right. \\ & \quad \left. + e^{-m_2} \left\{ 2 \sum_{k=2}^{\infty} \frac{m_2^{k-1}}{(k-3)!} + (2\alpha+3) \sum_{k=2}^{\infty} \frac{m_2^{k-1}}{(k-2)!} - \alpha \sum_{k=2}^{\infty} \frac{m_2^{k-1}}{(k-1)!} + \alpha \sum_{k=2}^{\infty} \frac{m_2^{k-1}}{k!} \right\} \right] \\ & = \frac{1}{6} \sum_{i=0}^n \binom{n}{k} k^{n-i} \left[\left\{ 2m_1^2 + (9-2\alpha)m_1 + (6-5\alpha)(1-e^{-m_1}) - \frac{\alpha}{m_1}(1-e^{-m_1}-m_1e^{-m_1}) \right\} \right. \\ & \quad \left. - \left\{ 2m_2^2 + (2\alpha+3)m_2 - \alpha(1-e^{-m_1}) + \frac{\alpha}{m_2}(1-e^{-m_2}-m_2e^{-m_2}) \right\} \right] \\ & \leq 1 - \alpha, \end{aligned}$$

by the given hypothesis.

This completes the proof of Theorem 3.3.

Theorem 3.4. If $0 \leq \alpha < 1$, $m_1, m_2 > 0$, then $I(\mathcal{TS}_{\mathcal{H}}^*(\alpha)) \subset \mathcal{TS}_{\mathcal{H}}^*(\alpha)$, if and only if the inequality

$$\sum_{i=0}^n \binom{n}{k} k^{n-i} \left[\frac{1}{m_1} (1 - e^{-m_1} - m_1 e^{-m_1}) + \frac{1}{m_2} (1 - e^{-m_2} - m_2 e^{-m_2}) \right] \leq 1 - \frac{1+\alpha}{1-\alpha} |b_1|, \quad (11)$$

is satisfied.

Proof. Let $f = h + \bar{g} \in \mathcal{TS}_{\mathcal{H}}^*(\alpha)$, where h and g are given by (2.2), we need to prove that the integral operator

$$I(f(z)) = z - \sum_{k=2}^{\infty} \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{k!} |a_k| z^k + |b_1| \bar{z} + \sum_{k=2}^{\infty} \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{k!} |b_k| \bar{z}^k$$

is in $\mathcal{TS}_{\mathcal{H}}^*(\alpha)$, if and only if,

$$\sum_{k=2}^{\infty} (k - \alpha) \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{k!} |a_k| + (1 + \alpha) |b_1| + \sum_{k=2}^{\infty} (k + \alpha) \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{k!} |b_k| \leq 1 - \alpha.$$

By using Lemma 2.3, we obtain

$$\begin{aligned}
 & \sum_{k=2}^{\infty} (k-\alpha) \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{k!} |a_k| + (1+\alpha) |b_1| + \sum_{k=2}^{\infty} (k+\alpha) \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{k!} |b_k| \\
 & \leq (1-\alpha) \left[\sum_{k=2}^{\infty} \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{k!} + \sum_{k=2}^{\infty} \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{k!} \right] + (1+\alpha) |b_1| \\
 & = (1-\alpha) \sum_{i=0}^n \binom{n}{k} k^{n-i} \left[\frac{1}{m_1} (1 - e^{-m_1} - m_1 e^{-m_1}) + \frac{1}{m_2} (1 - e^{-m_2} - m_2 e^{-m_2}) \right] + (1+\alpha) |b_1| \\
 & \leq 1 - \alpha,
 \end{aligned}$$

by the given condition and this completes the proof of the theorem.

We next explore a sufficient condition which ensure that I maps $\mathcal{C}_{\mathcal{H}}^0$ into $\mathcal{C}_{\mathcal{H}}(\alpha)$.

