On The Fuzzy Weak Complex Vector Spaces

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Abstract
The objective of this paper is to present for the first time the concept of fuzzy weak complex vector spaces depending on the ring of fuzzy weak complex numbers. Also, we examine some elementary algebraic properties of fuzzy weak complex vector spaces in terms of theorems.
On the other hand, we illustrate many related examples to explain the novelty of these spaces.

Keywords: Fuzzy weak complex number; fuzzy weak complex vector space; fuzzy weak complex ring; AH-subspace; AH-linear transformation.

Introduction

Fuzzy logic was presented by Zadeh [1] to deal with a degree of truth (T) and a degree of falsity (F), where a fuzzy set can be represented by values through the interval [0,1].

In [2], the concept of weak fuzzy complex numbers was defined for the first time as a new generalization of real numbers. This generalization is built in a similar way of split-complex numbers [3], neutrosophic numbers [4] and dual numbers [5].

In the literature, we find the concept of neutrosophic vector spaces which are defined by using neutrosophic numbers. These spaces form a module over the ring of neutrosophic real numbers.

Also, many algebraic substructures that describe the properties of neutrosophic vector spaces were studied widely such as AH-subspaces, AHS-subspaces, and AH-linear functions [6-9].

In this paper, we use the previous approach to define weak fuzzy complex vector spaces over the weak fuzzy complex ring of numbers, and we examine many elementary properties of these new spaces by proving some related theorems and illustrating some corresponding examples.

Main Discussion:

Definition:

Let $C_w = \{a + b\varepsilon; a, b \in R\}$ be the ring of weak fuzzy complex numbers with $t = \varepsilon^2 \in]0,1[$.
Let V be a vector space over the real field R, we define the fuzzy weak complex vector space V_w as follows:
$V_w = \{x + y\varepsilon\, ; x, y \in V\}$.
We define addition on V_w as follows:
$(x + y\varepsilon) + (z + t\varepsilon) = (x + z) + (y + t)\varepsilon$
We define the multiplication as follows:
$(a + b\varepsilon).(x + y\varepsilon) = ax + byt + \varepsilon(ay + bx)\, ; x, y \in V\, , a, b \in R$.
Theorem.
$(V_w, +, .)$ Is a module over the ring C_w.

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Proof.

Let $X = x_1 + x_2\varepsilon$, $Y = y_1 + y_2\varepsilon$ be two weak complex vectors and let $A = a_1 + a_2\varepsilon$, $B = b_1 + b_2\varepsilon$ be two weak complex numbers, we have:

1. $(A + B)X = A.X + B.X$
2. $A(X + Y) = A.X + A.Y$
3. $(A.B).X = A.(B.X)$
4. $1.X = X$
5. $(V_w, +)$ is abelian group.

Remark.

$(V_w, +, .)$ is a module not a vector space.

Definition.

Let W_w be a non empty subset of V_w,we call W_w a subspace of V_w if and only if:
$X - Y \in W_w$; $\forall X, Y \in W_w$
$A.X \in W_w$; $\forall A \in C_w$, $\forall X \in W_w$

Definition.

Let $S = \{V_1 , \dots , V_n\}$ be a subset of V_w, then it is called a basis of V_w over C_w if and only if:

1. , $\forall\ T \in V_w$, then $T = \sum_{i=1}^{n} A_i V_i$; $V_i \in S$, $A_i \in C_w$.
2. $\sum_{i=1}^{n} A_i V_i = 0$, then $A_i = 0$ for all $1 \leq i \leq n$.

Example.

Let $V = R^2$ be the Euclidean space over R.
$V_w = \{(x_1 + y_1\epsilon, x_2 + y_2\varepsilon) ; x_i, y_i \in R\} = \{(x_1, x_2) + (y_1, y_2)\varepsilon ; x_i, y_i \in R\}$
Is the corresponding weak fuzzy complex vector space.
Consider the following subset:
$W_w = \{(x, 0) + (z, y)\varepsilon ; x, y, z \in R\}$,
It is clear that $(W_w, +)$ is a subgroup of $(V_w, +)$.
Let $A = a_1 + a_2\varepsilon$; $a_i \in R$, then
$A[(x, 0) + (z, y)\varepsilon] = (a_1 + a_2\varepsilon)[(x, 0) + (z, y)\varepsilon] = a_1(x, 0) + (a_1 z, a_1 y)\varepsilon + a_2(x, 0)\varepsilon + (a_2 z, a_2 y)t = (a_1 x + a_2 zt, a_2 yt) + \varepsilon[(a_1 z + a_2 x, a_1 y)] \notin W_w$.
Thus W_w is not a subspace.

