On the Solutions of Fermat's Diophantine Equation in 2-cyclic Refined Neutrosophic Ring of Integers

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Abstract

The Diophantine equation $X^n + Y^n = Z^n$ is called the Fermat's Diophantine equation. Its solutions are called general Fermat's triples.The aim of this paper is to study the solutions of Fermat's Diophantine equation in the 2-cyclic refined neutrosophic ring of integers, where we determine all possible solutions of this Diophantine equation, as well as, the special case of Pythagoras triples.

Keywords: 2- cyclic refined Neutrosophic integer; 2-cyclic refined Neutrosophic Fermat's equation; Pythagoras triple

1. Introduction

Neutrosophy as a new branch of applied philosophy has lead to many improvements in both algebra and number theory [1-2,5-6], where many concepts were defined by symbolic neutrosophic sets to generalize the classical well known structures such as neutrosophic spaces, rings, and numbers [9-12,14-19].

In the history of number theory, we find the following non-linear Diophantine equation $X^n + Y^n = Z^n$ that plays a perfect role within the Fermat's last theorem [4].

The previous Diophantine equation was generalized to neutrosophic rings [7-8], and it had been solved completely in that case [7].

This work will be dedicated to study the Fermat's non-linear Diophantine equation $X^n + Y^n = Z^n$ in another neutrosophic structure (2-cyclic refined neutrosophic integer ring), where the full solutions of the previous equation will be presented and illustrated.

We must recall some basic definitions and concepts from [19].

(a) The 2-cyclic refined neutrosophic ring of integers is defined as follows:

$Z_2(I) = \{t_0 + t_1 I_1 + t_2 I_2;\ t_i \in Z\}$.

(b) Addition on $Z_2(I)$ can be clarified as follows:

$(a + bI_1 + cI_2) + (m + nI_1 + tI_2) = (a + m) + I_1(b + n) + I_2(c + t)$.

(c) Multiplication on $Z_2(I)$ can be clarified as follows:

$(a + bI_1 + cI_2)(m + nI_1 + tI_2) = am + anI_1 + atI_2 + bmI_1 + bnI_2 + btI_1 + cmI_2 + cnI_1 + ctI_2$

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$= am + I_1(an + bm + bt + cn) + I_2(at + bn + cm + ct)$.

Where $I_1 I_1 = I_{(1+1\ mod 2)} = I_2, I_2 I_2 = I_{(2+2\ mod\ 2)} = I_2, I_1 I_2 = I_{(1+2\ mod\ 2)} = I_1$.

2. **Main discussion**

Theorem.

Let $X = x_0 + x_1 I_1 + x_2 I_2$ be a 2-cyclic refined neutrosophic integer, then:

$$X^n = {x_0}^n + I_1\left[\frac{(x_0 + x_1 + x_2)^n - (x_0 - x_1 + x_2)^n}{2}\right] + I_2\left[\frac{(x_0 + x_1 + x_2)^n + (x_0 - x_1 + x_2)^n}{2} - {x_0}^n\right]$$

Proof.

For $n = 1$, it is true clearly.

We assume that it is true for $n = k$, we must prove for $n = k + 1$.

$$X^{n+1} = X.X^n = (x_0 + x_1 I_1 + x_2 I_2)\Big({x_0}^n + \frac{1}{2} I_1[(x_0 + x_1 + x_2)^n - (x_0 - x_1 + x_2)^n] + \frac{1}{2} I_2[(x_0 + x_1 + x_2)^n + (x_0 - x_1 + x_2)^n - 2{x_0}^n]\Big)$$

$$\begin{aligned} X^{n+1} = {x_0}^{n+1} &+ \frac{1}{2} I_1[x_0(x_0 + x_1 + x_2)^n - x_0(x_0 - x_1 + x_2)^n] \\ &+ \frac{1}{2} I_2[x_0(x_0 + x_1 + x_2)^n + x_0(x_0 - x_1 + x_2)^n - 2{x_0}^{n+1}] + x_1 {x_0}^n I_1 \\ &+ \frac{1}{2} I_2[x_1(x_0 + x_1 + x_2)^n - x_1(x_0 - x_1 + x_2)^n] \\ &+ \frac{1}{2} I_1[x_1(x_0 + x_1 + x_2)^n - x_1(x_0 - x_1 + x_2)^n - 2x_1{x_0}^n] + x_2{x_0}^n I_2 \\ &+ \frac{1}{2} I_1[x_2(x_0 + x_1 + x_2)^n - x_2(x_0 - x_1 + x_2)^n] \\ &+ \frac{1}{2} I_2[x_2(x_0 + x_1 + x_2)^n + x_2(x_0 - x_1 + x_2)^n - 2x_2{x_0}^n] \end{aligned}$$

