The Integration of Rational and Irrational Neutrosophic Functions

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Abstract

The purpose of this article is to study the integration of rational and irrational neutrosophic functions, where integration of standard rational neutrosophic functions discussed through three cases, in addition, integration of standard irrational neutrosophic functions were introduced through seven cases.

Keywords: neutrosophic proper rational function; neutrosophic integrals; improper rational functions..

1. Introduction

As In an attempt to replace the current logics, Smarandache introduced the neutrosophic logic to illustrate a mathematical model of redundancy, uncertainty, contradiction, unknown, ambiguity, undefined, inconsistency, vagueness, imprecision, and incompleteness. Smarandache defined neutrosophic real number [2-4], probabilities according to neutrosophic logic [3-5-12], the neutrosophic statistics [4][6], he has also introduced the concept of integration and differentiation in neutrosophic [1-7]. Madeleine Al- Taha presented results on single valued neutrosophic (weak) polygroups [8]. Chakraborty utilized pentagonal neutrosophic number in networking problems, and Shortest Path Problems [10-11]. Yaser Alhasan presented several papers in the field of nitro calculus [13-20-9]. Also, Mohamed Abdel-Basset presented a study TOPSIS technique in neutrosophic logic [14]. On the other hand, the neutrosophic sets played a significant role in the field of public life such as health, industry and others [15-16-17]. Recently, there are growing efforts to probe the generalized structures and spaces of neutrosophic such as the modules of refined neutrosophic [18-19].

Artica consists of four parts. In 1th part, provides an introduction, in which neutrosophic science review has given. In 2th part, some rules of the neutrosophic integrals and are discussed. The 3th part frames the integration of rational and irrational neutrosophic functions. In 4th part, a conclusion to the paper is given.

2. Preliminaries

2.1. Neutrosophic integration by substitution method [9]

Definition2.1.1

Let $f: D_f \subseteq R \to R_f \cup \{I\}$, to evaluate $\int f(x)dx$

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Put: $x = g(u) \quad \Rightarrow \quad dx = \acute{g}(u)du$

By substitution, we get:

$$\int f(x)dx = \int f(u)\acute{g}(u)du$$

then we can directly integral it.

Theorme2.1.1:

If $\int f(x,I)dx = \varphi(x,I)$ *then,*

$$\int f\big((a+bI)x + c + dI)\big)\,dx = \left(\frac{1}{a} - \frac{b}{a(a+b)}I\right)\varphi\big((a+bI)x + c + dI)\big) + C$$

where C is an indeterminate real constant, $a \neq 0\,, a \neq -b$ *and* b, c, d are real numbers, while I = indeterminacy.

Theorme2.1.2:

Let $f: D_f \subseteq R \to R_f \cup \{I\}$ then:

$$\int \frac{\acute{f}(x,I)}{f(x,I)}dx = ln|f(x,I)| + C$$

where C is an indeterminate real constant (i.e. constant of the form $a + bI$, where a, b are real numbers, while I = indeterminacy).

3. The integration of rational and irrational neutrosophic functions

3.1 Integration of standard rational neutrosophic functions

- **Standard rational neutrosophic integral I:**

$$\int \frac{r+\alpha I}{x^2 - (l+kI)^2}dx = \left(\frac{r}{2l} + \frac{l\alpha - rk}{2l(l+k)}\right) ln\left|\frac{x-l-kI}{x+l+kI}\right| + C \;\; ; l \neq 0\,, l \neq -k$$

Proof:

$$\frac{r+\alpha I}{x^2 - (l+kI)^2} = \frac{r+\alpha I}{(x-l-kI)(x+l+kI)}$$

$$= \frac{r+\alpha I}{2(l+kI)}\left[\frac{2(l+kI)}{(x-l-kI)(x+l+kI)}\right] = \frac{r+\alpha I}{2(l+kI)}\left[\frac{x+l+kI-x+l+kI}{(x-l-kI)(x+l+kI)}\right]$$

$$\frac{r+\alpha I}{2(l+kI)}\left[\frac{(x+l+kI)-(x-l-kI)}{(x-l-kI)(x+l+kI)}\right] = \frac{r+\alpha I}{2(l+kI)}\left[\frac{1}{x-l-kI} - \frac{1}{x+l+kI}\right]$$

$$\Rightarrow \int \frac{r+\alpha I}{x^2 - (l+kI)^2}dx = \int \frac{r+\alpha I}{2(l+kI)}\left[\frac{1}{x-l-kI} - \frac{1}{x+l+kI}\right]dx$$

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$$=\frac{r+\alpha I}{2(l+kI)}\left(\int\frac{1}{x-l-kI}dx-\int\frac{1}{x+l+kI}dx\right)$$

$$=\frac{r+\alpha I}{2(l+kI)}(ln|x-l-kI|-ln|x+l+kI|)+C$$

$$=\left(\frac{r}{2l}+\frac{l\alpha-rk}{2l(l+k)}\right)ln\left|\frac{x-l-kI}{x+l+kI}\right|+C$$