Definition.

Let V_w be a weak fuzzy complex vector space over C_w.
Let $W_w = V_1 + V_2\varepsilon$; $V_1, V_2$are two subspaces of V, then we call W_w an AH-subspace.

Example.

Let $V = R^2$ be the Euclidean space over R with three dimensions.
Let V_w be the corresponding weak fuzzy complex vector space over C_w.
Let $V_1 = \{(x_1, 0,0);\ x_1 \in R\}$, $V_2 = \{(0, x_2, 0);\ x_2 \in R\}$.

Theorem:

Let $W_w = V_1 + V_2\varepsilon$ be an AH-subspace of V_w, then W_w is a subspaces if and only if $V_1 = V_2$.
Proof:
Suppose that W_w is a subspace of V_w.
This is equivalent to the following statement:
For any $A = a_1 + a_2\varepsilon$ in C_w, then for any $X = x_1 + x_2\varepsilon \in W_w$, we have $A.X \in W_w$, so that $(a_1 + a_2\varepsilon)(x_1 + x_2\varepsilon) \in W_w$, thus $(a_1x_1 + a_2x_2t) + \varepsilon(a_1x_2 + a_2x_1) \in W_w$, this means that:
$a_1x_1 + a_2x_2t \in V_1$, $a_1x_2 + a_2x_1 \in V_2$.
On the other hand, from the definition of W_w, we have $x_1 \in V_1$, $x_2 \in V_2$
Hence, $a_2x_2t \in V_1$ and $a_2x_1 \in V_2$.
Which implies that $V_1 \subseteq V_2$ and $V_2 \subseteq V_1$, then $V_1 = V_2$.
So that $W_w = V_1 + V_1\varepsilon$.

Example.

For $W_w = R^2$ the Euclidean weak fuzzy complex vector space.
Let $W_w = \{(x_1, 0) + (x_2, 0)\varepsilon = (x_1 + x_2\varepsilon, 0);\ x_1, x_2 \in R\}$
Then W_w is a subspace of V_w.
Now, we will check the structure of the basis of V_w.
Let $V_w = V + V\varepsilon = \{x + y\varepsilon;\ x, y \in V\}$ be a weak fuzzy complex vector space.
Let $S = \{V_1 , \dots , V_n\}$ be the basis of V over R.

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Let $A = a_1 + a_2\varepsilon , B = b_1 + b_2\varepsilon$ be two weak fuzzy complex numbers with $A.X + BY = 0$, where $X = x_1 + x_2\varepsilon$, $Y = y_1 + y_2\varepsilon$. This means that:

$$(a_1x_1 + a_2x_2t) + \varepsilon(a_1x_2 + a_2x_1) + (b_1y_1 + b_2y_2t) + \varepsilon(b_1y_2 + b_2y_1) = 0$$

$$\Rightarrow \begin{cases} a_1x_1 + b_1y_1 + t(a_2x_2 + +b_2y_2) = 0 \ldots (1) \\ a_1x_2 + a_2x_1 + b_1y_2 + b_2y_1 = 0 \ldots \ (2) \end{cases}$$

Since S is a basis of V, we get:

$$x_1 = \sum_{i=1}^{n} t_iV_i \ , \qquad x_2 = \sum_{i=1}^{n} \acute{t}_iV_i$$

$$y_1 = \sum_{i=1}^{n} s_iV_i \ , \qquad y_2 = \sum_{i=1}^{n} \acute{s}_iV_i$$

; $t_i, \acute{t}_i, s_i, \acute{s}_i \in R$.

