Hyers - Ulam - Rassias Stability of Various Functional Equations in Non-Archimedean Neutrosophic Normed Spaces

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Abstract

In this paper, we introduce the notion of non- Archimedean neutrosophic normed space and also establish Hyers-Ulam-Rassias-type stability results concerning the Cauchy, Pexiderized Cauchy. We determine some stability results concerning the Cauchy, Jensen and its Pexiderized functional equations in the framework of non-Archimedean Neutrosophic Normed Space. This work indeed presents a relationship between four various disciplines, the theory of neutrosophic normed space, non – Archimedean, Hyers-Ulam-Rassias stability and functional equation.

Keywords: Non-Archimedean; Pexiderized Cauchy; Functional Equation; Pexiderized Jensen Functional Equation; Neutrosophic Normed Space.

1. Introduction

Stability problem of a functional equation was first posed by Ulam [22] which was answered by Hyers [4] and then generalized by Rassias [17] for addictive mappings and linear mappings respectively. Since then several stability problems for various functional equations have been investigated in [1,5,7,9,10,11] and various fuzzy stability results concerning Cauchy, Jensen and quadratic functional equations were discussed.

After a while, Smarandache [20] introduced the notion of Neutrosophic Sets [NS], which is the different kind of the notation of the classical set theory by adding an intermediate membership function. This set is a formal setting trying to measure the truth, indeterminacy and falsehood. Later on, the concepts of statistical convergence of double sequences have been analyzed in IFNS by Mursaleen and Mohiuddin [13]. Quite recently, Kirisci and Simsek [21] introduced the notion of neutrosophic metric space. In 2022, Jeyaraman et al. [5] generalized Hyers-Ulam-Rassias stability for functional equation in Neutrosophic Normed Spaces [NNS]. Since Neutrosophic Normed Space is a natural generalization of IFNS and statistical convergence.

In this paper, we introduce the notion of non-Archimedean Neutrosophic Normed Space and also establish Hyers-Ulam-Rassias-type stability results concerning the Cauchy, Pexiderized Cauchy, Jensen, and Pexiderized Jension functional equations in this new setup. This work indeed presents a

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relationship between four various disciplines: the theory of Neutrosophic Normed spaces, the theory of non-Archimedean spaces, the theory of Hyers-Ulam-Rassias stability, and the theory of functional equations.

2. Preliminaries

A valuation is a map $|\cdot|$ from a field $\mathbb{K}$ into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|k_1k_2| = \lceil k_1 \rceil \lceil k_2 \rceil$ and the triangle inequality holds, that is, $|k_1 + k_2| \leq \lceil k_1 \rceil + \lceil k_2 \rceil$, for all $k_1, k_2 \in \mathbb{K}$. We say that a field $\mathbb{K}$ is valued if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies stronger condition than the triangle inequality. If the triangle inequality is replaced by $|k_1 + k_2| \leq \max\{|k_1|, |k_2|\}$, for all $k_1, k_2 \in \mathbb{K}$ then, a map $|\cdot|$ is called Non-Archimedean $[\mathcal{NA}]$ or ultrametric valuation, and field is called a non-Archimedean field. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$, for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation (NAV) is the map $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$.

Let X be a vector space over a field $\mathbb{K}$ with a NAV $|\cdot|$ A non-Archimedean Normed Space (NANS) is a pair $(X, \|\cdot\|)$, where $\|\cdot\|: X \to [0,\infty)$ is such that

a. $\|x\| = 0$ iff $x = 0$,
b. $\|\alpha x\| = |\alpha|\|x\|$ for $\alpha \in \mathbb{K}$ and
c. The strong triangle inequality, $\|x + y\| \leq \max\{\|x\|, \|y\|\}$, for $x, y \in X$.