Example3.1.1:

Evaluate:

$$\int\frac{5I}{x^2-4+3I}dx$$

Solution:

to find the denominator factors

$$x^2-4+3I=x^2-\left(\sqrt{4-3I}\right)^2$$

Let's find $\sqrt{4-3I}$

$$\sqrt{4-3I}=m+nI$$

$$4-3I=m^2+2mnI+n^2I$$

$$4-3I=m^2+(2mn+n^2)I$$

then:

$$\begin{cases} m^2=4 \\ 2mn+n^2=-3 \end{cases}$$

$$\begin{cases} m=\pm 2 \\ m^2+2mn+3=0 \end{cases}$$

Find n:

- When $m=2 \quad \Longrightarrow n^2+4n+3=0$

$$(n+3)(n+1)=0 \quad \Longrightarrow n=-3\,, n=-1$$

$$(2,-3),(2,-1)$$

- When $m=-2 \quad \Longrightarrow n^2-4n+3=0$

$$(n-3)(n-1)=0 \quad \Longrightarrow n=3\,, n=1$$

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$$(-2,3),(-2,1)$$

Thus, the denominator factors can be written in two cases:

Case1:

$$x^2 - 4 + 3I = x^2 - (2 - 3I)^2$$

$$\int \frac{5I}{x^2 - 4 + 3I} dx = \frac{I5}{4 - 6I} ln \left|\frac{x - 2 + 3I}{x + 2 - 3I}\right|$$

$$= \left(-\frac{5}{2}I\right) ln \left|\frac{x - 2 + 3I}{x + 2 - 3I}\right| + C$$

Case2:

$$x^2 - 4 + 3I = x^2 - (2 - I)^2$$

$$\int \frac{5I}{x^2 - 4 + 3I} dx = \frac{I5}{4 - 2I} ln \left|\frac{x - 2 + I}{x + 2 - I}\right|$$

$$= \left(\frac{5}{2}I\right) ln \left|\frac{x - 2 + I}{x + 2 - I}\right| + C$$

Hence:

$$\int \frac{5I}{x^2 - 4 + 3I} dx = \begin{cases} \left(\frac{1}{4} - \frac{5}{2}I\right) ln \left|\frac{x - 2 + 3I}{x + 2 - 3I}\right| + C \\ \left(\frac{5}{4} + \frac{5}{2}I\right) ln \left|\frac{x - 2 + I}{x + 2 - I}\right| + C \end{cases}$$

- **Standard rational neutrosophic integral II:**

$$\int \frac{r + \alpha I}{(l + kI)^2 - x^2} dx = \left(\frac{r}{2l} + \frac{l\alpha - rk}{2l(l + k)}\right) ln \left|\frac{x - l - kI}{x + l + kI}\right| + C \ ; l \neq 0 \, , l \neq -k$$

Proof:

$$\frac{r + \alpha I}{(l + kI)^2 - x^2} = \frac{r + \alpha I}{(l + kI - x)(l + kI + x)}$$

$$= \frac{r + \alpha I}{2(l + kI)} \left[\frac{2(l + kI)}{(l + kI - x)(l + kI + x)}\right] = \frac{r + \alpha I}{2(l + kI)} \left[\frac{2(l + kI)}{(l + kI - x)(l + kI + x)}\right]$$

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$$\frac{r+\alpha I}{2(l+kI)}\left[\frac{(l+kI-x)+(l+kI+x)}{(l+kI-x)(l+kI+x)}\right]=\frac{r+\alpha I}{2(l+kI)}\left[\frac{1}{l+kI+x}+\frac{1}{l+kI-x}\right]$$

$$\Rightarrow \int \frac{p+qI}{(l+kI)^2-x^2}dx=\int \frac{r+\alpha I}{2(l+kI)}\left[\frac{1}{l+kI+x}+\frac{1}{l+kI-x}\right]dx$$

$$=\frac{r+\alpha I}{2(l+kI)}\left(\int \frac{1}{l+kI+x}dx+\int \frac{1}{l+kI-x}dx\right)$$

$$=\frac{r+\alpha I}{2(l+kI)}\left(\int \frac{1}{l+kI+x}dx-\int \frac{-1}{l+kI-x}dx\right)$$

$$=\frac{r+\alpha I}{2(l+kI)}(ln|l+kI+x|-ln|l+kI-x|)+C$$

$$=\left(\frac{r}{2l}+\frac{l\alpha-rk}{2l(l+k)}\right)ln\left|\frac{x-l-kI}{x+l+kI}\right|+C$$

Example3.1.2:

Evaluate:

$$\int \frac{2+3I}{16-15I-x^2}dx$$

Solution:

to find the denominator factors

$$16-15I-x^2=\sqrt{16-15I}-x^2$$

Let's find $\sqrt{16-15I}$

$$\sqrt{16-15I}=m+nI$$

$$16-15I=m^2+2mnI+n^2I$$

$$16-15I=m^2+(2mn+n^2)I$$

then:

