



An Introduction to M-Open Sets in Fuzzy Neutrosophic Topological Spaces

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Abstract

A new classes of sets called fuzzy neutrosophic M-open sets and fuzzy neutrosophic M-Closed sets in fuzzy neutrosophic topology are introduced some characterizations of these notions have been presented. After given the fundamental definitions of the new notions, we studied some theorems, propositions and some necessary relations with examples related to these notions had been discussed.

Keywords: Fuzzy Neutrosophic Sets (FNS); Fuzzy Neutrosophic M-Open Set(FNM-OS); Fuzzy Neutrosophic M-Closed Set(FNM-CS); Fuzzy Neutrosophic Topology (FNT).

1.Introduction

In 1995 F. Smarsndache [1,2] was introducing the significant neutrosophic logic "NSs" which was generalization of the term fuzzy sets "FSs" and the concept of intuitionistic fuzzy sets "IFSs" when he added the indeterminacy membership values between membership values of "FS" and non-membership values of "IFS" with the same unit interval [0,1]. Neutrosophic theory "NSs", has laid the foundation for a whole family of new mathematical theories. Then in 2012, A.Salama et.al. [3] studied the topology on "NSs" and simply "NTS". As generalization of "IFS". Before six years ago and exactly in 2017, .Y.Veerewari [4] gave the definition of fuzzy neutrosophic topological spaces "FNTSs".

In this paper, we introduced definition of new classes of sets called, fuzzy neutrosophic M-open sets and fuzzy neutrosophic M-Closed sets via fuzzy neutrosophic topological as generalization of many studied (see, F. Mohammed [5-12], and T.M. Nour et al. [13],[14]). Finally, we discussed some of the relationships between them and got several supply theories and propositions.

The remaining part of this paper is organized as the follows: section 2 provides preliminaries about the Neutrosophic fuzzy sets and M-Open Sets. Section 3 describes the M-Open Set and M-Closed Set in Fuzzy Neutrosophic Topology. Finally, section 4 concludes the paper.

2. Preliminaries

In this part, we recollect some useful basic concepts of neutrosophic set theory via fuzzy eutrosophic topology.

Definition 2.1[8] "Let ω_N be any non-empty set. The fuzzy neutrosophic set (FNS, for short), A_N is a topic has the form $A_N = \{ \langle \omega, \mu_{AN}(\omega), \sigma_{AN}(\omega), \nu_{AN}(\omega) \rangle : \omega \in \omega_N \}$ where each the functions $\mu_{AN}, \sigma_{AN}, \nu_{AN}: \omega_N \rightarrow [0,1]$ referse

to the degree of membership function (namely $\mu_{AN}(\omega)$), the degree of indeterminacy function (namely $\sigma_{AN}(\omega)$) and the degree of non-membership (namely $\nu_{AN}(\omega)$) respectively of any element $\omega \in \omega_N$ to the set A_N with $0 \leq \mu_{AN}(\omega) + \sigma_{AN}(\omega) + \nu_{AN}(\omega) \leq 3$, for any $\omega \in \omega_N$."

Remark 2.2 [8] The FNS $A_N = \{ \langle \omega, \mu_{AN}(\omega), \sigma_{AN}(\omega), \nu_{AN}(\omega) \rangle : \omega \in \omega_N \}$ can be identified to an ordered triple $\langle \omega, \mu_{AN}, \sigma_{AN}, \nu_{AN} \rangle$ in $[0, 1]$ on ω_N .

Definition 2.3 [8] "Let ω_N be a non-empty set and the FNSs A_N and β_F be in the form:

$$A_N = \{ \langle \omega, \mu_{AN}(\omega), \sigma_{AN}(\omega), \nu_{AN}(\omega) \rangle : \omega \in \omega_N \} \text{ and}$$

$$\beta_F = \{ \langle \omega, \mu_{\beta}(\omega), \sigma_{\beta_F}(\omega), \nu_{\beta_F}(\omega) \rangle : \omega \in \omega_N \} \text{ on } \omega_N.$$

Then,

- i. $A_N \subseteq \beta_F$ iff $\mu_{AN}(\omega) \leq \mu_{\beta_F}(\omega)$, $\sigma_{AN}(\omega) \leq \sigma_{\beta_F}(\omega)$ and $\nu_{AN}(\omega) \geq \nu_{\beta_F}(\omega)$ for all $\omega \in \omega_N$,
- ii. $A_N = \beta_F$ iff $A_N \subseteq \beta_F$ and $\beta_F \subseteq A_N$,
- iii. $(A_N)^c = 1_F - A_N = \{ \langle \omega, \nu_{AN}(\omega), 1 - \sigma_{AN}(\omega), \mu_{AN}(\omega) \rangle : \omega \in \omega_N \}$,
- iv. $A_N \cup \beta_F = \{ \langle \omega, \text{Max}(\mu_{AN}(\omega), \mu_{\beta_F}(\omega)), \text{Max}(\sigma_{AN}(\omega), \sigma_{\beta_F}(\omega)), \text{Min}(\nu_{AN}(\omega), \nu_{\beta_F}(\omega)) \rangle : \omega \in \omega_N \}$,
- v. $A_N \cap \beta_F = \{ \langle \omega, \text{Min}(\mu_{AN}(\omega), \mu_{\beta_F}(\omega)), \text{Min}(\sigma_{AN}(\omega), \sigma_{\beta_F}(\omega)), \text{Max}(\nu_{AN}(\omega), \nu_{\beta_F}(\omega)) \rangle : \omega \in \omega_N \}$,
- vi. $0_F = \langle \omega, 0, 0, 1 \rangle$ and $1_F = \langle \omega, 1, 1, 0 \rangle$."