Definition 2.1:

The 6-tuple $(X, \mu, \vartheta, \omega, *, \diamond, \odot)$ is said to be a Non -Archimedean Neutrosophic Normed Space ($\mathcal{NA} - \mathcal{NNS}$) if X is a vector space, * are the continuous t- norm , $\diamond$ and $\odot$ continuous t-conorm and μ, ϑ, ω are functions from on X x $(0,\infty)$ fulfilling the conditions below:
For each $x, y \in X$ and for each $s, t > 0$, $\emptyset \neq 0$,
(NNS-1) $0 \leq \mu(x,t) \leq 1$, $0 \leq v(x,t) \leq 1$, $0 \leq \omega(x,t) \leq 1$, for all $t \in (0,\infty)$;

(NNS-2) $\mu(x,t) + v(x,t) + \omega(x,t) \leq 3$;

(NNS-3) $\mu(x,t) > 0$;

(NNS-4) $\mu(x,t) = 1$ if and only if $x = 0$;

(NNS-5) $\mu(\emptyset x, t) = \mu\left(x, \frac{1}{|\emptyset|}\right)$, for each $\emptyset \neq 0$;

(NNS-6) $\mu(x,t) * \mu(y,s) \leq \mu(x + y, \max\{t + s\})$;

(NNS-7) $\mu(x,.): (0,\infty) \to [0,1]$ is continuous and increasing;

(NNS-8) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$;

(NNS-9) $v(x,t) < 1$;

(NNS-10) $v(x,t) = 0$ if and only if $\vartheta = 0$;

(NNS-11) $v(\emptyset x, t) = \lambda\left(x, \frac{t}{|\emptyset|}\right)$ for each $\emptyset \neq 0$;

(NNS-12) $v(x,t) \diamond v(y,s) \geq v(x + y, \max\{t + s\})$;

(NNS-13) $v(x,.): (0,\infty) \to [0,1]$ is continuous and increasing;

(NNS-14) $\lim_{t\to\infty} v(x,t) = 0$ and $\lim_{t\to 0} v(x,t) = 1$;

(NNS-15) $\omega(x,t) < 1$;

(NNS-16) $\omega(x,t) = 0$ if and only if $\omega = 0$;

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(NNS-17) $\omega(\emptyset x, t) = \omega\left(x, \frac{t}{|\emptyset|}\right)$,for each $\emptyset \neq 0$;

(NNS-18) $\omega(x,t) \odot \omega(y,s) \geq \omega(x+y, \max\{t+s\})$;

(NNS-19) $\omega(x,.) : (0,\infty) \to [0,1]$ is continuous and increasing;

(NNS-20) $\lim_{t\to\infty} \omega(x,t) = 0$ and $\lim_{t\to 0} \omega(x,t) = 1$;

Then (μ, ϑ, ω) is called Non – Archimedean Neutrosophic Norm ($\mathcal{NA} - \mathcal{NN}$).

Example 2.2 :

Let $(X, \|.\|)$ be a $\mathcal{NA} - \mathcal{NN}$, $a * b = ab$, $a \diamond b = min\{a+b, 1\}$ and $a \odot b = min\{a+b, 1\}$, for all $a, b \in [0,1]$. For all $x \in X$ and every t > 0. Consider

$$\mu_k(x,t) = \begin{cases} \frac{t}{t+\|x\|} & if\ t > 0 \\ 0 & if\ t \leq 0 \end{cases}, \ \ v_K(x,t) = \begin{cases} \frac{\|x\|}{t+\|x\|} & if\ t > 0 \\ 1 & if\ t \leq 0 \end{cases} \text{ and } \omega_K(x,t) = \begin{cases} \frac{\|x\|}{t} & if\ t > 0 \\ 1 & if\ t \leq 0 \end{cases}.$$

Then $(X, \mu, \vartheta, \omega, *, \diamond, \odot)$ is an $\mathcal{NA} - \mathcal{NN}$.

The concepts of convergence and Cauchy sequences in an $\mathcal{NA} - \mathcal{NNS}$ are studied .

