Approximate Solution of Boundary Value Problem for Heat Equation after Represented by Volterra Integral Equation of the First Kind

H.K. Al-Mahdawi [1], Mostafa Abotaleb [2], Hussein Alkattan [2], El-Sayed M El-kenawy [*3]

[1] Electronic Computer Centre, University of Diyala, Diyala ,32001, Iraq
[2] Department of System Programming, South Ural State University, 454080 Chelyabinsk, Russia
[3] Department of Communications and Electronics, Delta Higher Institute of Engineering and Technology, Mansoura 35111, Egypt
Emails: hssnkd@gmail.com; abotalebmostafa@bk.ru; alkattan.hussein92@gmail.com; skenawy@ieee.org

Abstract

In this work, we study the regularization method for solving the Boundary Value Problem (BVP) for heat equation. The discretization method applied with two variables on Volterra integral equation in order to covert the problem into a linear operator equation after applied the separation of variables method to solve the partial differential equation. The regularization way used to obtain the estimate solution by using the Lavrentiev regularization method.

Keywords: Ill-posed problem; Lavrentiev regularization; Inverse problem; Heat conduction

1. Introduction

Resolving of a cost of thermal amounts is converted to resolutions of inverse problems (IPs) of conductivity for thermal and it is a slightly problem. Hence, in numerous IPs of heat transmission are the base key of getting the required observation data when solving the problems. Identification and diagnostics of heat processes for transferring case can be connected with solving IPs of many kinds [1]–[5], nevertheless, boundary value problems BVP , which are the one of most greatest, significant and common classes of IPs for thermal form. The preparation and study of same problems are dedicated to the works [6]–[13].

Many works studied the discretization algorithms as an significant way for reducing the integral equations to system equations of linear algebra [14]–[17].The estimated error with the discretization of an integral equation [18][19] and with other method such as Fourier transform method [20]. The finite-dimensional approximation domain with residual methods for regularization solutions as studied in [7], [8],[22]–[25]. There are numerous papers which explained and used the regularization methods for solving IPs , especially when the IPs in the integral equation for the first kind form [26], [27]. The achievement of these regularization methods are based on thoughtful and investigating the mathematical IPs related to the statements of the belongings for problems and classifying explicit problems in solving them [28]–[33].

The core goal in this paper is solving BVP in the inverse heat equation by applying separation of the variable on it as a first step in the process of solution. Then we obtain an integral equation of fist kind after end the separation of the variable step. Hence, the solution for that problems in form of integral equation does not base on the information in mathematical spaces, so that cause the solution has unstable property, consequently for all that this problem known as an ill-posed. For solving the ill-posed, the significant part is worked for the error approximations of the solution. In this work, we get approximations solution by using the modified Lavrentiev regularization method on linear operator equation or system of linear algebra equations these results are new and practical interest.

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All steps are planned over the next parts in the article as following. Section 2 describes the system of partial differential equation PDE for heat equation and labels the solution of that problem as the integral equation of the first kind. Section 3 pronounced the general discretization process for changing the integral equation to the linear operator equation. Section 4 the statement of the IP is considered by defining the ill-posed IP with special conditions to the integral equation and the Lavrentiev regularization methods and finding the estimate solution for linear operator form with a non-injective operator. After that, in section 5 the case study offered to verify the estimated result. Lastly, the description of the proposed scheme summarized in the sector of conclusion.

2. Problem Statement

Consider inverse BVP

$$\frac{\partial w(x,t)}{\partial t}=\frac{\partial^2 w(x,t)}{\partial x^2}, \qquad 0<x<1, \quad t\in(0,T], \tag{1}$$

$$w(x,0)=0, \; x\in[0,1], \tag{2}$$

$$\frac{\partial w(0,t)}{\partial t}=0, \; t\in(0,T], \tag{3}$$

$$w(1,t)=w(t), \; t\in[0,T], \tag{4}$$

let's assume that $w(t)$ - is function with following condition

$$w(t)\in H^4[0,T], w(0)=w'(0)=w''(0)=w'(T)=w''(T)=0 \tag{5}$$

and $\int_0^T w''(t)dt \le r^2$, $r-$ given value. By applying method "separation of variables" [20], we get