Definition 2.4 [8]"The fuzzy neutrosophic topology (FNT, for short) on a non-empty set ω_N is a family Γ_N of fuzzy neutrosophic subsets in ω_N satisfying:

- i. $0_F, 1_F \in \Gamma_N$,
- ii. $A_{N1} \cap A_{N2} \in \mathcal{T}_F$ for any $A_{N1}, A_{N2} \in \Gamma_N$,
- iii. $\cup A_{Nj} \in \Gamma_N, \forall \{ A_{Nj} : j \in J \} \subseteq \Gamma_N$,

Then (ω_N, Γ_N) is called fuzzy neutrosophic topological space (FNTPS, for short). Every elements of Γ_N are fuzzy neutrosophic-open set (FN-OS, for short) and the complement of FN-open set is fuzzy neutrosophic-closed set (FN-CS, for short)."

Definition 2.5 [8]" Let (ω_N, Γ_N) is FNTPS and $A_N = \langle \omega, \mu_{AN}, \sigma_{AN}, \nu_{AN} \rangle$ is FNS in ω_N . Then, the fuzzy neutrosophic-closure (FN-cl, for short) and the fuzzy neutrosophic-interior (FN-int, for short) of A_N are defined by:

$$\text{FN-cl}(A_N) = \cap \{ \beta_F : \beta_F \text{ is FN-CS in } \omega_N \text{ and } A_N \subseteq \beta_F \},$$

$$\text{FN-int}(A_N) = \cup \{ \beta_F : \beta_F \text{ is FN-OS in } \omega_N \text{ and } \beta_F \subseteq A_N \}.$$

Now, the $\text{FNcl}(A_N)$ is FN-CS and $\text{FNint}(A_N)$ is FN-OS in ω_N .

Further,

- i. A_N is FN-CS in ω_N iff $\text{FN-cl}(A_N) = A_N$.
- ii. A_N is FN-OS in ω_N iff $\text{FN-int}(A_N) = A_N$."

Proposition 2.6 [8] "Let (ω_N, Γ_N) is FNTPS and λ_F, β_F are FNSs in ω_N . Then the following properties hold:

- i. $\text{FN-int}(A_N) \subseteq A_N$ and $A_N \subseteq \text{FN-cl}(A_N)$,
- ii. $A_N \subseteq \beta_F \Rightarrow \text{FN-int}(A_N) \subseteq \text{FN-int}(\beta_F)$ and $A_N \subseteq \beta_F \Rightarrow \text{FN-cl}(A_N) \subseteq \text{FN-cl}(\beta_F)$,
- iii. $\text{FN-int}(\text{FN-int}(A_N)) = \text{FN-int}(A_N)$ and $\text{FN-cl}(\text{FN-cl}(A_N)) = \text{FN-cl}(A_N)$,
- iv. $\text{FN-int}(A_N \cap \beta_F) = \text{FN-int}(A_N) \cap \text{FN-int}(\beta_F)$ and $\text{FN-cl}(A_N \cup \beta_F) = \text{FN-cl}(A_N) \cup \text{FN-cl}(\beta_F)$,
- v. $\text{FN-int}(1_F) = 1_F$ and $\text{FN-cl}(0_F) = 0_F$."

Definition 2.7 [9,10] " FNS A_N in FNTS (ω_N, Γ_N) is called:

- i. Fuzzy neutrosophic regular-open set (FNR-open set, for short) if $A_N = \text{FN-int}(\text{FN-cl}(A_N))$.
- ii. Fuzzy neutrosophic regular-closed set (FNR-closed set, for short) if $A_N = \text{FN-cl}(\text{FN-int}(A_N))$.
- iii. Fuzzy neutrosophic pre-open set (FNP-open set, for short) if $A_N \subseteq \text{FN-int}(\text{FN-cl}(A_N))$.
- iv. Fuzzy neutrosophic pre-closed set (FNP-closed set, for short) if $\text{FN-cl}(\text{FN-int}(A_N)) \subseteq A_N$."

Definition 2.8 [12] " FNS A_N in FNTS (ω_N, Γ_N) is called:

- i. $\text{FN-int}_\delta(A_N) = \{ \omega_N : A_N \cup \text{FN-cl}(\text{FN-int}(U_N)) \neq 0_N, U_N \in \tau_N \text{ and } \omega_N \in U_N \}$
 $= \cup \{ A_N : A_N \text{ is FN-R-OS} \}$
 $= \cup \{ \text{FN-int}(U_N) \mid \text{FN-int}(U_N) \subseteq A_N, U_N \in \Gamma_N \}$.
- ii. $\text{FN-cl}_\delta(A_N) = \{ \omega_N : A_N \cap \text{FN-int}(\text{FN-cl}(U_N)) \neq 0_N, U_N \in \Gamma_N \text{ and } \omega_N \in U_N \}$
 $= \cap \{ A_N : A_N \text{ is FN-R-CS} \}$
 $= \cap \{ \text{FN-cl}(U_N) \mid A_N \subseteq \text{FN-cl}(U_N), U_N \in \Gamma_N \}$.
- iii. $\text{FN-int}^\circ(A_N) = \cup \{ A_N \in \tau : \text{FN-cl}(A_N) \subseteq A_N \}$.
- iv. $\text{FN-cl}^\circ(A_N) = \cap \{ A_N : A_N \text{ is FN-clopen} \}$.

3. M-Open Set and M-Closed Set in Fuzzy Neutrosophic Topology

In this part we have been presented new classes of sets called M-open set and its complement M-closed set. As well as, some of its relationships have been proposed in the same space.