$$\begin{aligned} X^{n+1} = {x_0}^{n+1} &+ \frac{1}{2} I_1[(x_0 + x_1 + x_2)(x_0 + x_1 + x_2)^n - (x_0 - x_1 + x_2)(x_0 - x_1 + x_2)^n] \\ &+ \frac{1}{2} I_2[(x_0 + x_1 + x_2)(x_0 + x_1 + x_2)^n + (x_0 - x_1 + x_2)(x_0 - x_1 + x_2)^n - 2{x_0}^{n+1}] \end{aligned}$$

$$\begin{aligned} X^{n+1} = {x_0}^{n+1} &+ \frac{1}{2} I_1[(x_0 + x_1 + x_2)^{n+1} - (x_0 - x_1 + x_2)^{n+1}] \\ &+ \frac{1}{2} I_2[(x_0 + x_1 + x_2)^{n+1} + (x_0 - x_1 + x_2)^{n+1} - 2{x_0}^{n+1}] \end{aligned}$$

So that the proof is complete.

Pythagoras triples.

A triple (X, Y, Z) is called a Pythagoras if $X^2 + Y^2 = Z^2$.

Theorem.

Let $X = x_0 + x_1 I_1 + x_2 I_2, Y = y_0 + y_1 I_1 + y_2 I_2, Z = z_0 + z_1 I_1 + z_2 I_2$ be three 2-cyclic refined neutrosophic integers, then (X, Y, Z) is a Pythagoras triple if and only if:

$P_1 = (x_0, y_0, z_0), P_2 = (x_0 + x_1 + x_2, y_0 + y_1 + y_2, z_0 + z_1 + z_2)$,

$P_3 = (x_0 - x_1 + x_2, y_0 - y_1 + y_2, z_0 - z_1 + z_2)$

Are three Pythagoras triples.

Proof.

We have:

$$X^2 = {x_0}^2 + \frac{1}{2} I_1[(x_0 + x_1 + x_2)^2 - (x_0 - x_1 + x_2)^2] + \frac{1}{2} I_2[(x_0 + x_1 + x_2)^2 + (x_0 - x_1 + x_2)^2 - 2{x_0}^2]$$

$$Y^2 = {y_0}^2 + \frac{1}{2} I_1[(y_0 + y_1 + y_2)^2 - (y_0 - y_1 + y_2)^2] + \frac{1}{2} I_2[(y_0 + y_1 + y_2)^2 + (y_0 - y_1 + y_2)^2 - 2{y_0}^2]$$

$$Z^2 = {z_0}^2 + \frac{1}{2} I_1[(z_0 + z_1 + z_2)^2 - (z_0 - z_1 + z_2)^2] + \frac{1}{2} I_2[(z_0 + z_1 + z_2)^2 + (z_0 - z_1 + z_2)^2 - 2{z_0}^2]$$

The equation $X^2 + Y^2 = Z^2$ is equivalent to:

$${x_0}^2 + {y_0}^2 = {z_0}^2, (x_0 + x_1 + x_2)^2 + (y_0 + y_1 + y_2)^2 = (z_0 + z_1 + z_2)^2, (x_0 - x_1 + x_2)^2 + (y_0 - y_1 + y_2)^2 = (z_0 - z_1 + z_2)^2$$

Thus the proof is complete.

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Example.

$X = 1 + I_1 + 6I_2, Y = -I_1 + 7I_2, Z = 1 + 9I_2$, (X, Y, Z) is a Pythagoras triple, that is because:
$X^2 + Y^2 = (1 + 14I_1 + 49I_2) + (-14I_1 + 50I_2) = 1 + 99I_2 = Z^2$
It is clear that $(x_0, y_0, z_0) = (1,0,1), (x_0 + x_1 + x_2, y_0 + y_1 + y_2, z_0 + z_1 + z_2) = (8,6,10)$, $(x_0 - x_1 + x_2, y_0 - y_1 + y_2, z_0 - z_1 + z_2) = (6,8,10)$ are three Pythagoras triples.