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$$\begin{cases} m^2 = 16 \\ 2mn + n^2 = -15 \end{cases}$$

$$\begin{cases} m = \pm 4 \\ n^2 + 2mn + 15 = 0 \end{cases}$$

Find n:

- When $m = 4 \quad \Longrightarrow n^2 + 8n + 15 = 0$

$$(n+3)(n+5) \quad \Longrightarrow n = -3\,, n = -5$$

$$(4,-3),(4,-5)$$

- When $m = -4 \quad \Longrightarrow n^2 - 8n + 15 = 0$

$$(n-3)(n-5) \quad \Longrightarrow n = 3\,, n = 5$$

$$(-4,3),(-4,5)$$

Thus, the denominator factors can be written in two cases:

Case1:

$$16 - 15I - x^2 = (4 - 3I)^2 - x^2$$

$$\int \frac{2+3I}{16-15I-x^2}dx = \frac{2+3I}{8-6I} ln\left|\frac{4-3I+x}{4-3I-x}\right|$$

$$= \left(\frac{1}{4} + \frac{9}{4}I\right) ln\left|\frac{4-3I+x}{4-3I-x}\right| + C$$

Case2:

$$16 - 15I - x^2 = (4 - 5I)^2 - x^2$$

$$\int \frac{2+3I}{16-15I-x^2}dx = \frac{2+3I}{8-10I} ln\left|\frac{4-5I+x}{4-5I-x}\right|$$

$$= \left(\frac{1}{4} - \frac{11}{4}I\right) ln\left|\frac{4-5I+x}{4-5I-x}\right| + C$$

Hence:

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$$\int \frac{5I}{x^2-4+3I}dx = \begin{cases} \left(\frac{1}{4}+\frac{9}{4}I\right) ln\left|\frac{4-3I+x}{4-3I-x}\right| + C \\ \left(\frac{1}{4}-\frac{11}{4}I\right) ln\left|\frac{4-3I+x}{4-3I-x}\right| + C \end{cases}$$

- **Standard rational neutrosophic integral III:**

$$\int \frac{r+\alpha I}{x^2+(l+kI)^2}dx = \left(\frac{r}{2l}+\frac{l\alpha - rk}{2l(l+k)}\right) tan^{-1}\left(\left(\frac{1}{l}-\frac{k}{l(l+k)}I\right)x\right) + C \ ; l \neq 0 , l \neq -k$$

Proof:

Let's put: $x = (l+kI)tan\theta \quad \Rightarrow dx = (l+kI)sec^2\theta \, d\theta$

Then:

$$x^2+(l+kI)^2 = (l+kI)^2tan^2\theta + (l+kI)^2$$

$$= (l+kI)^2(tan^2\theta + 1)$$

$$= (l+kI)^2sec^2\theta$$

$$\Rightarrow \quad \int \frac{r+\alpha I}{x^2+(l+kI)^2}dx = \int \frac{r+\alpha I}{(l+kI)^2sec^2\theta}(r+sI)sec^2\theta \, d\theta$$

$$= \int \frac{r+\alpha I}{l+kI}d\theta = \frac{r+\alpha I}{l+kI}\int d\theta$$

$$= \frac{r+\alpha I}{l+kI}\theta + C = \frac{r+\alpha I}{l+kI}tan^{-1}\left(\frac{x}{l+kI}\right) + C$$

where

$$\theta = tan^{-1}\left(\frac{x}{l+kI}\right)$$

hence:

$$\int \frac{r+\alpha I}{x^2+(l+kI)^2}dx = \left(\frac{r}{2l}+\frac{l\alpha - rk}{2l(l+k)}\right) tan^{-1}\left(\left(\frac{1}{l}-\frac{k}{l(l+k)}I\right)x\right) + C$$

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Example3.1.3:

Evaluate:

$$\int \frac{4+I}{x^2+9+7I}dx$$

Solution:

$$x^2+9+7I = x^2+\left(\sqrt{9+7I}\right)^2$$

Let's find $\sqrt{9+7I}$

$$\sqrt{9+7I} = m+nI$$

$$9+7I = m^2+2mnI+n^2I$$

$$9+7I = m^2+(2mn+n^2)I$$

then:

$$\begin{cases} n^2 = 9 \\ 2mn+n^2 = 7 \end{cases}$$

$$\begin{cases} m = \pm 3 \\ n^2+2mn-7 = 0 \end{cases}$$

Find n:

- When $m = 3 \quad \Longrightarrow n^2+6n-7 = 0$

$$(n+7)(n-1) = 0 \quad \Longrightarrow n = -7 \,, n = 1$$

$$(3,-7),(3,1)$$

- When $n = -3 \quad \Longrightarrow n^2-6n-7 = 0$

$$(n-7)(n+1) = 0 \quad \Longrightarrow n = 7 \,, n = -1$$

$$(-3,7),(-3,-1)$$

$$(m,n) = (3,-7),(3,1),(-3,7),(-3,-1)$$

$$\sqrt{9+7I} = 3-7I \text{ } or \text{ } 3+I \text{ } or -3+7I \text{ } or -3-I$$