Definition 3.1: Let (ω_N, Γ_N) be FNTS, then a subset A_N of ω_N is said to be:

- i. FNM-open set if $A_N \subseteq \text{FN-cl}(\text{FN-int}^\circ(A_N)) \cup \text{FN-int}(\text{FN-cl}_\delta(A_N))$.
- ii. FNM-closed set if $A_N \supseteq (\text{FN-int}(\text{FN-cl}^\circ(A_N)) \cap \text{FN-cl}(\text{FN-int}_\delta(A_N)))$

The family of all FNM-open (FNM-closed) subsets of a FNTS (ω_N, Γ_N) will be as always denoted by $\text{FNMO}(\omega_N)$ ($\text{FNMC}(\omega_N)$).

Example 3.2: Let (ω_N, Γ_N) be FNTS and $\omega_N = \{a, b, c\}$ where,

$\Gamma_N = \{0_N, 1_N, A_N, B_N, U_N\}$ wherevers,

$0.3, 0.5, 0.3), b(0.4, 0.2, 0.2), c(0.5, 0.3, 0.2) >$, $(a A_N = < \omega,$
 $B_N = < \omega, a(0.3, 0.5, 0.3), b(0.2, 0.8, 0.4), c(0.2, 0.7, 0.5) >$,
 $U_N = < \omega, a(0.3, 0.5, 0.3), b(0.4, 0.8, 0.2), c(0.5, 0.7, 0.2) >$.

So, $\delta_N = \{0_N, 1_N, A_N, B_N\}$.

Then, $\text{FN-int}^\circ(A_N) = \cup \{ A_N : A_N \text{ is FN-clopen set} \}$,

$$= < \omega, a(0.3, 0.5, 0.3), b(0.4, 0.8, 0.2) c(0.5, 0.7, 0.2) >$$

Also,

$$\text{FN-cl}^\circ(U_N) = \cap \{ A_N : A_N \text{ is FN-clopen set} \} = < \omega, a(0.3, 0.5, 0.3), b(0.2, 0.2, 0.4), c(0.2, 0.3, 0.5) >$$

$$\text{And, FN-int}_\delta(U_N) = \cup \{ A_N : A_N \text{ is FNR-open set} \} = < \omega, a(0.3, 0.5, 0.3), b(0.4, 0.8, 0.2), c(0.5, 0.7, 0.2) >$$

Also,

$$\text{FN-cl}_\delta(U_N) = \cap \{A_N : A_N \text{ is FNR-closed set}\} = \langle \omega, a(0.3, 0.5, 0.3), b(0.2, 0.2, 0.4), c(0.2, 0.3, 0.5) \rangle.$$

Now, FNM-open set where, $U_N \subseteq (\text{FN-cl}(\text{FN-int}^\circ(U_N)) \cup \text{FN-int}(\text{FN-cl}_\delta(U_N)))$

$$\begin{aligned} &\subseteq \text{FN-cl}(\delta_N) \cup \text{FN-int}(\delta_N) \text{ where } \delta_N = \{0_N, 1_N, A_N, B_N\} \\ &\subseteq \delta_N \end{aligned}$$

So, δ_N is FNM-open set

Also, δ_N is FNM-closed set where, $U_N \supseteq (\text{FN-int}(\text{FN-cl}^\circ(U_N))) \cap (\text{FN-cl}(\text{FN-int}_\delta(U_N)))$

$$\begin{aligned} &\supseteq \text{FN-int}(\delta_N) \cap \text{FN-cl}(\delta_N) \\ &\supseteq \delta_N \end{aligned}$$

So, δ_N is FNM – closed set.

Example 3.3: Let (ω_N, Γ_N) be FNTS and $\omega_N = \{a, b, c\}$ where,

$$\Gamma_N = \{0_N, 1_N, A_N, B_N\} \text{ wherever,}$$

$$A_N = \langle \omega, a(0.3, 0.2, 0.5), b(0.4, 0.3, 0.5), c(0.5, 0.2, 0.5) \rangle,$$

$$B_N = \langle \omega, a(0.5, 0.5, 0.5), b(0.5, 0.5, 0.5), c(0.5, 0.5, 0.5) \rangle.$$

So, $\delta_N = \{1_N, 0_N, B_N\}$ where,

$$\begin{aligned} \delta_N &\subseteq \text{FN-cl}(\text{FN-int}^\circ(\delta_N)) \cup \text{FN-int}(\text{FN-cl}_\delta(\delta_N)) \\ &\subseteq \delta_N. \end{aligned}$$

Then, δ_N is FNM-open set.

Theorem 3.4: Let (ω_N, Γ_N) be FNTS then, the union of any FNM-open sets is again FNM-open set.

Proof: Let $\{A_i, i \in I\}$ be a collection of FNM-open sets.

Then, $A_i \subseteq \text{FN-cl}(\text{FN-int}^\circ(A_N)) \cup \text{FN-int}(\text{FN-cl}_\delta(A_N))$, hence

$$\begin{aligned} \cup_i A_i &\subseteq \cup_i (\text{FN-cl}(\text{FN-int}^\circ(A_N)) \cup \text{FN-int}(\text{FN-cl}_\delta(A_N))) \\ &\subseteq \text{FN-cl}(\text{FN-int}^\circ(\cup_i A_i)) \cup \text{FN-int}(\text{FN-cl}_\delta(\cup_i A_i)) \text{ For all } i \in I. \end{aligned}$$

Thus, $\cup_i A_i$ is FNM-open set .