$$w(x_0,t)=w(t)+\sum_{n=1}^{\infty} g_n(t)\cos(n+0.5)\pi x_0, \tag{6}$$

where $x_0\in(0,1), t\in[0,T]$,

$$g_n(t)=\frac{2e^{-(n+0.5)^2\pi^2 t}}{(n+0.5)\pi}\int_0^t w'(\tau)e^{(n+0.5)^2\pi^2\tau}d\tau \tag{7}$$

integrating twice by parts the right part for (7), we get

$$\begin{aligned} g_n(t)=&\frac{2}{(n+0.5)\pi}\int_0^t w'''(\tau)e^{-(n+0.5)^2\pi^2(t-\tau)}d\tau+\frac{2}{(n+0.5)^3\pi^3}w'(t)\\ &-\left[\frac{2}{(n+0.5)^3\pi^3}w'(0)+\frac{2}{(n+0.5)^5\pi^5}w''(t)+\frac{2}{(n+0.5)^5\pi^5}w''(0)\right]\end{aligned} \tag{8}$$

and

$$\begin{aligned} g_n(t)=&\frac{2}{(n+0.5)^5\pi^5}\int_0^t w'''(\tau)e^{-(n+0.5)^2\pi^2(t-\tau)}d\tau\\ &-\left[\frac{2}{(n+0.5)^3\pi^3}w'(0)+\frac{2}{(n+0.5)^5\pi^5}w''(0)\right]\end{aligned} \tag{9}$$

where $w'(0)=w''(0)=0$ according to (5)

$$g_n(t)=\frac{2}{(n+0.5)^5\pi^5}\int_0^t w'''(\tau)e^{-(n+0.5)^2\pi^2(t-\tau)}d\tau \tag{10}$$

by (6) and (10) we obtain the next integral equation

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$$f(t)=w(x_0,t)=\int_0^t\left[w(t)+\sum_{n=1}^{\infty}\frac{2\cos(n+0.5)\pi x_0}{(n+0.5)^5\pi^5}e^{-(n+0.5)^2\pi^2(t-\tau)}\right]w'''(\tau)d\tau. \quad (11)$$

3. Discretization steps for integral equation

The integral equation is careful in the next form to shorten the multi-time discretization algorithm for the derivative for the kernel [19].

$$Aw(s)=\int_a^b L(s,t)w(s)ds=f(t), c\le t\le d, \quad (12)$$

where $w(s)\in L_2[a,b], f(t)\in L_2[c,d]$, kernel $L(s,t)$ the A operator is bounded, $L(s,t)$ bounded by $[a,b]\times[c,d]$, rectangle, $t\in[c,d]$, $L(s,t)$ is piecewise nonstop on $[a,b]$, and for $l\ge 2$, $D'_t(s,t)\in C([a,b]\times[c,d])$

$$D(s,t)=\int_a^s\frac{(\theta-s)^{l-1}}{(l-1)!}L(\theta,t)d\theta. \quad (13)$$

Assume that $f(t)=f_0(t)$ there is an exact solution $w_0(s)$ of equation (12) belonging to the following set M_r:

$$M_r=\left\{w(s): w^{[l]}(s)\in L_2[a,b], w(a)=w'(a)\text{K } w^{[l-1]}(a)=0, \int_a^b\left|w^{[l-1]}(s)\right|^2 ds\le r^2\right\}. \quad (14)$$

Since the kernel $L(s,t)$ is closed, the solution $w_0(s)$ of equation (12) is unique.

We introduce the operator B, which maps from the space $L_2[a,b]$ to $L_2[a,b]$, using the formula

$$w(s)=Bq(s)=\int_a^s\frac{(s-\theta)^{l-1}}{(l-1)!}q(\theta)d\theta, Bq(s)\in L_2[a,b] \quad (15)$$

and introduce the operator C,

$$Cq(s)=ABq(s); q(s)\in L_2[a,b], Cq(s)\in L_2[c,d]. \quad (16)$$

From (15) and (16) follow that

$$Cq(s)=\int_a^b D(s,t)q(s)ds, c\le t\le d, \quad (17)$$

where $D(s,t)$ is determined by the formula (13).