Thus, the denominator factors can be written in two cases:

Case1:

$$x^2-4+3I = x^2+(3-7I)^2$$

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$$\int \frac{4+I}{x^2+9+7I}dx = \int \frac{4+I}{x^2+(3-7I)^2}dx$$

$$= \left(\frac{4+I}{3-7I}\right) tan^{-1}\left(\frac{x}{3-7I}\right) + C$$

$$= \left(\frac{4}{3} - \frac{31}{12}I\right) tan^{-1}\left(\left(\frac{1}{3} - \frac{7}{12}\right)x\right) + C$$

Case2:

$$x^2 - 4 + 3I = x^2 + (3+I)^2$$

$$\int \frac{4+I}{x^2+9+7I}dx = \int \frac{4+I}{x^2+(3+I)^2}dx$$

$$= \left(\frac{4+I}{3+I}\right) tan^{-1}\left(\frac{x}{3+I}\right) + C$$

$$= \left(\frac{4}{3} - \frac{1}{12}I\right) tan^{-1}\left(\left(\frac{1}{3} - \frac{1}{12}I\right)x\right) + C$$

Hence:

$$\int \frac{4+I}{x^2+9+7I}dx = \begin{cases} \left(\frac{4}{3} - \frac{31}{12}I\right) tan^{-1}\left(\left(\frac{1}{3} - \frac{7}{12}I\right)x\right) + C \\ \left(\frac{4}{3} - \frac{1}{12}I\right) tan^{-1}\left(\left(\frac{1}{3} - \frac{1}{12}I\right)x\right) + C \end{cases}$$

3.2 Integration of standard irrational neutrosophic functions

- **Standard irrational neutrosophic integral I:**

$$\int \frac{r+\alpha I}{\sqrt{(l+kI)^2 - x^2}}dx = (r+\alpha I)sin^{-1}\left(\left(\frac{1}{l} - \frac{k}{l(l+k)}I\right)x\right) + C\,; l \neq 0\,, l \neq -k$$

Proof:

Let's put: $x = (l+kI)sin\theta \quad \Longrightarrow dx = (l+kI)cos\theta\ d\theta$

Then:

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$$(l+kI)^2 - x^2 = (l+kI)^2 - (l+kI)^2 sin^2\theta$$

$$= (l+kI)^2(1 - sin^2\theta)$$

$$= (l+kI)^2 cos^2\theta$$

$$\Rightarrow \quad \int \frac{r+\alpha I}{\sqrt{(l+kI)^2 - x^2}} dx = \int \frac{r+\alpha I}{\sqrt{((l+kI)^2 cos^2\theta}} ((l+kI)cos\theta \, d\theta$$

$$= \int (r+\alpha I) d\theta = (r+\alpha I) \int d\theta$$

$$= (r+\alpha I)\theta + C = (r+\alpha I) sin^{-1}\left(\frac{x}{l+kI}\right) + C$$

where

$$\theta = sin^{-1}\left(\frac{x}{l+kI}\right)$$

hence:

$$\int \frac{r+\alpha I}{\sqrt{(l+kI)^2 - x^2}} dx = (r+\alpha I) sin^{-1}\left(\left(\frac{1}{l} - \frac{k}{l(l+k)} I\right) x\right) + C$$

Example3.2.1:

Evaluate:

$$\int \frac{2-2I}{\sqrt{1+3I-x^2}} dx$$

Solution:

$$1 + 3I - x^2 = \left(\sqrt{1+3I}\right)^2 - x^2$$

Let's find $\sqrt{1+3I}$

$$\sqrt{1+3I} = m + nI$$

$$1 + 3I = m^2 + 2mnI + n^2 I$$

$$1 + 3I = m^2 + (2mn + n^2)I$$

then:

$$\begin{cases} m^2 = 1 \\ 2mn + n^2 = 3 \end{cases}$$

$$\begin{cases} m = \pm 1 \\ n^2 + 2mn - 3 = 0 \end{cases}$$

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Find n:

- When $m = 1 \quad \Longrightarrow n^2 + 2n - 3 = 0$

$$(n+3)(n-1) = 0 \quad \Longrightarrow n = -3\,, n = 1$$

$$(1,-3), (1,1)$$

- When $m = -1 \quad \Longrightarrow n^2 - 2n - 3 = 0$

$$(n-3)(n+1) = 0 \quad \Longrightarrow n = 3\,, n = -1$$

$$(-1,3), (-1,-1)$$

$$(m,n) = (1,-3), (1,1), (-1,3), (-1,-1)$$

$$\sqrt{1+3I} = 1 - 3I \text{ or } 1 + I \text{ or } -1 + 3I \text{ or } -1 - I$$

Thus, the denominator factors can be written in two cases:

Case1:

$$1 + 3I - x^2 = (1-3I)^2 - x^2$$

$$\int \frac{2-2I}{1+3I-x^2}\,dx = \int \frac{2-2I}{(1-3I)^2 - x^2}\,dx$$

$$= (2-2I)sin^{-1}\left(\left(1 - \frac{3}{2}I\right)x\right) + C$$

Case2:

$$1 + 3I - x^2 = (1+I)^2 - x^2$$

$$\int \frac{2-2I}{1+3I-x^2}\,dx = \int \frac{2-2I}{(1+I)^2 - x^2}\,dx$$

$$= (2-2I)sin^{-1}\big((2-2I)x\big) + C$$

Hence:

$$\int \frac{2-2I}{1+3I-x^2}\,dx = \begin{cases} (2-2I)sin^{-1}\left(\left(1 - \frac{3}{2}I\right)x\right) + C \\ (2-2I)sin^{-1}\big((2-2I)x\big) + C \end{cases}$$

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- **Standard irrational neutrosophic integral II:**

$$\int \frac{r+\alpha I}{x\sqrt{x^2-(l+kI)^2}}dx = \left(\frac{r}{2l}+\frac{l\alpha - rk}{2l(l+k)}\right)sec^{-1}\left(\left(\frac{1}{l}-\frac{k}{l(l+k)}I\right)x\right)+C \ ; l\neq 0\,, l\neq -k$$

Proof:

Let's put: $x=(l+kI)sec\theta \quad \Rightarrow dx=(l+kI)sec\theta \ tan\theta \ d\theta$

Then:

$$x\sqrt{x^2-(l+kI)^2}=(l+kI)sec\theta\ \sqrt{(l+kI)^2sec^2\theta-(l+kI)^2}$$

$$=(l+kI)sec\theta\sqrt{(l+kI)^2(sec^2\theta-1)}$$

$$=(l+kI)^2sec\theta\ tan\theta$$

$$\Rightarrow \int \frac{r+\alpha I}{x\sqrt{x^2-(l+kI)^2}}dx=\int\frac{r+\alpha I}{(l+kI)^2sec\theta\ tan\theta}(l+kI)sec\theta\ tan\theta\ d\theta$$

$$=\int\frac{r+\alpha I}{l+kI}d\theta=\frac{r+\alpha I}{l+kI}\int d\theta$$

$$=\frac{r+\alpha I}{l+kI}\theta+C=\frac{r+\alpha I}{l+kI}sec^{-1}\left(\frac{x}{l+kI}\right)+C$$

where

$$\theta=sec^{-1}\left(\frac{x}{l+kI}\right)$$

hence:

$$\int \frac{r+\alpha I}{x\sqrt{x^2-(l+kI)^2}}dx = \left(\frac{r}{2l}+\frac{l\alpha - rk}{2l(l+k)}\right)sec^{-1}\left(\left(\frac{1}{l}-\frac{k}{l(l+k)}I\right)x\right)+C$$

Example3.2.2:

Evaluate:

$$\int\frac{5-3I}{x\sqrt{x^2-(1+I)^2}}dx$$

Solution:

$$\int\frac{5-3I}{x\sqrt{x^2-(1+I)^2}}dx=\left(\frac{5}{2}-2I\right)sec^{-1}\left(\left(1-\frac{1}{4}I\right)x\right)+C$$

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- **Standard irrational neutrosophic integral III:**

$$\int \frac{r+\alpha I}{\sqrt{x^2-(l+kI)^2}}dx = (r+\alpha I)\ ln\left|x+\sqrt{x^2-(l+kI)^2}\right| + C$$

Proof:

Let's put: $x=(l+kI)sec\theta \quad \Rightarrow dx=(l+kI)sec\theta\ tan\theta\ d\theta$

Then:

$$\sqrt{x^2-(l+kI)^2}=\sqrt{(l+kI)^2sec^2\theta-(l+kI)^2}$$

$$=\sqrt{(l+kI)^2(sec^2\theta-1)}$$

$$=(l+kI)tan\theta$$

$$\Rightarrow \int \frac{r+\alpha I}{\sqrt{x^2-(l+kI)^2}}dx = \int \frac{r+\alpha I}{(l+kI)tan\theta}(l+kI)sec\theta\ tan\theta\ d\theta$$

$$=\int (r+\alpha I)sec\theta d\theta = (r+\alpha I)\int sec\theta d\theta$$

$$=(r+\alpha I)ln|sec\theta+tan\theta|+C_1=(r+\alpha I)ln\left|sec\theta+\sqrt{sec^2\theta-1}\right|+C_1$$

$$=(r+\alpha I)ln\left|\frac{x}{l+kI}+\sqrt{\left(\frac{x}{l+kI}\right)^2-1}\right|+C_1$$

$$=(r+\alpha I)ln\left|\frac{x}{l+kI}+\frac{1}{l+kI}\sqrt{x^2-(l+kI)^2}\right|+C_1$$

$$=(r+\alpha I)ln\left|\frac{1}{r+sI}\left(x+\sqrt{x^2-(l+kI)^2}\right)\right|+C_1$$

$$=(r+\alpha I)\left[ln\left|\left(x+\sqrt{x^2-(l+kI)^2}\right)\right|-ln|l+kI|+C_1\right]$$

hence:

$$\int \frac{r+\alpha I}{\sqrt{x^2-(l+kI)^2}}dx = (r+\alpha I)\ ln\left|x+\sqrt{x^2-(l+kI)^2}\right| + C$$

where

$$sec\theta=\frac{x}{l+kI}\ and\ \ C=-ln|l+kI|+C_1$$

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Example3.2.3:

Evaluate:

$$\int \frac{9I}{\sqrt{x^2-(6+8I)^2}}dx$$

Solution:

$$\int \frac{9I}{\sqrt{x^2-(6-8I)^2}}dx = 9I\ ln\left|x+\sqrt{x^2-(6+8I)^2}\right|+C$$

$$= 9I\ ln\left|x+\sqrt{x^2-36-160I}\right|+C$$

- **Standard irrational neutrosophic integral IV:**

$$\int \frac{r+\alpha I}{\sqrt{x^2+(l+kI)^2}}dx = (r+\alpha I)\ ln\left|x+\sqrt{x^2+(l+kI)^2}\right|+C$$

Proof:

Let's put: $x=(l+kI)tan\theta \quad \Rightarrow dx=(l+kI)sec^2\theta\ d\theta$

Then:

$$\sqrt{x^2+(l+kI)^2} = \sqrt{(l+kI)^2tan^2\theta+(r+sI)^2}$$

$$= \sqrt{(l+kI)^2(tan^2\theta+1)}$$

$$= (l+kI)sec\theta$$

$$\Rightarrow \int \frac{r+\alpha I}{\sqrt{x^2+(l+kI)^2}}dx = \int \frac{r+\alpha I}{(l+kI)sec\theta}(l+kI)sec^2\theta\ d\theta$$

$$= \int (r+\alpha I)sec\theta d\theta = (r+\alpha I)\int sec\theta d\theta$$

$$= (r+\alpha I)ln|sec\theta+tan\theta|+C_1 = (r+\alpha I)ln\left|\sqrt{tan^2\theta+1}+tan\theta\right|+C_1$$

$$= (r+\alpha I)ln\left|\frac{x}{l+kI}+\sqrt{\left(\frac{x}{l+kI}\right)^2+1}\right|+C_1$$

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$$= (r + \alpha I) ln \left| \frac{x}{l + kI} + \frac{1}{l + kI} \sqrt{x^2 + (l + kI)^2} \right| + C_1$$

$$= (r + \alpha I) ln \left| \frac{1}{l + kI} \left(x + \sqrt{x^2 + (l + kI)^2} \right) \right| + C_1$$

$$= (r + \alpha I) \left[ln \left| \left(x + \sqrt{x^2 + (l + kI)^2} \right) \right| - ln|l + kI| + C_1 \right]$$

hence:

$$\int \frac{r + \alpha I}{\sqrt{x^2 + (l + kI)^2}} dx = (r + \alpha I)\ ln \left| x + \sqrt{x^2 + (l + kI)^2} \right| + C$$

where

$$sec\theta = \frac{x}{l + kI}\ and\ \ C = -ln|l + kI| + C_1$$

Example3.2.4:

Evaluate:

$$\int \frac{6 + 3I}{\sqrt{x^2 + (4 - 2I)^2}} dx$$

Solution:

$$\int \frac{6 + 3I}{\sqrt{x^2 + (4 - 2I)^2}} dx = (6 + 3I)\ ln \left| x + \sqrt{x^2 + (4 - 2I)^2} \right| + C$$

$$= (6 + 3I)\ ln \left| x + \sqrt{x^2 + 16 - 12I} \right| + C$$

- **Standard irrational neutrosophic integral V:**

$$\int \sqrt{(l + kI)^2 - x^2}\, dx = \frac{x}{2} \sqrt{(l + kI^2 - x^2} + \frac{(l + kI)^2}{2} sin^{-1} \left(\left(\frac{1}{l} - \frac{k}{l(l + k)} I \right) x \right) + C\ ; l \neq 0\ , l \neq -k$$

Proof:

Let's put: $x = (l + kI)sin\theta \quad \Rightarrow dx = (l + kI)cos\theta\ d\theta$