Remark 3.5: The intersection of any two FNM-open sets, is not necessary to be FNM-open set.

Example 3.6: Let $\delta_{N1} = \{0_N, 1_N, A_{N1}\}$

$$\text{Where, } A_{N1} = \langle \omega, a(0.5, 0.5, 0.5), b(0.5, 0.5, 0.5), c(0.5, 0.5, 0.5) \rangle.$$

$$\text{And let } \delta_{N2} = \{0_N, 1_N, A_{N2}, B_{N2}\}$$

$$\text{Where, } A_{N2} = \langle \omega, a(0.3, 0.5, 0.3), b(0.4, 0.2, 0.2), c(0.5, 0.3, 0.2) \rangle,$$

$$\text{And, } B_{N2} = \langle \omega, a(0.3, 0.5, 0.3), b(0.2, 0.8, 0.4), c(0.2, 0.7, 0.5) \rangle. \text{ So,}$$

Now,

$$\begin{aligned} A_{N1} \cap A_{N2} &= \langle \omega, (A_{N1} \cap A_{N2}), (A_{N1} \cap A_{N2}), (A_{N1} \cup A_{N2}) \rangle \\ &= \langle \omega, \min(A_{N1}, A_{N2}), \min(A_{N1}, A_{N2}), \max(A_{N1}, A_{N2}) \rangle, \\ &= \langle \omega, a(0.3, 0.5, 0.5), b(0.4, 0.2, 0.5), c(0.5, 0.3, 0.4) \rangle. \\ &= \langle \omega, a(0.3, 0.5, 0.5), b(0.2, 0.5, 0.5), c(0.2, 0.5, 0.5) \rangle. \end{aligned}$$

That is $A_{N1} \cap A_{N2}$ is not FNM-open set.

Definition 3.7: Let (ω_N, Γ_N) be FNTS and B_N is any fuzzy neutrosophic subset of ω_N then,

i. The fuzzy neutrosophic –semi interior (FN-sint(B_N)), for short) is defined by:

$$\begin{aligned} \text{FN-sint}(B_N) &= \cup \{A_N: A_N \text{ is FNS-CS in } U_N \text{ and } B_N \subseteq A_N\} \\ &= B_N \cap \text{FN-cl}(\text{FN-int}(B_N)). \end{aligned}$$

ii. The fuzzy neutrosophic –semi closure (FNs-cl(B_N)), for short) is defined by:

$$\begin{aligned} \text{FN-scl}(B_N) &= \cap \{A_N: A_N \text{ is FNS-CS in } U_N \text{ and } B_N \subseteq A_N\} \\ &= B_N \cup \text{FN-int}(\text{FN-cl}(B_N)). \end{aligned}$$

iii. The fuzzy neutrosophic –pre interior (FN-pint(B_N)), for short) is defined by:

$$\begin{aligned} \text{FN-pint}(B_N) &= \cup \{A_N: A_N \text{ is FNS-CS in } U_N \text{ and } B_N \subseteq A_N\} \\ &= B_N \cap \text{FN-int}(\text{FN-cl}(B_N)). \end{aligned}$$

iv. The fuzzy neutrosophic-pre closure (FN-pcl(B_N)), for short) is defined by:

$$\begin{aligned} \text{FN-pcl}(B_N) &= \cap \{A_N: A_N \text{ is FNP-CS in } U_N \text{ and } B_N \subseteq A_N\} \\ &= B_N \cup \text{FN-cl}(\text{FN-int}(B_N)). \end{aligned}$$

Remark 3.8: For any subsets A_N and B_N of a FNTS (ω_N, Γ_N) , the following statements hold :

- i. $\text{FN-scl}^{\circ}(A_N) = A_N \cup \text{FN-int}(\text{FN-cl}^{\circ}(A_N))$,
 $\text{FN-sint}^{\circ}(A_N) = A_N \cap \text{FN-cl}(\text{FN-int}^{\circ}(A_N))$.
- ii. $\text{FN-pcl}_{\delta}(A_N) = A_N \cup \text{FN-cl}(\text{FN-int}_{\delta}(A_N))$,
 $\text{FN-pint}_{\delta}(A_N) = A_N \cap \text{FN-int}(\text{FN-cl}_{\delta}(A_N))$.
- iii. $\text{FN}_{\delta}\text{-scl}^{\circ}(\omega_N \setminus A_N) = \omega_N \setminus \text{FNsint}^{\circ}(A_N)$,
 $\text{FN}_{\delta}\text{-scl}^{\circ}(A_N \cup B_N) \subset \text{FN}_{\delta}\text{-scl}^{\circ}(A_N) \cup \text{FN}_{\delta}\text{-scl}^{\circ}(B_N)$.
- iv. $\text{FN}_{\delta}\text{-pcl}^{\circ}(\omega_N \setminus A_N) = \omega_N \setminus \text{FNsint}^{\circ}(A_N)$,
 $\text{FN}_{\delta}\text{-pcl}^{\circ}(A_N \cup B_N) \subset \text{FN}_{\delta}\text{-pcl}^{\circ}(A_N) \cup \text{FN}_{\delta}\text{-pcl}^{\circ}(B_N)$.
- v. $\omega_N \setminus \text{FN-int}_{\delta}(A_N) = \text{FN-cl}_{\delta}(\omega_N \setminus A_N)$ and $\text{FN-int}_{\delta}(\omega_N \setminus A_N) = \omega_N \setminus \text{FN-cl}_{\delta}(A_N)$.