In order to substitute the C operator with a finite-dimensional, we present $Z(t)$ as the function:

$$Z(t)=\max_{a\le s\le b}\left|D'_s(s,t)\right|, c\le t\le d, \quad (18)$$

and number Z:

$$Z=\max\left\{\left|D'_t(s,t)\right|; c\le t\le d, a\le s\le b\right\} \quad (19)$$

Split $[a,b]$ interval into n equal parts with points $s_i=a+\frac{i(b-a)}{n}, i=0,1,\text{K }, n-1$, and interval $[c,d]$, into m equal parts with points $t_i=c+\frac{j(d-c)}{m}, i=0,1,\text{K }, m-1$. We present the next features:

$$\bar{D}_i(t)=D(s_i,t), \quad (20)$$

$$\hat{D}_n(s,t)=\bar{D}_i(t); s_i\le s\le s_{i+1}, t\in[c,d], i=0,1,\text{K }, n-1, \quad (21)$$

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$$\breve{D}_{nm}(s,t)=\bar{D}_i(t_j);s_i\le s\le s_{i+1},t_j\le t\le t_{j+1},i=0,1,\ldots,n-1,\ j=0,1,\ldots,m-1, \tag{22}$$

Using formulas (20) - (22), we define operators C_n and C_{nm} as follows:

$$C_nq(s)=\int_a^b\breve{D}_n(s,t)q(s)ds,t\in[c,d],$$

$$C_{nm}q(s)=\int_a^b\breve{D}_{nm}(s,t)q(s)ds,t\in[c,d],$$

and suppose that these operators C_n and C_{nm} map $L_2[a,b]$ to $L_2[c,d]$,
Now we decrease the integral formula (12) to the following linear operator equation

$$C_{nm}q(s)=f_\delta^m(t) \tag{23}$$

Where $f_\delta^m=H_m\left|f_\delta(t)\right|$ from (23), $H_m:L_2(c,d)\to Y_m$ where Y_m subspaces under spaces $L_2(c,d)$.

4. Statement of IP for (1) - (4)

The inverse BVP for the heat equation is considered. We want to discovery $w(t)$, where $w(t)\in H^4[0,T]$, and let defines the function $f(t)$ by,

$$f(t)=w(x_0,t),0\le t\le T \tag{24}$$

Where $w(x_0,t)$ solution of the problem (1) - (4), and $x_0\in[0,1]$.
It is well known that the inverse boundary value problem (1) - (4), (24) is ill-posed. Therefore, suppose that for $f(t)=f_0(t)$ There exists a $w_0(t)$, filling condition (5).

$$f_0(t)=w(x_0,t) \tag{25}$$

Suppose that $f_0(t)$ is unidentified. Instead we have $f_\delta(t)\in L_2[0,T]$ and $\delta>0$ such that

$$\|f_\delta(t)-f_0(t)\|\le\delta \tag{26}$$

The problem is as follows: given f_δ,δ, M_r and an approximate solution $w_\delta(t)\in L_2[0,T]$ of the inverse boundary value problem (1) - (4), (24) and estimate the deviation $\|w_\delta(t)-w_0(t)\|_{L_2}$ of the estimated $w_\delta(t)$ from the exact solution $w_0(t)$
Assume $w'''(\tau)=q(\tau)$

$$Cq(s)=\int_0^t D(\tau,t)q(\tau)d\tau=f_\delta(t) \tag{27}$$

where $w(t)=Bq(\tau)=\int_a^t\frac{(t-\tau)^2}{2}q(\tau)d\tau,q(\tau)Bq(\tau)\in L_2[0,T]$ and $C=AB$,

$$D(\tau,t)=w(t)+\sum_{n=1}^{\infty}\frac{2\cos(n+0.5)\pi x_0}{(n+0.5)^5\pi^5}e^{-(n+0.5)^2\pi^2(t-\tau)} \tag{28}$$