Let $(X, \mu, \vartheta, \omega, *, \diamond, \odot)$ be an $\mathcal{NA} - \mathcal{NNS}$. Then, a sequence $x = (x_k)$ is said to be neutrosophic convergent to $L \in X$ if $\lim \mu(x_k - L, t) = 1, \lim v(x_k - L, t) = 0$ and $\omega(x_k - L, t) = 0$, for all $t > 0$. In this case we write $x_k \to L$ as $k \to \infty$.

Let $(X, \mu, \vartheta, \omega, *, \diamond)$ be an $\mathcal{NA} - \mathcal{NNS}$. Then $x = (x_k)$ is said to be neutrosophic Cauchy sequences if $\lim \mu(x_{k+p} - x_k, t) = 1$, $\lim \vartheta(x_{k+p} - x_k, t) = 0$ and $\omega(x_{k+p} - x_k, t) = 0$ for all $t > 0$ and $p = 1,2 \ldots$

Let $(X, \mu, \vartheta, \omega, *, \diamond, \odot)$ be an $\mathcal{NA} - \mathcal{NNS}$. Then $(X, \mu, \vartheta, \omega, *, \diamond, \odot)$ is said to be complete if every neutrosophic Cauchy sequences if $(X, \mu, \vartheta, \omega, *, \diamond, \odot)$ neutrosophic convergent in $(X, \mu, \vartheta, \omega, *, \diamond, \odot)$.

3. Stability of Cauchy Functional Equation

In this section, we determine stability result concerning the Cauchy functional equation $f(x+y) = f(x) + f(y)$ in $\mathcal{NA} - \mathcal{NNS}$.

Theorem 3.1:

Let X be a linear space over a $\mathcal{NA}$ field $\mathbb{K}$ and let $(Z, \mu', \vartheta', \omega')$ be a $\mathcal{NA} - \mathcal{NNS}$. Suppose that $\varphi: X \times X \to Z$ is a function such that for some $\alpha > 0$ and some positive integer k with $|k| < \alpha$

$$\left\{\begin{array}{c} \mu'(\varphi(k^{-1}x, k^{-1}y), t) \geq \mu'(\varphi(x,y), \alpha t), \\ \vartheta'(\varphi(k^{-1}x, k^{-1}y), t) \leq \vartheta'(\varphi(x,y), \alpha t) \text{ and} \\ \omega'(\varphi(k^{-1}x, k^{-1}y), t) \leq \omega'(\varphi(x,y), \alpha t), \end{array}\right\} \quad (3.1.1)$$

for all $x, y \in X$ and $t > 0$. Let $(Y, \mu, \vartheta, \omega)$ be a Non-Archimedean Neutrosophic Banach Space [$\mathcal{NA} - \mathcal{NBS}$] over $\mathbb{K}$ and let $f: X \to Y$ be a φ-approximately Cauchy in the sense that

$$\left\{\begin{array}{l} \mu(f(x+y) - f(x) - f(y), t) \geq \mu'(\varphi(x,y), t), \\ \vartheta(f(x+y) - f(x) - f(y), t) \leq \vartheta'(\varphi(x,y), t) \text{ and} \\ \omega(f(x+y) - f(x) - f(y), t) \leq \omega'(\varphi(x,y), t), \end{array}\right\} \quad (3.1.2)$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique additive mapping $C: X \to Y$ such that

$$\left\{\begin{array}{c} \mu(f(x) - C(x), t) \geq M(x, \alpha t), \\ \vartheta(f(x) - C(x), t) \leq N(x, \alpha t) \text{ and} \\ \omega(f(x) - C(x), t) \leq P(x, \alpha t), \end{array}\right\} \quad (3.1.3)$$

for all $x \in X$ and $t > 0$, where

$$\left\{\begin{array}{c} M(x,t) = \mu'(\varphi(x,x),t) * \mu'(\varphi(x,2x),t) * \ldots * \mu'(\varphi(x,(k-1)x),t), \\ N(x,t) = \vartheta'(\varphi(x,x),t) \diamond \vartheta'(\varphi(x,2x),t) \diamond \ldots \diamond \vartheta'(\varphi(x,(k-1)x),t) \text{ and} \\ P(x,t) = \omega'(\varphi(x,x),t) \odot \omega'(\varphi(x,2x),t) \odot \ldots \odot \omega'(\varphi(x,(k-1)x),t), \end{array}\right\} (3.1.4)$$