Proposition 3.9: The following hold for any subset A_N of a (ω_N, Γ_N) :

- i. $\text{FN-cl}(\text{FN-int}_{\delta}(A_N)) = \text{FN-cl}_{\delta}(\text{FN-int}_{\delta}(A_N))$,
- ii. $\text{FN-int}(\text{FN-cl}_{\delta}(A_N)) = \text{FN-int}_{\delta}(\text{FN-cl}_{\delta}(A_N))$.

Definition 3.10: Let (ω_N, Γ_N) be FNTS. A subset A_N is said to be FN θ -semi open (resp. FN δ semi open, FN-semi open) If there exist a FN θ -open (resp. FN δ -open, FN-open) set U of ω_N such that $U \subseteq A_N \subseteq \text{FN-cl}(U)$ which mean is the union of all FNS-o(semi open) (A_N) of ω_N whose closures are contained in A_N .

Remark 3.11: In general topology the equality part does not true but, in FNT satisfying both equality case and subset case.

Proposition 3.12: Let A_N be any subset of a FNTS (ω_N, Γ_N) then, every FN- θ semiopen set is FNM-open.

Proof: By the definition of FN θ -semiopen.

Example 3.13: Let $\omega_N = \{a, b, c\}$ and (ω_N, Γ_N) be FNTS where,

$$\Gamma_N = \{0_N, 1_N, A_N, B_N\} \text{ whenever,}$$

$$A_N = \langle \omega, a(0.3, 0.2, 0.5), b(0.4, 0.3, 0.5), c(0.5, 0.2, 0.5) \rangle, \text{ and}$$

$$B_N = \langle \omega, a(0.5, 0.5, 0.5), b(0.5, 0.5, 0.5), c(0.5, 0.5, 0.5) \rangle.$$

$$\text{So, } \delta_N = \{1_N, 0_N, B_N\}.$$

$$\text{Where, } B_N \subseteq \text{FN-cl}(\text{FN-int}^\theta(B_N)) \cup \text{FN-int}(\text{FN-cl}_\delta(B_N)).$$

Then, δ_N is FNM-open set and FN θ -semiopen whenever, $U_N \subseteq A \subseteq \text{FN-cl}(U_N) \subseteq B_N \subseteq \text{FN-cl}(B_N)$.

So, B_N is FN θ -semiopen set.

Definition 3.14: Let (ω_N, Γ_N) be FNTS then,

- i. The union of all FNM-open sets of ω_N contained in FNS A_N is called the FNM-interior of A_N and is denoted by $\text{FNM-int}(A_N)$.
- ii. The intersection of all FNM-closed sets of ω_N containing FNS A_N is called the FNM-closure

Remark 3.15: For any subsets A_N and B_N of a FNTS (ω_N, Γ_N) then, the following statements hold:

- i. $\text{FN-scl}^\theta(A_N) = A_N \cup \text{FN-int}(\text{FN-cl}^\theta(A_N))$, $\text{FN-sint}^\theta(A_N) = A_N \cap \text{FN-cl}(\text{FN-int}^\theta(A_N))$.
- ii. $\text{FN-pcl}_\delta(A_N) = A_N \cup \text{FN-cl}(\text{FN-int}_\delta(A_N))$, $\text{FN-pint}_\delta(A_N) = A_N \cap \text{FN-int}(\text{FN-cl}_\delta(A_N))$.
- iii. $\text{FN}_\delta\text{-scl}^\theta(\omega_N \setminus A_N) = \omega_N \setminus \text{FN-sint}^\theta(A_N)$,
 $\text{FN}_\delta\text{-scl}^\theta(A_N \cup B_N) \subseteq \text{FN}_\delta\text{-scl}^\theta(A_N) \cup \text{FN}_\delta\text{-scl}^\theta(B_N)$.
- iv. $\text{FN}_\delta\text{-pcl}^\theta(\omega_N \setminus A_N) = \omega_N \setminus \text{FN-sint}^\theta(A_N)$,
 $\text{FN}_\delta\text{-pcl}^\theta(A_N \cup B_N) \subseteq \text{FN}_\delta\text{-pcl}^\theta(A_N) \cup \text{FN}_\delta\text{-pcl}^\theta(B_N)$.
- v. $\omega_N \setminus \text{FN-int}_\delta(A_N) = \text{FN-cl}_\delta(\omega_N \setminus A_N)$ and $\text{FN-int}_\delta(\omega_N \setminus A_N) = \omega_N \setminus \text{FN-cl}_\delta(A_N)$.

Proposition 3.16: The following hold for a subsets A_N and B_N of a FNTS (ω_N, Γ_N) :

- i. $\text{FN-pcl}_\delta(B_N) = A_N \cup \text{FN-cl}(\text{FN-int}_\delta(A_N))$ and,
 $\text{FN-pint}_\delta(A_N) = B_N \cap \text{FN-int}(\text{cl}_\delta(A_N))$.
- ii. $\text{FN-pcl}_\delta(\text{FN-pint}_\delta(A_N)) = \text{FN-pint}_\delta(A_N) \cup \text{FN-cl}(\text{FN-int}_\delta(A_N))$ and,
 $\text{FN-pint}_\delta(\text{FN-pcl}_\delta(A_N)) = \text{FN-pcl}_\delta(A_N) \cap \text{FN-int}(\text{FN-cl}_\delta(A_N))$.
- iii. $\text{FN-sint}(A_N) = A_N \cap \text{FN-cl}(\text{FN-int}_\delta(A_N))$ and,
 $\text{FN-scl}_\delta(A_N) = A_N \cup \text{FN-int}(\text{cl}_\delta(A_N))$. of A_N .