The general case.

A triple (X, Y, Z) is called a general Fermat's triple if and only if it is a solution of the Fermat's Diophantine equation $X^n + Y^n = Z^n \,; n \geq 3$.

Theorem.

Let $X = x_0 + x_1 I_1 + x_2 I_2, Y = y_0 + y_1 I_1 + y_2 I_2, Z = z_0 + z_1 I_1 + z_2 I_2$ be three 2-cyclic refined neutrosophic integers, then (X, Y, Z) is Fermat's triple if and only if:
$P_1 = (x_0, y_0, z_0), P_2 = (x_0 + x_1 + x_2, y_0 + y_1 + y_2, z_0 + z_1 + z_2)$,
$P_3 = (x_0 - x_1 + x_2, y_0 - y_1 + y_2, z_0 - z_1 + z_2)$
Are Fermat's triples.

Proof.

We have:

$$X^n = x_0{}^n + \frac{1}{2} I_1[(x_0 + x_1 + x_2)^n - (x_0 - x_1 + x_2)^n] + \frac{1}{2} I_2[(x_0 + x_1 + x_2)^n + (x_0 - x_1 + x_2)^n - 2x_0{}^n]$$

$$Y^2 = y_0{}^n + \frac{1}{2} I_1[(y_0 + y_1 + y_2)^n - (y_0 - y_1 + y_2)^n] + \frac{1}{2} I_2[(y_0 + y_1 + y_2)^n + (y_0 - y_1 + y_2)^n - 2y_0{}^n]$$

$$Z^2 = z_0{}^n + \frac{1}{2} I_1[(z_0 + z_1 + z_2)^n - (z_0 - z_1 + z_2)^n] + \frac{1}{2} I_2[(z_0 + z_1 + z_2)^n + (z_0 - z_1 + z_2)^n - 2z_0{}^n]$$

The equation $X^n + Y^n = Z^n$ is equivalent to:

$$\begin{cases} x_0{}^n + y_0{}^n = z_0{}^n \\ (x_0 + x_1 + x_2)^n + (y_0 + y_1 + y_2)^n = (z_0 + z_1 + z_2)^n \\ (x_0 - x_1 + x_2)^n + (y_0 - y_1 + y_2)^n = (z_0 - z_1 + z_2)^n \end{cases}$$

Thus proof is complete.

Now, we well find the solutions of Fermat's Diophantine equation $X^n + Y^n = Z^n$ in the ring of 2-cyclic refined neutrosophic ring of integers.

It is known that the solutions of $X^n + Y^n = Z^n$ for all $n \geq 3$ are:

$$E_1 = (x, 0, x), E_2 = (0, x, x) \,; x \in Z$$

To find the solutions of 2-cyclic refined neutrosophic equation $X^n + Y^n = Z^n$; $n \geq 3$, we must find the formula of transforming classical Fermat's triples to the corresponding 2-cyclic refined neutrosophic Fermat's triples.

Theorem.

Let $T_1 = (a_0, b_0, c_0), T_2 = (a_1, b_1, c_1), T_3 = (a_2, b_2, c_2)$ are three Fermat's triples in Z, then exists a corresponding 2-cyclic refined neutrosophic Fermat's triple if and only if the following system of Diophantine equations is solvable:

$$\begin{cases} (x_0, x_1, x_2) = (a_0, b_0, c_0) \\ x_0 + x_1 + x_2 = a_1, x_0 - x_1 + x_2 = a_2 \\ y_0 + y_1 + y_2 = b_1 \,, y_0 - y_1 + y_2 = b_2 \\ z_0 + z_1 + z_2 = c_1 \,, z_0 - z_1 + z_2 = c_2 \end{cases} \quad \dots (I)$$

The proof holds directly from the previous theorem.

Remark.

If the system (I) is solvable in Z, then:

$$\begin{cases} x_0 = a_0, x_1 = b_0, x_2 = c_0 \\ x_1 = \frac{a_1 - a_2}{2}, x_2 = \frac{a_1 + a_2}{2} - a_0 \\ y_1 = \frac{b_1 - b_2}{2}, x_2 = \frac{b_1 + b_2}{2} - b_0 \\ z_1 = \frac{c_1 - c_2}{2}, x_2 = \frac{c_1 + c_2}{2} - c_0 \end{cases}$$

so that $a_1 - a_2 \in 2Z, b_1 - b_2 \in 2Z, c_1 - c_2 \in 2Z$, i.e. $(a_1, a_2), (b_1, b_2), (c_1, c_2)$ are odd or even to gother.