We define a finite-dimensional operator C_n, using steps (20) - (23), we get:

$$\bar{D}_i(t)=D(\tau_i,t), \tag{29}$$

$$\bar{D}_i(t),\tau_i\le\tau\le\tau_{i+1},t\in[0,T],i=0,1,\ldots,n-1, \tag{30}$$

$$\bar{D}_i(t_j),\tau_i\le\tau\le\tau_{i+1},t_j\le t\le t_{j+1},j,i=0,1,\ldots,n-1, \tag{31}$$

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$$D_n(\tau,t) = \bar{D}_i(t_j) \tag{32}$$

$$C_n q(\tau) = \int_0^t D_n(\tau,t) q(\tau) d\tau \tag{33}$$

the kernel value depended on the next condition

$$\bar{D}_i(t_j) = \begin{cases} D(\tau_i, t_j) & i \le j \\ 0 & i > j \end{cases}, \tag{34}$$

from above, we convert the BVP for heat PDE into a linear operator equivalence

$$C_n \begin{bmatrix} q(\tau_0) \\ q(\tau_1) \\ \vdots \\ q(\tau_{n-1}) \end{bmatrix} = \begin{bmatrix} f(t_0) \\ f(t_2) \\ \vdots \\ f(t_{n-1}) \end{bmatrix}, \tag{35}$$

where

$$C_n = \frac{1}{n} \begin{bmatrix} D(\tau_0, t_0) & 0 & & \dots & 0 \\ D(\tau_0, t_1) & D(\tau_1, t_1) & & \dots & 0 \\ \vdots & \vdots & \dots & & \vdots \\ D(\tau_0, t_{n-1}) & D(\tau_1, t_{n-1}) & \dots & D(\tau_{n-1}, t_{n-1}) & \end{bmatrix},$$

The operator C_n in (35) is a **non-injective** because the matrix of this operator is a triangular matrix..

5. Solution of the inverse problem by the Lavrentiev method

The operator C_n in (35) has non-injective property, since the matrix of this operator is a lower down triangular matrix and $N(C_n) \neq \{\bar{0}\}$., Tikhonov regularization is not appropriate in this case, because

$$N(C_n^*) \neq \{\bar{0}\},\ f_\delta \perp N(C_n^*) \Rightarrow C_n^* f_\delta = \bar{0}.$$

Then, to solve problem (35) the Lavrentiev method is proposed. This method is founded on replacing the (35) by a family of operator equations of the second kind dependent on the parameter α>0 as follows

$$Au + \alpha Qu = f_\delta, \tag{36}$$

where $Q-$ defined as a unitary operator, using a polar expansion

$$A = Q\bar{A} \tag{37}$$

where $\bar{A} = \sqrt{A^* A}$. We can find Q ,by using the SVD “singular value decomposition” on operator A [34] (page. 367–385),

$$A = U\Sigma V^*, \tag{38}$$

$$Q = UV^*,\ \bar{A} = VUV^*. \tag{39}$$

The Lavrentiev method explained in [35], the operators regularizing family $\{R_\alpha : 0 < \alpha < \alpha_0\}$, is used, mapping from $L_2[0,T]$ into $L_2[0,T]$ and defined by the formula

$$R_\alpha = \bar{B}\left(\bar{C} + \alpha E\right)^{-1} Q^*,\ 0 < \alpha \le \alpha_0, \tag{40}$$

$\bar{C} = \bar{A}\bar{B}$ and $E-$ is the identification operator, but we have a finite-dimensional operator $\bar{C}_n$ as an alternative of $\bar{C}$. The finite-dimensional regularizing operators’ family of $\{R_{n,\alpha} : 0 < \alpha < \alpha_0\}, R_{n,\alpha} : L_2[0,T] \to L_2[0,T]$, defined by the formula

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$$R_{n,\alpha} = \bar{B}_n \left(\bar{C}_n + \alpha E\right)^{-1} Q_n^*, \ 0 < \alpha \le \alpha_0, \tag{41}$$

where $\bar{C}_n = \bar{A}_n \bar{B}_n$.