Then:

$$(l + kI)^2 - x^2 = (l + kI)^2 - (l + kI)^2 sin^2\theta$$

$$= (l + kI)^2 (1 - sin^2\theta)$$

$$= (l + kI)^2 cos^2\theta$$

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$$\Rightarrow \int \sqrt{(l+kI)^2 - x^2}\, dx = \int \sqrt{(l+kI)^2 cos^2\theta}\, (l+kI)cos\theta\, d\theta$$

$$= \int (l+kI)^2 cos^2\theta\, d\theta = \frac{(l+kI)^2}{2}\int (\cos 2\theta + 1)d\theta$$

$$= \frac{(l+kI)^2}{2}\int (\cos 2\theta + 1)d\theta = \frac{(l+kI)^2}{2}\left(\frac{1}{2}sin2\theta + \theta\right)$$

$$= \frac{(l+kI)^2}{2}(sin\theta\, cos\theta + \theta)$$

$$= \frac{(l+kI)^2}{2}\left(\frac{x}{l+kI}\,\frac{\sqrt{(l+kI)^2 - x^2}}{l+kI} + sin^{-1}\left(\left(\frac{1}{l} - \frac{k}{l(l+k)}I\right)x\right)\right)$$

where

$$\theta = sin^{-1}\left(\frac{x}{l+kI}\right)$$

hence:

$$\int \sqrt{(l+kI)^2 - x^2}\, dx = \frac{x}{2}\sqrt{(l+kI^2 - x^2} + \frac{(l+kI)^2}{2} sin^{-1}\left(\left(\frac{1}{l} - \frac{k}{l(l+k)}I\right)x\right) + C$$

Example3.2.5:

$$\int \sqrt{(4+6I)^2 - x^2}\, dx = \frac{x}{2}\sqrt{(4+6I)^2 - x^2} + \frac{(4+6I)^2}{2} sin^{-1}\left(\left(\frac{1}{4} - \frac{6}{4(4+6)}I\right)x\right) + C$$

$$= \frac{x}{2}\sqrt{(4+6I)^2 - x^2} + (8+42I)sin^{-1}\left(\left(\frac{1}{4} - \frac{3}{20}I\right)x\right) + C$$

- **Standard irrational neutrosophic integral VI:**

$$\int \sqrt{x^2 - (l+kI)^2}dx = \frac{x}{2}\sqrt{x^2 - (l+kI)^2} - \frac{(l+kI)^2}{2}\, ln\left|x + \sqrt{x^2 - (l+kI)^2}\right| + C$$

Proof:

Let's put: $x = (l+kI)sec\theta \quad \Rightarrow dx = (l+kI)sec\theta\, tan\theta\, d\theta$

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Then:

$$\sqrt{x^2-(l+kI)^2}=\sqrt{(l+kI)^2sec^2\theta-(l+kI)^2}$$

$$=\sqrt{(l+kI)^2(sec^2\theta-1)}$$

$$=(l+kI)tan\theta$$

$$\Rightarrow \int\sqrt{x^2-(l+kI)^2}dx=\int(l+kI)tan\theta(l+kI)sec\theta\ tan\theta\ d\theta$$

$$=(l+kI)^2\int sec\theta tan^2\theta d\theta=(l+kI)^2\int sec\theta(sec^2\theta-1)d\theta$$

$$=(l+kI)^2\int(sec^3\theta-sec\theta)d\theta$$

$$=(l+kI)^2\left[\frac{1}{2}sec\theta\ tan\theta+\frac{1}{2}ln|sec\theta+tan\theta|-ln|sec\theta+tan\theta|\right]+C_1$$

$$=(l+kI)^2\left[\frac{1}{2}sec\theta\sqrt{sec^2\theta-1}-\frac{1}{2}ln\left|sec\theta+\sqrt{sec^2\theta-1}\right|\right]+C_1$$

$$=(l+kI)^2\left[\frac{1}{2}sec\theta\sqrt{sec^2\theta-1}-\frac{1}{2}ln|sec\theta+tan\theta|\right]+C_1\quad(*)$$

But:

$$sec\theta=\frac{x}{l+kI}$$

$$\sqrt{sec^2\theta-1}=\sqrt{\left(\frac{x}{l+kI}\right)^2-1}=\frac{1}{r+sI}\sqrt{x^2-(l+kI)^2}$$

By substitution in (*), we get:

$$\int\sqrt{x^2-(l+kI)^2}dx=(l+kI)^2\left[\frac{1}{2}\frac{x}{(l+kI)^2}\sqrt{x^2-(l+kI)^2}-\frac{1}{2}ln\left|\frac{x}{l+kI}+\frac{1}{l+kI}\sqrt{x^2-(l+kI)^2}\right|\right]+C_1$$

$$=\frac{x}{2}\sqrt{x^2-(l+kI)^2}-\frac{(l+kI)^2}{2}ln\left|\frac{1}{l+kI}\left(x+\sqrt{x^2-(l+kI)^2}\right)\right|+C_1$$