Proof:

By induction on j we will show that for each $x \in X, t > 0$ and $j \geq 2$

$$\left\{\begin{array}{c} \mu(f(jx) - jf(x), t) \geq M_j(x,t) = \mu'(\varphi(x,x),t) * \ldots * \mu'(\varphi(x,(j-1)x,t), \\ \vartheta(f(jx) - jf(x), t) \leq N_j(x,t) = \vartheta'(\varphi(x,x),t) \diamond \ldots \diamond \vartheta'(\varphi(x,(j-1)x,t) \text{and} \\ \omega(f(jx) - jf(x), t) \leq P_j(x,t) = \omega'(\varphi(x,x),t) \odot \ldots \odot \omega'(\varphi(x,(j-1)x,t). \end{array}\right\} \quad (3.1.5)$$

Putting $x = y$ in (3.1.2), we obtain

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$$\begin{cases} \mu(f(2x)-2f(x),t) \geq \mu'(\varphi(x,x),t), \\ \vartheta(f(2x)-2f(x),t) \leq \vartheta'(\varphi(x,x),t) \text{and} \\ \omega(f(2x)-2f(x),t) \leq \omega'(\varphi(x,x),t), \end{cases} \tag{3.1.6}$$

for all $x \in$ and $t > 0$. This proves (3.1.5) for $j = 2$. Let (3.1.5) hold for some $j > 2$. Replacing y by jx in (3.1.2), we get

$$\begin{cases} \mu(f(j+1)x) - f(x) - f(jx),t) \geq \mu'(\varphi(x,jx),t), \\ \vartheta(f(j+1)x) - f(x) - f(jx),t) \leq \vartheta'(\varphi(x,jx),t) \text{ and} \\ \omega(f(j+1)x) - f(x) - f(jx),t) \leq \omega'(\varphi(x,jx),t), \end{cases} \tag{3.1.7}$$

for each $x \in X$ and $t > 0$. Thus

$$\begin{cases} \mu(f((j+1)x) - (j+1)f(x),t) = \mu(f((j+1)x) - f(x) - f(jx) + f(jx) - jf(x),t) \\ \quad \geq \mu(f((j+1)x) - f(x) - f(jx),t) * \mu(f(jx) - jf(x),t) \\ \quad \geq \mu'(\varphi(x,jx),t) * M_j(x,t) = M_{j+1}(x,t), \\ \vartheta(f((j+1)x) - (j+1)f(x),t) = \vartheta(f((j+1)x) - f(x) - f(jx) + f(jx) - jf(x),t) \\ \quad \leq \vartheta(f((j+1)x) - f(x) - f(jx),t) \diamond \vartheta(f(jx) - jf(x),t) \\ \quad \leq \vartheta'(\varphi(x,jx),t) \diamond N_j(x,t) = N_{j+1}(x,t) \text{ and} \\ \omega(f((j+1)x) - (j+1)f(x),t) = \omega(f((j+1)x) - f(x) - f(jx) + f(jx) - jf(x),t) \\ \quad \leq \omega(f((j+1)x) - f(x) - f(jx),t) \odot \omega(f(jx) - jf(x),t) \\ \quad \leq \omega'(\varphi(x,jx),t) \odot P_j(x,t) = P_{j+1}(x,t), \end{cases} \tag{3.1.8}$$

for each $x \in X$ and $t > 0$. Hence (3.1.5) holds for all $j \geq 2$. In particular

$$\begin{cases} \mu(f(kx) - kf(x),t) \geq M(x,t), \\ \vartheta(f(kx) - kf(x),t) \leq N(x,t) \text{ and} \\ \omega(f(kx) - kf(x),t) \leq P(x,t). \end{cases} \tag{3.1.9}$$