The regularized solution of IP is demarcated by

$$u_\delta^\alpha = R_{n,\alpha} f_\delta, \tag{42}$$

we define the variational problem for considering the finite-dimensional approximation with Lavrentiev method.

$$\inf \left\{ \left\| \bar{C}_n v + \alpha Q \bar{B}_n v - f_\delta \right\| : v \in M_r \right\}, \tag{43}$$

Now let estimate the deviation $\left\| u_\delta^\alpha - u_0 \right\|$ for u_δ^α from u_0

$$\begin{aligned} \left\| u_\delta^\alpha - u_0 \right\| &\le \sup \left\{ \left\| u_\delta^\alpha - u_0^\alpha \right\| : u_0 \in L_2[0,T], \left\| f_\delta - Au_0 \right\| \le \delta \right\} \\ &+ \sup \left\{ \left\| u_0^\alpha - u_0 \right\| : u_0 \in L_2[0,T] \right\}, \end{aligned} \tag{44}$$

where $u_0^\alpha = R_{n,\alpha} f_0$, from (43)

$$\left\| u_\delta^\alpha - u_0 \right\| \le \left\| R_{n,\alpha} \right\| \delta + \sup \left\{ \left\| R_{n,\alpha} \bar{C}_n v_0 - \bar{B}_n v_0 \right\| \right\}, \tag{45}$$

the next we need to determine the value of the parameter $\alpha(\delta)$ in (42) using V.N. Strakhov way from the conditional equation.

$$\inf \left\{ \left\| R_{n,\alpha} \right\| \delta + \sup_{\|v_0\|^2 \le r^2} \left\| R_{n,\alpha} \bar{C}_n v_0 - \bar{B}_n v_0 \right\| \right\}, \tag{46}$$

where $\left\| R_{n,\alpha} \right\| \delta = \left\| u_\delta^\alpha - u_0^\alpha \right\|$, define $\Delta_1(\alpha(\delta))$ and $\Delta_2(\alpha)$

$$\Delta_1(\alpha(\delta)) = \left\| R_{n,\alpha} \right\| \delta, \tag{47}$$

$$\Delta_2(\alpha) = \sup_{\|v_0\|^2 \le r^2} \left\| R_{n,\alpha} \bar{C}_n v_0 - \bar{B}_n v_0 \right\|, \tag{48}$$

where $\Delta_1(\alpha(\delta)) = \Delta_2(\alpha)$ for $\alpha = \delta(\alpha)$ and $\delta(\alpha) \to 0$ for $\delta \to 0$, whence equation (45) takes the form

$$\left\| u_\delta^\alpha - u_0 \right\| \le 2\Delta_1(\alpha(\delta)), \tag{49}$$

from (47)

$$2\Delta_1(\alpha(\delta)) = 2\left\| R_{n,\alpha(\delta)} \right\| \delta, \tag{50}$$

from (49) and (50)

$$\left\| u_\delta^\alpha - u_0 \right\| \le 2\left\| R_{n,\alpha(\delta)} \right\| \delta, \tag{51}$$

Using (51), one can choose the best regularization parameter $\alpha(\delta)$ in (42).

6. Case study with BVP

Considering BVP (1) - (4) for the PDE of heat transfer, it is necessary, using the Lavrentiev method, to find $u_\delta^\alpha(t) \in H^4\left[0,T\right]$, the true solution of $u(t)$, as shown in fig. 1.

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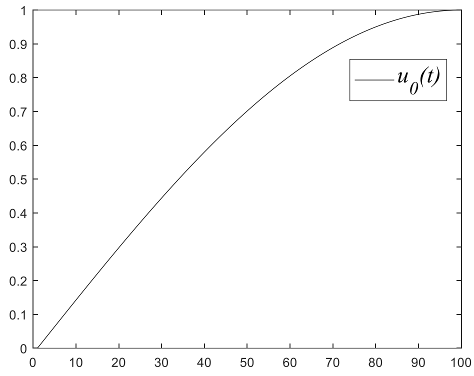

Figure 1: Exact solution of the inverse problem $u_0(t)$

Input function for $u(x_0,t) = f_0(t)$, where $x_0 \in (,1), t \in [0,T], x_0 = 0.05$ and $T = 5$, as shown in fig. 2.