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$$= \frac{x}{2}\sqrt{x^2 - (l + kI)^2} - \frac{(l + kI)^2}{2} ln\left|\left(x + \sqrt{x^2 - (l + kI)^2}\right)\right| - ln|l + kI| + C_1$$

hence:

$$\int \sqrt{x^2 - (l + kI)^2} dx = \frac{x}{2}\sqrt{x^2 - (l + kI)^2} - \frac{(l + kI)^2}{2}\ ln\left|x + \sqrt{x^2 - (l + kI)^2}\right| + C$$

where

$$C = - - ln|l + kI| + C_1$$

Example3.2.6:

$$\int \sqrt{x^2 - (5 - 7I)^2} dx = \frac{x}{2}\sqrt{x^2 - (5 - 7I)^2} - \frac{(5 - 7I)^2}{2} ln\left|\left(x + \sqrt{x^2 - (5 - 7I)^2}\right)\right| + C$$

$$= \frac{x}{2}\sqrt{x^2 - (5 - 7I)^2} - \left(\frac{25}{2} - \frac{21}{2}I\right) ln\left|\left(x + \sqrt{x^2 - (5 - 7I)^2}\right)\right| + C$$

- **Standard irrational neutrosophic integral VII:**

$$\int \sqrt{x^2 + (l + kI)^2} dx = \frac{x}{2}\sqrt{x^2 + (l + kI)^2} + \frac{(l + kI)^2}{2}\ ln\left|x + \sqrt{x^2 + (l + kI)^2}\right| + C$$

Proof:

Let's put: $x = (l + kI)tan\theta \quad \Rightarrow dx = (l + kI)sec^2\theta\ d\theta$

Then:

$$\sqrt{x^2 + (l + kI)^2} = \sqrt{(l + kI)^2 tan^2\theta + (l + kI)^2}$$

$$= \sqrt{(l + kI)^2(tan^2\theta + 1)}$$

$$= (l + kI)sec\theta$$

$$\Rightarrow \int \sqrt{x^2 + (l + kI)^2} dx = \int (l + kI)sec\theta(l + kI)sec^2\theta\ d\theta$$

$$= (l + kI)^2 \int sec^3\theta\ d\theta$$

$$= (l + kI)^2\left[\frac{1}{2}sec\theta\ tan\theta + \frac{1}{2}ln|sec\theta + tan\theta|\right] + C_1$$

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$$= (l + kI)^2 \left[\frac{1}{2} tan\theta\sqrt{tan^2\theta + 1} + \frac{1}{2} ln \left|tan\theta + \sqrt{tan^2\theta + 1}\right|\right] + C_1$$

$$= (l + kI)^2 \left[\frac{1}{2} tan\theta\sqrt{tan^2\theta + 1} + \frac{1}{2} ln \left|tan\theta + \sqrt{tan^2\theta + 1}\right|\right] + C_1 \quad (*)$$

But:

$$tan\theta = \frac{x}{l + kI}$$

$$\sqrt{tan^2\theta + 1} = \sqrt{\left(\frac{x}{l + kI}\right)^2 + 1} = \frac{1}{l + kI}\sqrt{x^2 + (l + kI)^2}$$

By substitution in (*), we get:

$$\int \sqrt{x^2 + (l + kI)^2}dx = (l + kI)^2 \left[\frac{1}{2}\frac{x}{(l + kI)^2}\sqrt{x^2 + (l + kI)^2} + \frac{1}{2} ln \left|\frac{x}{l + kI} + \frac{1}{l + kI}\sqrt{x^2 + (l + kI)^2}\right|\right] + C_1$$

$$= \frac{x}{2}\sqrt{x^2 + (l + kI)^2} + \frac{(l + kI)^2}{2} ln \left|\frac{1}{l + kI}\left(x + \sqrt{x^2 + (l + kI)^2}\right)\right| + C_1$$

$$= \frac{x}{2}\sqrt{x^2 + (l + kI)^2} + \frac{(l + kI)^2}{2} ln \left|\left(x + \sqrt{x^2 + (l + kI)^2}\right)\right| - ln|l + kI| + C_1$$

hence:

$$\int \sqrt{x^2 + (l + kI)^2}dx = \frac{x}{2}\sqrt{x^2 + (l + kI)^2} + \frac{(l + kI)^2}{2}\ ln \left|x + \sqrt{x^2 + (l + kI)^2}\right| + C$$

where

$$C = -ln|l + kI| + C_1$$

Example3.2.7:

$$\int \sqrt{x^2 + (3 + I)^2}dx = \frac{x}{2}\sqrt{x^2 + (3 + I)^2} + \frac{(3 + I)^2}{2} ln \left|\left(x + \sqrt{x^2 + (3 + I)^2}\right)\right| + C$$

$$= \frac{x}{2}\sqrt{x^2 + (3 + I)^2} + \left(\frac{9}{2} + \frac{7}{2}I\right) ln \left|\left(x + \sqrt{x^2 + 9 + 7I}\right)\right| + C =$$

5. Conclusions

This artical expands on the studies we previously published on neutrosophic integrals. Whereas included three cases of integration of standard rational neutrosophic functions, and seven cases of integrals integration of standard irrational neutrosophic functions. These cases make it easy to derive direct rules for calculating integrals, without resorting to following the rules for neutrosophic integrals that studied in a paper previously published in NSS, herein lies the importance of this paper.

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