Discussion the solutions.

Case(1).

If $E_1 = (x, 0, x), E_2 = (y, 0, y), E_3 = (z, 0, z)$; x is odd , y, z are even, then:

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$$\begin{cases} x_0 = x, y_0 = 0, z_0 = x \\ x_1 = \frac{y-z}{2}, x_2 = \frac{y+z}{2} - x \\ y_1 = 0, y_2 = 0 \\ z_1 = \frac{y-z}{2}, z_2 = \frac{y+z}{2} - x \end{cases} \Rightarrow \begin{cases} X = x + \frac{y-z}{2} I_1 + \left(\frac{y+z}{2} - x\right) I_2 \\ Y = 0 , Z = X \end{cases}$$

Case(2).

If $E_1 = (x, 0, x), E_2 = (0, y, y), E_3 = (z, 0, z)$; x is odd , y, z are even, then:

$$\begin{cases} x_0 = x, y_0 = 0, z_0 = x \\ x_1 = \frac{-z}{2}, x_2 = \frac{z}{2} - x \\ y_1 = \frac{y}{2}, y_2 = \frac{y}{2} \\ z_1 = \frac{y-z}{2}, z_2 = \frac{y+z}{2} - x \end{cases} \Rightarrow \begin{cases} X = x - \frac{z}{2} I_1 + \left(\frac{z}{2} - x\right) I_2 \\ Y = \frac{y}{2} I_1 + \frac{y}{2} I_2 \\ Z = x + \frac{y-z}{2} I_1 + \left(\frac{y+z}{2} - x\right) I_2 \end{cases}$$

Case(3).

If $E_1 = (x, 0, x), E_2 = (y, 0, y), E_3 = (0, z, z)$; x is odd , y, z are even, then:

$$\begin{cases} x_0 = x, y_0 = 0, z_0 = x \\ x_1 = \frac{y}{2}, x_2 = \frac{y}{2} - x \\ y_1 = \frac{-z}{2}, y_2 = \frac{z}{2} \\ z_1 = \frac{y-z}{2}, z_2 = \frac{y+z}{2} - x \end{cases} \Rightarrow \begin{cases} X = x + \frac{y}{2} I_1 + \left(\frac{y}{2} - x\right) I_2 \\ Y = -\frac{z}{2} I_1 + \frac{z}{2} I_2 \\ Z = x + \frac{y-z}{2} I_1 + \left(\frac{y+z}{2} - x\right) I_2 \end{cases}$$

Case(4).

If $E_1 = (x, 0, x), E_2 = (0, y, y), E_3 = (0, z, z)$; x is odd , y, z are even, then:

$$\begin{cases} x_0 = x, y_0 = 0, z_0 = x \\ x_1 = 0, x_2 = -x \\ y_1 = \frac{y-z}{2}, y_2 = \frac{y+z}{2} \\ z_1 = \frac{y-z}{2}, z_2 = \frac{y+z}{2} - x \end{cases} \Rightarrow \begin{cases} X = x - xI_2 \\ Y = \frac{y-z}{2} I_1 + \frac{y+z}{2} I_2 \\ Z = x + \frac{y-z}{2} I_1 + \left(\frac{y+z}{2} - x\right) I_2 \end{cases}$$

Case(5).

If $E_1 = (x, 0, x), E_2 = (y, 0, y), E_3 = (z, 0, z)$; x, y, z are even, we get a similar formula of **case(1)**.

Case(6).

If $E_1 = (x, 0, x), E_2 = (0, y, y), E_3 = (z, 0, z)$; x, y, z are even, we get a similar formula of **case(2)**.

Case(7).

If $E_1 = (x, 0, x), E_2 = (y, 0, y), E_3 = (0, z, z)$; x, y, z are even, we get a similar formula of **case(3)**.

Case(8).

If $E_1 = (x, 0, x), E_2 = (0, y, y), E_3 = (0, z, z)$; x, y, z are even, we get a similar formula of **case(4)**.

Case(9).