Replacing x by $k^{-n-1}x$ in (3.1.9) and using (3.1.1), we get

$$\begin{cases} \mu(f(k^{-n}x) - kf(k^{-(n+1)}x),t) \geq M(x,\alpha^{n+1}t), \\ \vartheta(f(k^{-n}x) - kf(k^{-(n+1)}x),t) \leq N(x,\alpha^{n+1}t) \text{ and} \\ \omega(f(k^{-n}x) - kf(k^{-(n+1)}x),t) \leq P(x,\alpha^{n+1}t), \end{cases} \tag{3.1.10}$$

for all $x \in X, t > 0$ and $n = 0,1,2,\ldots$. Therefore

$$\begin{cases} \mu(k^n f(k^{-n}x) - k^{n+1} f(k^{-(n+1)}x),t) \geq M\left(x, \dfrac{\alpha^{n+1}t}{|k|^n}\right), \\ \vartheta(k^n f(k^{-n}x) - k^{n+1} f(k^{-(n+1)}x),t) \leq N\left(x, \dfrac{\alpha^{n+1}t}{|k|^n}\right) \text{ and} \\ \omega(k^n f(k^{-n}x) - k^{n+1} f(k^{-(n+1)}x),t) \leq P\left(x, \dfrac{\alpha^{n+1}t}{|k|^n}\right), \end{cases} \tag{3.1.11}$$

for all $x \in X, t > 0$ and $n = 0,1,2,\ldots$. Since

$$\left\{ \lim_{m\to\infty} M\left(x, \frac{\alpha^{m+1}t}{|k|^m}\right) = 1, \lim_{m\to\infty} N\left(x, \frac{\alpha^{m+1}t}{|k|^m}\right) = 0 \text{ and } \lim_{m\to\infty} P\left(x, \frac{\alpha^{m+1}t}{|k|^m}\right) = 0. \right\} \tag{3.1.12}$$

So, (3.1.11) shows that $(k^n f(k^{-n}x))$ is a Cauchy sequence in $\mathcal{NA} - \mathcal{NBS}$ (Y,μ,ϑ,ω). Therefore, we can define a mapping $C: X \to Y$ by $Cx = (\mu,\vartheta,\omega) - \lim_{n\to\infty} k^n f(k^{-n}x)$. Hence

$$\begin{cases} \lim_{n\to\infty} \mu(k^n f(k^{-n}x) - C(x),t) = 1, \\ \lim_{n\to\infty} \vartheta(k^n f(k^{-n}x) - C(x),t) = 0 \text{ and} \\ \lim_{n\to\infty} \omega(k^n f(k^{-n}x) - C(x),t) = 0. \end{cases} \tag{3.1.13}$$

For each $n \geq 1, x \in X$ and $t > 0$,

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$$\left\{\begin{array}{c}\mu(f(x)-k^nf(k^{-n}x),t)=\mu\left(\sum_{i=0}^{n-1}k^if(k^{-i}x)-k^{i+1}f\left(k^{-(i+1)}x\right),t\right)\\ \geq\prod_{i=0}^{n-1}\mu\left(k^if(k^{-i}x)-k^{i+1}f\left(k^{-(i+1)}x\right),t\right)=M(x,\alpha t),\\ \vartheta(f(x)-k^nf(k^{-n}x),t)=\vartheta\left(\sum_{i=0}^{n-1}k^if(k^{-i}x)-k^{i+1}f\left(k^{-(i+1)}x\right),t\right)\\ \leq\coprod_{i=0}^{n-1}\vartheta\left(k^if(k^{-i}x)-k^{i+1}f\left(k^{-(i+1)}x\right),t\right)=N(x,\alpha t)\text{ and}\\ \omega(f(x)-k^nf(k^{-n}x),t)=\omega\left(\sum_{i=0}^{n-1}k^if(k^{-i}x)-k^{i+1}f\left(k^{-(i+1)}x\right),t\right)\\ \leq\coprod_{i=0}^{n-1}\omega\left(k^if(k^{-i}x)-k^{i+1}f\left(k^{-(i+1)}x\right),t\right)=P(x,\alpha t),\end{array}\right\}\quad(3.1.14)$$