Theorem 3.17: Let (ω_N, Γ_N) be FNTS and $A_N \subseteq \omega_N$, then the following statements are equivalent:

- i. A_N is FNM-open set,
- ii. $A_N = \text{FN-sint}^\theta(A_N) \cup \text{FN-pint}_\delta(A_N)$.

Proof: i \rightarrow ii . Let A_N be FNM-open set. Then, $A_N \subseteq \text{FN-cl}(\text{FN-int}^\theta(A_N)) \cup \text{FN-int}(\text{FN-cl}_\delta(A_N))$.

Hence, by Proposition 3.16 and Remark 3.15, we have

$$\begin{aligned} \text{FN-sint}_\delta(A_N) \cup \text{FN-pint}_\delta(A_N) &= (A_N \cap \text{FN-cl}(\text{FN-int}_\delta(A_N))) \cup (A_N \cap \text{FN-int}(\text{FN-cl}_\delta(A_N))) \\ &= A_N \cap (\text{FN-cl}(\text{int}_\delta(A_N)) \cup \text{FN-int}(\text{cl}_\delta(A_N))) = A_N. \end{aligned}$$

ii \rightarrow i Suppose that $A_N = \text{FN-sint}_\delta(A_N) \cup \text{FN-pint}_\delta(A_N)$. Then,

By Proposition 3.16 and Remark 3.15 we have,

$$A_N = (A_N \cap \text{FN-cl}(\text{FN-int}_\delta(A_N))) \cup (A_N \cap \text{FN-int}(\text{FN-cl}_\delta(A_N))) \subset \text{FN-cl}(\text{FN-int}_\delta(A_N)) \cup \text{FN-int}(\text{FN-cl}_\delta(A_N)).$$

Therefore, A_N is FNM-open set.

Proposition 3.18: Let (ω_N, Γ_N) be FNTS and $A_N \subset \omega_N$ then, the following statements are equivalent:

- i. A_N is FNM-closed set.
- ii. $A_N = \text{FN-scl}_\delta(A_N) \cap \text{FN-pcl}_\delta(A_N)$.

Proof: By the definition.

Theorem 3.19: Let A_N be a subset of a FNTS (ω_N, Γ_N) , then:

- i. $\text{FNM-cl}(A_N) = \text{FN-scl}_\delta(A_N) \cap \text{FN-pcl}_\delta(A_N)$.
- ii. $\text{FNM-int}(A_N) = \text{FN-sint}_\delta(A_N) \cup \text{FN-pint}_\delta(A_N)$

Proof : i. $\text{FNM-cl}(A_N) \subseteq \text{FN-scl}_\delta(A_N) \cap \text{FN-pcl}_\delta(A_N)$

$$\begin{aligned} \text{Also, } \text{FN-scl}_\delta(A_N) \cap \text{FN-pcl}_\delta(A_N) &= (A_N \cap \text{FN-int}(\text{FN-cl}_\delta(A_N))) \cap (A_N \cup \text{FN-cl}(\text{FN-int}_\delta(A_N))) \\ &= A_N \cup (\text{FN-int}(\text{FN-cl}_\delta(A_N)) \cap \text{FN-cl}(\text{FN-int}_\delta(A_N))). \end{aligned}$$

But, $\text{FNM-cl}(A_N)$ is FNM-closed.

Hence, $\text{FNM-cl}(A_N) \supset \text{FN-int}(\text{FN-cl}_\delta(A_N)) \cap \text{FN-cl}(\text{FN-int}_\delta(\text{FNM-cl}(A_N))) \supset \text{FN-int}(\text{FN-cl}_\delta(A_N)) \cap \text{FN-cl}(\text{FN-int}_\delta(A_N))$.

Thus, $A_N \cup (\text{FN-int}(\text{FN-cl}_\delta(A_N)) \cap \text{FN-cl}(\text{FN-int}_\delta(A_N))) = A_N \cup \text{FNM-cl}(A_N) = \text{FNM-cl}(A_N)$,

Therefore, $\text{FN-scl}_\delta(A_N) \cap \text{FN-pcl}_\delta(A_N) \subset \text{FNM-cl}(A_N)$.

So, $\text{FNM-cl}(A_N) = \text{FN-scl}_\delta(A_N) \cap \text{FN-pcl}_\delta(A_N)$.

By the same way .) ii(

Theorem 3.20: Let A_N be a subset of a FNTS (ω_N, Γ_N) , then:

- i. $\text{FNM-cl}(A_N) = \text{FN-scl}_\delta(A_N) \cap \text{FN-pcl}_\delta(A_N)$.
- ii. $\text{FNM-int}(A_N) = \text{FN-sint}_\delta(A_N) \cup \text{FN-pint}_\delta(A_N)$.

Proof : i. $\text{FNM-cl}(A_N) \subseteq \text{FN-scl}_\delta(A_N) \cap \text{FN-pcl}_\delta(A_N)$

$$\begin{aligned} \text{Also, } \text{FN-scl}_\delta(A_N) \cap \text{FN-pcl}_\delta(A_N) &= (A_N \cap \text{FN-int}(\text{FN-cl}_\delta(A_N))) \cap (A_N \cup \text{FN-cl}(\text{FN-int}_\delta(A_N))) \\ &= A_N \cup (\text{FN-int}(\text{FN-cl}_\delta(A_N)) \cap \text{FN-cl}(\text{FN-int}_\delta(A_N))). \end{aligned}$$

But, $\text{FNM-cl}(A_N)$ is FNM-closed.