If $E_1 = (0, x, x), E_2 = (y, 0, y), E_3 = (z, 0, z)$; x is odd , y, z are even, then:

$$\begin{cases} x_0 = 0, y_0 = x, z_0 = x \\ x_1 = \frac{y-z}{2}, x_2 = \frac{y+z}{2} \\ y_1 = 0, y_2 = -x \\ z_1 = \frac{y-z}{2}, z_2 = \frac{y+z}{2} - x \end{cases} \Rightarrow \begin{cases} X = \left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2}\right) I_2 \\ Y = x - xI_2 \\ Z = x + \left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2} - x\right) I_2 \end{cases}$$

Case(10).

If $E_1 = (0, x, x), E_2 = (0, y, y), E_3 = (z, 0, z)$; x is odd , y, z are even, then:

$$\begin{cases} x_0 = 0, y_0 = x, z_0 = x \\ x_1 = \frac{z}{2}, x_2 = \frac{z}{2} \\ y_1 = \frac{y}{2}, y_2 = \frac{y}{2} - x \\ z_1 = \frac{y-z}{2}, z_2 = \frac{y+z}{2} - x \end{cases} \Rightarrow \begin{cases} X = \frac{z}{2} I_1 + \frac{z}{2} I_2 \\ Y = x + \frac{y}{2} I_1 + \left(\frac{y}{2} - x\right) I_2 \\ Z = x + \left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2} - x\right) I_2 \end{cases}$$

Case(11).

If $E_1 = (0, x, x), E_2 = (y, 0, y), E_3 = (0, z, z)$; x is odd , y, z are even, then:

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$$\begin{cases} x_0 = 0, y_0 = x, z_0 = x \\ x_1 = \frac{y}{2}, x_2 = \frac{z}{2} \\ y_1 = \frac{-z}{2}, y_2 = \frac{z}{2} - x \\ z_1 = \frac{y-z}{2}, z_2 = \frac{y+z}{2} - x \end{cases} \Rightarrow \begin{cases} X = \frac{y}{2} I_1 + \frac{y}{2} I_2 \\ Y = x - \frac{z}{2} I_1 + \left(\frac{z}{2} - x\right) I_2 \\ Z = x + \left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2} - x\right) I_2 \end{cases}$$

Case(12).

If $E_1 = (0, x, x), E_2 = (0, y, y), E_3 = (0, z, z)$; x is odd , y, z are even, then:

$$\begin{cases} x_0 = 0, y_0 = x, z_0 = x \\ x_1 = 0, x_2 = 0 \\ y_1 = \frac{y-z}{2}, y_2 = \frac{y+z}{2} - x \\ z_1 = \frac{y-z}{2}, z_2 = \frac{y+z}{2} - x \end{cases} \Rightarrow \begin{cases} X = 0 \\ Y = x + \left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2} - x\right) I_2 \\ Z = x + \left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2} - x\right) I_2 \end{cases}$$

Case(13).

If $E_1 = (0, x, x), E_2 = (y, 0, y), E_3 = (z, 0, z)$; x, y, z are even, we get a similar formula of **case(9)**.

Case(14).

If $E_1 = (0, x, x), E_2 = (0, y, y), E_3 = (z, 0, z)$; x, y, z are even, we get a similar formula of **case(10)**.

Case(15).

If $E_1 = (0, x, x), E_2 = (y, 0, y), E_3 = (0, z, z)$; x, y, z are even, we get a similar formula of **case(11)**.

Case(16).

If $E_1 = (0, x, x), E_2 = (0, y, y), E_3 = (0, z, z)$; x, y, z are even, we get a similar formula of **case(12)**.

Case(17).

If $E_1 = (x, 0, x), E_2 = (y, 0, y), E_3 = (z, 0, z)$; x, y, z are odd, then:

$$\begin{cases} x_0 = x, y_0 = 0, z_0 = x \\ x_1 = \frac{y-z}{2}, x_2 = \frac{y+z}{2} - x \\ y_1 = 0, y_2 = 0 \\ z_1 = \frac{y-z}{2}, z_2 = \frac{y+z}{2} - x \end{cases} \Rightarrow \begin{cases} X = x + \left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2} - x\right) I_2 \\ Y = 0 \\ Z = x + \left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2} - x\right) I_2 \end{cases}$$

Case(18).