By using equation (1) and (2), we get:

$$\sum_{i=1}^{n} (a_1t_i + ta_2\acute{t}_i + b_1s_i + tb_2\acute{s}_i)V_i = 0$$

And

$$\sum_{i=1}^{n} (a_1\acute{t}_i + a_2t_i + b_1\acute{s}_i + b_2s_i)V_i = 0$$

Since S is linearly independent, hence:

$$\begin{cases} a_1t_i + ta_2\acute{t}_i + b_1s_i + tb_2\acute{s}_i = 0 \\ a_1\acute{t}_i + a_2t_i + b_1\acute{s}_i + b_2s_i = 0 \end{cases}$$

For all $1 \leq i \leq n$.

Since x_1, x_2, y_1, y_2 are arbitrary elements in V, then we can chose $s_i, \acute{s}_i, t_i, \acute{t}_i$ to get $a_1 = a_2 = b_1 = b_2 = 0$.

But, we can not ensure that for all $X, Y \in V_w$ if we get the same results.

Remark.

The finding of a general basis of V_w is still an open problem in general.

Definition.

Let V_w, W_w be two weak fuzzy complex vector spaces over C_w.

Let $f: V_w \to W_w$ be a mapping, than f is called a linear transformation if and only if:

$$f(x + y) = f(x) + f(y)$$
$$f(Ax) = Af(x) \ ; \ \forall x, y \in V_w \ , A \in C_w$$

Definition.

Let V, W be two vector spaces over R, and V_w, W_w be the corresponding weak fuzzy complex vector spaces.

Let $f_1, f_2: V_w \to W_w$ be two linear transformations, then we define $f = (f_1, f_2): V_w \to W_w$ such that $f(x + y\varepsilon) = f_1(x) + f_2(y)\varepsilon$,

Is called AH-linear transformation.

Now, we will try to check if an AH-linear transformation can be an ordinary linear transformation.

For this goal, we assume that $f = (f_1, f_2)$ is a linear transformation.

Then $\forall x_1, y_1 \in V$, then $x_1 + 0\varepsilon \ , y_1 + 0\ \varepsilon \in V_w$

$$f(x_1 + 0\varepsilon + y_1 + 0\ \varepsilon) = f_1(x_1, y_1) = f_1(x_1) + f_2(y_1)$$

Also, $x_1 + 0\varepsilon \ , y_1 + 0\ \varepsilon \in V_w$, we have:

$f(x_1 + 0\varepsilon + y_1 + 0\ \varepsilon) = f_2(x_1 + y_1)\varepsilon = [f_2(x_1) + f_2(y_1)]\varepsilon$.

Thus $f_2(x_1 + y_1) = f_2(x_1) + f_2(y_1)$.

On other hand, for every $a \in R$, we have: $A_1 = a + 0\varepsilon \in C_w$

$f[A_1(0 + x_1\varepsilon)] = af_1(x_1)$ and $f[A_1(0 + x_1\varepsilon)] = f_2(ax_1)\varepsilon$, hence $f_2(ax_1) = af_2(x_1)$

The previous discussion ensures that every AH-transformation $f = (f_1, f_2)$ is an ordinary linear transformation implies that f_1, f_2 are linear transformation between V and W.

For the converse, we assume that $f = (f_1, f_2)$ is an AH-linear transformation, we will check if f is an ordinary linear transformation.

Let $X = x_1 + x_2\varepsilon, Y = y_1 + y_2\varepsilon$, then:

$f(X + Y) = f_1(x_1 + y_1) + f_2(x_2 + y_2\varepsilon) = [f_1(x_1) + f_2(x_2)\varepsilon] + [f_1(y_1) + f_2(y)\varepsilon] = f(X) + f(Y)$

Take $A = a_1 + a_2\varepsilon \in C_w$, then:

$f(AX) = f_1[a_1x_1 + a_2x_2]\varepsilon + f_2[a_1x_2 + a_2x_1]\varepsilon = a_1f_1(x_1) + a_2tf_2(x_2) + [a_1f_2(x_2) + a_2f_2(x_1)]\varepsilon$

$Af(X) = a_1f_1(x_1) + a_2f_2(x_2)t + \varepsilon[a_1f_2(x_2) + a_2f_1(x_1)]$

$f(AX) = Af(X)$ if and only if $f_1(x_2) = f_2(x_2)$; $\forall x_2 \in V$. This means that $f_1 = f_2$.

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