Hence, $\text{FNM-cl}(A_N) \supset \text{FN-int}(\text{FN-cl}_\delta(A_N)) \cap \text{FN-cl}(\text{FN-int}_\delta(\text{FNM-cl}(A_N))) \supset \text{FN-int}(\text{FN-cl}_\delta(A_N)) \cap \text{FN-cl}(\text{FN-int}_\delta(A_N))$.

Thus, $A_N \cup (\text{FN-int}(\text{FN-cl}_\delta(A_N)) \cap \text{FN-cl}(\text{FN-int}_\delta(A_N))) = A_N \cup \text{FNM-cl}(A_N) = \text{FNM-cl}(A_N)$,

Therefore, $\text{FN-scl}^*(A_N) \cap \text{FN-pcl}_\delta(A_N) \subset \text{FNM-cl}(A_N)$.

So, $\text{FNM-cl}(A_N) = \text{FN-scl}^*(A_N) \cap \text{FN-pcl}_\delta(A_N)$.

ii. By the same way .

Theorem 3.21: Let A_N be a subset of a FNTS (ω_N, Γ_N) . Then,

- i. A_N is FNM-open set if and only if $A_N = \text{FNM-int}(A_N)$.
- ii. A_N is FNM-closed set if and only if $A_N = \text{FNM-cl}(A_N)$.

Proof: (i) Let A_N be FNM-open set . Then, by Theorem 3.18,

$$A_N = \text{FN-sint}^*(A_N) \cup \text{FN-pint}_\delta(A_N) \text{ and, we have, } A_N = \text{FNM-int}(A_N) .$$

Conversely, let $A_N = \text{FNM-int}(A_N)$ then, by Theorem 3.20,

$$A_N = \text{FN-sint}^*(A_N) \cup \text{FN-pint}_\delta(A_N) \text{ and } A_N \text{ is FNM-open .}$$

(ii) by the same way.

Theorem 3.22: Let A_N and B_N be subsets of a FNTS (ω_N, Γ_N) . Then, the following are hold:

- i. $\text{FNM-cl}(\omega_N \setminus A_N) = \omega_N \setminus \text{FNM-int}(A_N)$.
- iii. $\text{FNM-int}(\omega_N \setminus A_N) = \omega_N \setminus \text{FNM-cl}(A_N)$.
- iv. If $A_N \subseteq B_N$, then $\text{FNM-cl}(A_N) \subseteq \text{FNM-cl}(B_N)$ and $\text{FNM-int}(A_N) \subseteq \text{FNM-int}(B_N)$.
- v. $x \in \text{FNM-cl}(A_N)$ if and only if there exist FNM-open set U_N and $x \in U_N$ such that $U_N \cap A_N \neq \emptyset$.
- vi. $\text{FNM-cl}(\text{FNM-cl}(A_N)) = \text{FNM-cl}(A_N)$ and $\text{FNM-int}(\text{FNM-int}(A_N)) = \text{FNM-int}(A_N)$.
- vii. $\text{FNM-cl}(A_N) \cup \text{FNM-cl}(B_N) \subset \text{FNM-cl}(A_N \cup B_N)$ and $\text{FNM-int}(A_N) \cup \text{FNM-int}(B_N) \subset \text{FNM-int}(A_N \cup B_N)$.
- viii. $\text{FNM-int}(A_N \cap B_N) \subset \text{FNM-int}(A_N) \cap \text{FNM-int}(B_N)$ and $\text{FNM-cl}(A_N \cap B_N) \subset \text{FNM-cl}(A_N) \cap \text{FNM-cl}(B_N)$.

Proof: By Theorem 3.18 we have

$$\begin{aligned} \text{FNM-cl}(\omega_N \setminus A_N) &= \text{FN-scl}^*(\omega_N \setminus A_N) \cap \text{FN-pcl}_\delta(\omega_N \setminus A_N) \text{ and by Remark 3.8 we have,} \\ \text{FNM-cl}(\omega_N \setminus A_N) &= (\omega_N \setminus \text{FN-sint}^*(A_N)) \cap (\omega_N \setminus \text{FN-pint}_\delta(A_N)) = \omega_N \setminus (\text{FN-sint}^*(A_N) \cup \text{FN-pint}_\delta(A_N)). \end{aligned}$$

Hence by Theorem 3.19, $\text{FNM-cl}(\omega_N \setminus A_N) = \omega_N \setminus \text{FNM-int}(A_N)$.

(ii) by the same way.

(iii) hence, $\text{FNM-cl}(A_N) = \text{FN-scl}^*(A_N) \cap \text{FN-pcl}_\delta(A_N)$ and $A_N \subseteq B_N$,

$$\begin{aligned} \text{FNM-cl}(A_N) &= \text{FN-scl}^*(A_N) \cap \text{FN-pcl}_\delta(A_N) \\ &\subseteq \text{FN-scl}^*(B_N) \cap \text{FN-pcl}_\delta(B_N) = \text{FNM-cl}(B_N) . \end{aligned}$$

(iv) By the definitions .