If $E_1 = (x, 0, x), E_2 = (0, y, y), E_3 = (0, z, z)$; x, y, z are odd, then:

$$\begin{cases} x_0 = x, y_0 = 0, z_0 = x \\ x_1 = 0, x_2 = -x \\ y_1 = \frac{y-z}{2}, y_2 = \frac{y+z}{2} - x \\ z_1 = \frac{y-z}{2}, z_2 = \frac{y+z}{2} - x \end{cases} \Rightarrow \begin{cases} X = x - xI_2 \\ Y = \left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2}\right) I_2 \\ Z = x + \left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2} - x\right) I_2 \end{cases}$$

Case(19).

If $E_1 = (0, x, x), E_2 = (y, 0, y), E_3 = (z, 0, z)$; x, y, z are odd, then:

$$\begin{cases} x_0 = 0, y_0 = x, z_0 = x \\ x_1 = \frac{y-z}{2}, x_2 = \frac{y+z}{2} \\ y_1 = 0, y_2 = -x \\ z_1 = \frac{y-z}{2}, z_2 = \frac{y+z}{2} - x \end{cases} \Rightarrow \begin{cases} X = \left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2}\right) I_2 \\ Y = x - xI_2 \\ Z = x + \left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2} - x\right) I_2 \end{cases}$$

Case(20).

If $E_1 = (0, x, x), E_2 = (0, y, y), E_3 = (0, z, z)$; x, y, z are odd, then:

$$\begin{cases} x_0 = 0, y_0 = x, z_0 = x \\ x_1 = 0, x_2 = 0 \\ y_1 = \frac{y-z}{2}, y_2 = \frac{y+z}{2} - x \\ z_1 = \frac{y-z}{2}, z_2 = \frac{y+z}{2} - x \end{cases} \Rightarrow \begin{cases} X = 0 \\ Y = x + \left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2}\right) I_2 \\ Z = x + \left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2} - x\right) I_2 \end{cases}$$

Case(21).

If $E_1 = (x, 0, x), E_2 = (y, 0, y), E_3 = (z, 0, z)$; x is odd y, z are even, we get **case(17)**.

Case(22).

If $E_1 = (x, 0, x), E_2 = (0, y, y), E_3 = (0, z, z)$; x is odd y, z are even, we get **case(18)**.

Case(23).

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If $E_1 = (0, x, x), E_2 = (y, 0, y), E_3 = (z, 0, z)$; x is odd y, z are even, we get **case(19)**.

Case(24).

If $E_1 = (0, x, x), E_2 = (0, y, y), E_3 = (0, z, z)$; x is odd y, z are even, we get **case(20)**.

Result.

The solutions of Fermat's Diophantine equation $X^n + Y^n = Z^n$; $n \geq 3$ in $Z_3(I)$ are:

$$
\begin{aligned}
&\left\{\left(x + \left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2} - x\right) I_2, 0, x + \left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2} - x\right) I_2\right), \left(x - \frac{z}{2} I_1 + \left(\frac{z}{2} - x\right) I_2, \frac{y}{2} I_1\right.\right. \\
&\quad + \frac{y}{2} I_2, x + \left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2} - x\right) I_2\Big), \left(x + \frac{y}{2} I_1 + \left(\frac{y}{2} - x\right) I_2, -\frac{z}{2} I_1 + \frac{z}{2} I_2, x\right. \\
&\quad + \left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2} - x\right) I_2\Big), \left(x - x I_2, \left(\frac{y-z}{2}\right) I_1 + \frac{y+z}{2} I_2, x + \left(\frac{y-z}{2}\right) I_1\right. \\
&\quad + \left(\frac{y+z}{2} - x\right) I_2\Big), \left(\left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2}\right) I_2, x - x I_2, x + \left(\frac{y-z}{2}\right) I_1\right. \\
&\quad + \left(\frac{y+z}{2} - x\right) I_2\Big), \left(\frac{z}{2} I_1 + \frac{z}{2} I_2, x + \left(\frac{y}{2}\right) I_1 + \left(\frac{y}{2} - x\right) I_2, x + \left(\frac{y-z}{2}\right) I_1\right. \\
&\quad + \left(\frac{y+z}{2} - x\right) I_2\Big), \left(\frac{y}{2} I_1 + \frac{y}{2} I_2, x - \left(\frac{y}{2}\right) I_1 + \left(\frac{z}{2} - x\right) I_2, x + \left(\frac{y-z}{2}\right) I_1\right. \\
&\quad + \left(\frac{y+z}{2} - x\right) I_2\Big), \left(0, x + \left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2} - x\right) I_2, x + \left(\frac{y-z}{2}\right) I_1\right. \\
&\quad + \left(\frac{y+z}{2} - x\right) I_2\Big), \left(x + \left(\frac{y-z}{2}\right) I_1 + \left(\frac{y+z}{2} - x\right) I_2, 0, x + \left(\frac{y-z}{2}\right) I_1\right. \\
&\quad \left.\left. + \left(\frac{y+z}{2} - x\right) I_2\right)\right\}
\end{aligned}
$$

Under the conditions of theorem which discussion when y, z are odd or even.