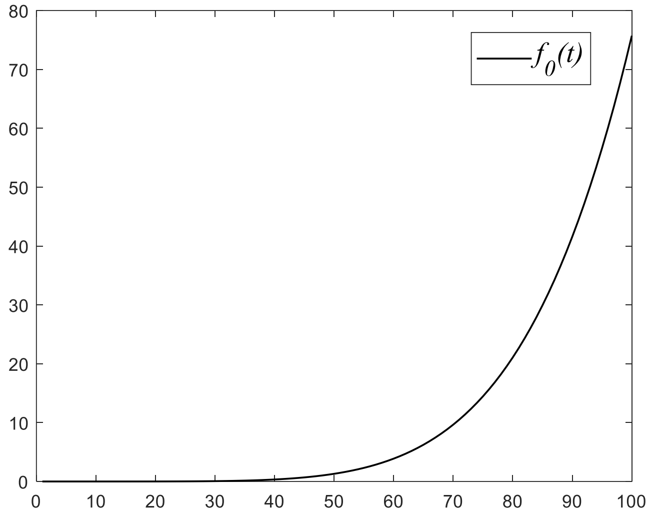

Figure 2: $u(x_0,t) = f_0(t)$, $x_0 = 0.05$ and $T = 5$

We have separated the intervals for the kernel in (31) by $n = 100$. We have considered δ, having implemented the Lavrentiev method, we have obtained some solutions shown in fig. 3. and fig. 4.

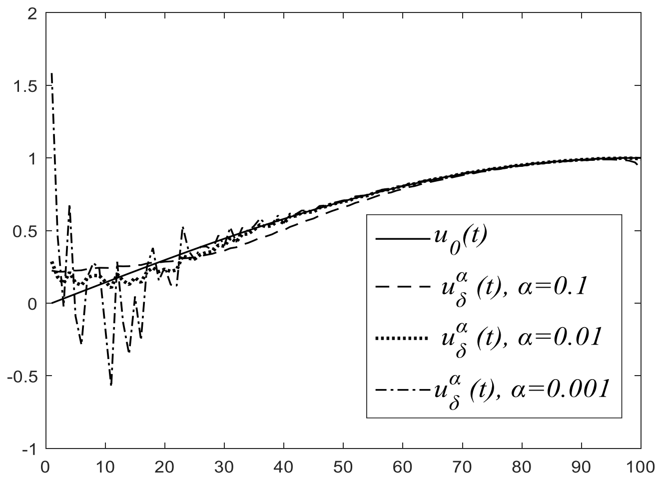

Figure 3: $u_\delta^\alpha(t), \delta = 0.006$

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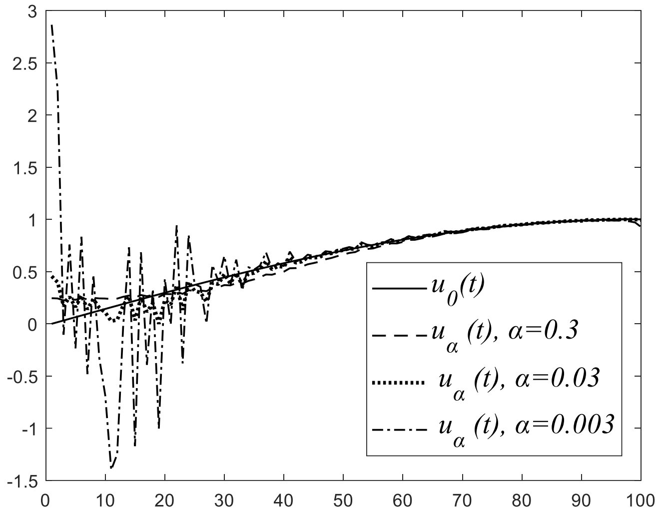

Figure 4: $u_\delta^\alpha(t), \delta = 0.016$

8. Conclusions

The inverse BVP for the heat PDE is expressed and solved in this paper. The BVP can be signified as a first kind of an integral equation by applying the separation of variables way. The discretization of the integral equation made it probable to decrease the integral equation to a system of linear algebraic equations or a linear operator equation. We applied a regularization algorithm to obtain the estimated solution. In equations of this kind and at the stage of regularization, we encountered the problem of a non-injective operator. The Lavrentiev regularization method applied to get an approximate solution with a regularization parameter based on the Strakhov scheme.

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