(v) Since , $\text{FNM-cl}(\text{FNM-cl}(A_N)) = \text{FN-scl}^*(\text{FNM-cl}(A_N)) \cap \text{FN-pcl}_\delta(\text{FNM-cl}(A_N))$,
by Theorem 3.20,

$$\begin{aligned} \text{FN-scl}^*(\text{FN-scl}^*(A_N) \cap \text{FN-pcl}_\delta(A_N)) \cap \text{FN-pcl}_\delta(\text{FN-scl}^*(A_N) \cap \text{FN-pcl}_\delta(A_N)) &\subseteq \\ \text{FN-scl}^*(A_N) \cap \text{FN-scl}^*(\text{FN-pcl}_\delta(A_N)) \cap \text{FN-pcl}_\delta(\text{FN-scl}^*(A_N) \cap \text{FN-pcl}_\delta(A_N)) &= \\ \text{FN-scl}^*(A_N) \cap \text{FN-pcl}_\delta(A_N) &= \text{FNM-cl}(A_N). \end{aligned}$$

Hence, $\text{FNM-cl}(\text{FNM-cl}(A_N)) \subseteq \text{FNM-cl}(A_N)$. But ,

$\text{FNM-cl}(A_N) \subseteq \text{FNM-cl}(\text{FNM-cl}(A_N))$. Therefore,

$$\text{FNM-cl}(\text{FNM-cl}(A_N)) = \text{FNM-cl}(A_N) .$$

(vi),(vii) by the definition.

Proposition 3.23: Let A_N be a subset of a FNTS (ω_N, Γ_N) , then:

- i. $\text{FNM-cl}(A_N) = A_N \cup \text{FN-pint}^*(\text{FN-pcl}_\delta(A_N))$,
- ii. $\text{FNM-int}(A_N) = A_N \cap \text{FN-pcl}^*(\text{FN-pint}_\delta(A_N))$.

Proof: By Proposition 3.19 we have,

$$\begin{aligned} A_N \cup \text{FN-pint}^*(\text{FN-pcl}_\delta(A_N)) &= A_N \cup \text{FN-pcl}_\delta(A_N) \cap \text{FN-int}(\text{FN-cl}^*(A_N)) \\ &= (A_N \cup \text{FN-pcl}_\delta(A_N)) \cap (A_N \cap \text{FN-int}(\text{FN-cl}^*(A_N))) \\ &= \text{FN-pcl}_\delta(A_N) \cap \text{FN-scl}^*(A_N). \end{aligned}$$

Proposition 3.24: Let A_N be a subset of a FNTS (X_N, τ_N) . Then, the following statements are hold:

- i. $\text{FN-pint}_\delta(\text{FN-pcl}(A_N)) = \text{FN}_\delta(\text{FN-pcl}(A_N)) \cap \text{FN-int}(\text{FN-cl}(A_N))$ and $\text{FN-pcl}_\delta(\text{FN-pint}(A_N)) = \text{FN}_\delta(\text{FN-pint}(A_N)) \cup \text{FN-cl}(\text{FN-int}(A_N))$.
- ii. $\text{FN-pint}^*(\text{FN-pcl}_\delta(A_N)) = \text{FN-pcl}_\delta(A_N) \cap \text{FN-int}(\text{FN-cl}^*(A_N))$ and, $\text{FN-pcl}^*(\text{FN-pint}_\delta(A_N)) = \text{FN-pint}_\delta(A_N) \cup \text{FN-cl}(\text{FN-int}^*(A_N))$.
- iii. $\text{FN-scl}^*(\text{FN-int}(A_N)) = \text{FN-scl}(\text{FN-int}^*(A_N)) = \text{FN-int}(\text{FN-cl}(\text{int}^*(A_N)))$.

Proof: By the definitions .

Theorem 3.25: Let A_N be a subset of a FNTS (ω_N, Γ_N) . then the following are equivalent:

- i. A_N is FNM-open set .
- ii. $A_N \subseteq \text{FN}(\text{pcl}^*(\text{pint}_\delta(A_N)))$.
- iii. $\text{FN}(\text{pcl}^*(A_N)) = \text{FN}(\text{pcl}^*(\text{pint}_\delta(A_N)))$.

Proof: $i \rightarrow ii$. Let A_N be FNM-open set, then, $A_N = \text{FN}(\text{M-int}(A_N))$ and by Proposition 3.24, $A_N = A_N \cap \text{FN-pcl}^*(\text{FN-pint}_\delta(A_N))$ and hence , $A_N \subseteq \text{FN-pcl}^*(\text{FN-pint}_\delta(A_N))$

$ii \rightarrow i$.Let $A_N \subseteq \text{FN-pcl}^*(\text{FN-pint}_\delta(A_N))$. Then by proposition 3.24,

$A_N \subseteq A_N \cap \text{FN-pcl}^*(\text{FN-pint}_\delta(A_N)) = \text{FNM-int}(A_N)$.

So, $A_N \subseteq \text{FNM-int}(A_N)$. Then $A_N = \text{FNM-int}(A_N)$ and hence ,

A_N is FNM-open set .

$ii \rightarrow iii$. Let $A_N \subseteq \text{FN-pcl}^*(\text{FN-pint}_\delta(A_N))$. Then $\text{FN-pcl}^*(A_N) \subseteq \text{FN-pcl}^*(\text{FN-pint}_\delta(A_N))$ and hence, $\text{FN-pcl}^*(A_N) = \text{FN-pcl}^*(\text{FN-pint}_\delta(A_N))$.

$iii \rightarrow ii$. by the same way.

Theorem 3.26: Let A_N be a subset of FNTS (ω_N, Γ_N) , then the following are equivalent :

- i. A_N is FNM-closed set.
- ii. $\text{FN-pint}^*(\text{FN-pcl}_\delta(A_N)) \subseteq A_N$.
- iii. $\text{FN-pint}^*(A_N) = \text{FN-pint}^*(\text{FN-pcl}_\delta(A_N))$.

Proof: By the same way to Theorem 3.21.

4. Conclusions

In this study, we introduced the concept new classes of sets which is including fuzzy neutrosophic M-open set and fuzzy neutrosophic M-closed set with some of their properties were discussed. This study can be extended and generalized to several functions such as M-continuous and irresolute M- functions.

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