3. Conclusion

In this work, we have presented a full solution of Fermat's Diophantine equation in the ring of 2-cyclic integers, where we have shown that it has exactly 9 different forms of solutions with respect to the ring of 2-cyclic refined neutrosophic integers. As an important topic in number theory, we aim to solve Fermat's Diophantine equation in many other algebraic rings, especially fuzzy rings, and generalized neutrosophic rings.

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References

[1] Smarandache, F., " A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability", American Research Press. Rehoboth, 2003.

[2] Sankari, H., and Abobala, M., "Neutrosophic Linear Diophantine Equations With two Variables", Neutrosophic Sets and Systems, Vol. 38, pp. 22-30, 2020.

[3] Adeleke, E.O., Agboola, A.A.A.,and Smarandache, F., "Refined Neutrosophic Rings I", International Journal of Neutrosophic Science, Vol. 2(2), pp. 77-81, 2020.

[4] Ali, R., "A Short Note On The Solution of n-Refined Neutrosophic Linear Diophantine Equations", International Journal Of Neutrosophic Science, Vol. 15, 2021.

[5] Ibrahim, M., and Abobala, M., "An Introduction To Refined Neutrosophic Number Theory", Neutrosophic Sets and Systems, Vol. 45, 2021.

[6] Ceven, Y., and Tekin, S., " Some Properties of Neutrosophic Integers", Kırklareli University Journal of Engineering and Science, Vol. 6, 2020.

[7] Ahmad, K., Bal, M., and Aswad, M., " A Short Note On The Solutions Of Fermat's Diophantine Equation In Some Neutrosophic Rings", Journal Of Neutrosophic and Fuzzy Systems, 2022.

Doi: https://doi.org/10.54216/IJNS.200301
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[8] Abobala, M., Partial Foundation of Neutrosophic Number Theory, Neutrosophic Sets and Systems, Vol. 39 , 2021.

[9] Celik, M., and Olgun, N., " An Introduction To Neutrosophic Real Banach And Hillbert Spaces", Galoitica Journal Of Mathematical Structures And Applications, 2022.

[10] Celik, M., and Olgun, N., " On The Classification Of Neutrosophic Complex Inner Product Spaces", Galoitica Journal Of Mathematical Structures And Applications, 2022.

[11] Celik, M., and Hatip, A., " On The Refined AH-Isometry And Its Applications In Refined Neutrosophic Surfaces", Galoitica Journal Of Mathematical Structures And Applications, 2022

[12] Smarandache, F., "Symbolic Neutrosophic Theory", EuropaNova asbl, Bruxelles, 2015.

[13] Merkepci, H., and Ahmad, K., " On The Conditions Of Imperfect Neutrosophic Duplets and Imperfect Neutrosophic Triplets", Galoitica Journal Of Mathematical Structures And Applications, Vol.2, 2022.

[14] Merkepci, H., and Abobala, M., " The Application of AH-isometry In The Study Of Neutrosophic Conic Sections", Galoitica Journal Of Mathematical Structures And Applications, Vol.2, 2022.

[15] Abobala, M., and Zeina, M.B., " A Study Of Neutrosophic Real Analysis By Using One Dimensional Geometric AH-Isometry", Galoitica Journal Of Mathematical Structures And Applications, Vol.3, 2023

[16] Hatip, A., "An Introduction To Weak Fuzzy Complex Numbers ", Galoitica Journal Of Mathematical Structures and Applications, Vol.3, 2023

[17] Kandasamy, V.W.B., and Smarandache, F., "Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures", Hexis, Phonex, Arizona 2006.

[18] Smarandache, F., *n-Valued Refined Neutrosophic Logic and Its Applications in Physics*, Progress in Physics, 143-146, Vol. 4, 2013.

[19] Sadiq, B., " A Contribution To The Group Of Units' Problem In Some 2-Cyclic Refined Neutrosophic Rings", International Journal Of Neutrosophic Science, 2022.

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