Theorem 3.5. *If $0 \leq \alpha < 1$, $m_1, m_2 > 0$, then $I(\mathcal{C}_{\mathcal{H}}^0) \subset \mathcal{C}_{\mathcal{H}}(\alpha)$, if the inequality*

$$\sum_{i=0}^n \binom{n}{k} k^{n-i} \left[m_1^2 + m_2^2 + (4-\alpha)m_1 + (2-\alpha)m_2 \right] \leq 2(1-\alpha).$$

is satisfied.

Proof. Let $f = h + \bar{g} \in \mathcal{C}_{\mathcal{H}}^0$, where h and g are given by (2.2) with $b_1 = 0$, we need to prove that the integral operator $I(f(z)) = H + \bar{G} \in \mathcal{C}_{\mathcal{H}}(\alpha)$, where H and G are given by (1.4) with $b_1 = 0$ are analytic function in $\mathbb{U}$. In view of Lemma 2.4, it is enough to show that

$$\sum_{k=2}^{\infty} k(k-\alpha) \left| \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{k!} a_k \right| + \sum_{k=2}^{\infty} k(k+\alpha) \left| \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{k!} b_k \right| \leq 1 - \alpha.$$

By applying Lemma 2.1, we have

$$\begin{aligned}
 & \sum_{k=2}^{\infty} k(k-\alpha) \left| \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{k!} a_k \right| + \sum_{k=2}^{\infty} k(k+\alpha) \left| \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{k!} b_k \right| \\
 & \leq \frac{1}{2} \left[\sum_{k=2}^{\infty} (k-\alpha)(k+1) \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{(k-1)!} + \sum_{k=2}^{\infty} (k+\alpha) \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{(k-2)!} \right] \\
 & = \frac{1}{2} \left[\sum_{k=2}^{\infty} \{ (k-1)(k-2) + (4-\alpha)(k-1) + 2(1-\alpha) \} \frac{e^{-m_1} m_1^{k-1} (k-1)^n}{(k-1)!} \right. \\
 & \quad \left. + \sum_{k=2}^{\infty} \{ (k-2) + (2+\alpha) \} \frac{e^{-m_2} m_2^{k-1} (k-1)^n}{(k-2)!} \right] \\
 & = \frac{1}{2} \sum_{i=0}^n \binom{n}{k} k^{n-i} \left[m_1^2 + (4-\alpha)m_1 + 2(1-\alpha)(1 - e^{-m_1}) + m_2^2 + (2-\alpha)m_2 \right] \\
 & \leq 1 - \alpha,
 \end{aligned}$$

by the given hypothesis. Thus the proof of Theorem 3.5 is established.

The proof of the following theorems are similar to previous theorems so we state only the results.

Theorem 3.6. *If $0 \leq \alpha < 1$, $m_1, m_2 > 0$, then $I(\mathcal{TS}_{\mathcal{H}}^*(\alpha)) \subset \mathcal{TC}_{\mathcal{H}}(\alpha)$, if and only if the inequality*

$$\sum_{i=0}^n \binom{n}{k} k^{n-i} [e^{-m_1} + e^{-m_2}] \leq 1 + \frac{1+\alpha}{1-\alpha} |b_1|,$$

is satisfied.

Theorem 3.7. If $0 \leq \alpha < 1$, $m_1, m_2 > 0$, then $I(\mathcal{TC}_{\mathcal{H}}(\alpha)) \subset \mathcal{TC}_{\mathcal{H}}(\alpha)$, if and only if the inequality (3.4) is satisfied.

Theorem 3.8. If $0 \leq \alpha < 1$, $m_1, m_2 > 0$ and if the inequality

$$\sum_{i=0}^n \binom{n}{k} k^{n-i} e^{m_1} \left[2(m_1^3 + m_2^3) + (15 - 2\alpha)m_1^2 + 3(8 - 3\alpha)m_1 + (2\alpha + 9)m_2^2 + 3(2 + \alpha)m_2 \right] \leq 6(1 - \alpha),$$

is satisfied, then $I(\mathcal{S}_{\mathcal{H}}^{*,0}) \subset \mathcal{C}_{\mathcal{H}}(\alpha)$ and $I(\mathcal{C}_{\mathcal{H}}^0) \subset \mathcal{K}_{\mathcal{H}}(\alpha)$.

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