where $\prod_{j=1}^{n}a_j=a_1*a_2*\dots*a_n$, $\coprod_{j=1}^{n}a_j=a_1\diamond a_2\diamond\dots\diamond a_n$ and $\coprod_{j=1}^{n}a_j=a_1\odot a_2\odot\dots\odot a_n$. It follows from (3.1.13) and (3.1.14) that

$$\left\{\begin{array}{c}\mu(f(x)-C(x),t)\geq\mu(f(x)-k^nf(k^{-n}x),t)*\mu(k^nf(k^{-n}x)-C(x),t)\geq M(x,\alpha t),\\ \vartheta(f(x)-C(x),t)\leq\vartheta(f(x)-k^nf(k^{-n}x),t)\diamond\vartheta(k^nf(k^{-n}x)-C(x),t)\leq N(x,\alpha t),\\ \omega(f(x)-C(x),t)\leq\omega(f(x)-k^nf(k^{-n}x),t)\odot\omega(k^nf(k^{-n}x)-C(x),t)\leq P(x,\alpha t),\end{array}\right\}\quad(3.1.15)$$

for each $x\in X,t>0$ and for sufficiently large n; that is, (3.1.3) holds. Also (3.1.1), (3.1.2) and (3.1.13), we have

$$\left\{\begin{array}{c}\mu(C(x+y)-C(x)-C(y),t)\\ \geq\mu(C(x+y)-k^nf(k^{-n}(x+y),t)*\mu(k^nf(k^{-n}x)-C(x),t)*\mu(k^nf(k^{-n}y)-C(y),t)\\ *\,\mu\left(k^nf\left(k^{-n}(x+y)\right)-k^nf(k^{-n}x)-k^nf(k^{-n}y),t\right)\\ \geq\mu'\left(\varphi(k^{-n}x,k^{-n}y),\frac{t}{|k|^n}\right)\geq\mu'\left(\varphi(x,y),\frac{\alpha^nt}{|k|^n}\right),\\ \vartheta(C(x+y)-C(x)-C(y),t)\\ \leq\vartheta(C(x+y)-k^nf(k^{-n}(x+y),t)\diamond\vartheta(k^nf(k^{-n}x)-C(x),t)\diamond\vartheta(k^nf(k^{-n}y)-C(y),t)\\ \diamond\,\vartheta\left(k^nf\left(k^{-n}(x+y)\right)-k^nf(k^{-n}x)-k^nf(k^{-n}y),t\right)\\ \leq\vartheta'\left(\varphi(k^{-n}x,k^{-n}y),\frac{t}{|k|^n}\right)\leq\vartheta'\left(\varphi(x,y),\frac{\alpha^nt}{|k|^n}\right)\text{ and}\\ \omega(C(x+y)-C(x)-C(y),t)\\ \leq\omega(C(x+y)-k^nf(k^{-n}(x+y),t)\odot\omega(k^nf(k^{-n}x)-C(x),t)\odot\omega(k^nf(k^{-n}y)-C(y),t)\\ \odot\,\omega\left(k^nf\left(k^{-n}(x+y)\right)-k^nf(k^{-n}x)-k^nf(k^{-n}y),t\right)\\ \leq\omega'\left(\varphi(k^{-n}x,k^{-n}y),\frac{t}{|k|^n}\right)\leq\omega'\left(\varphi(x,y),\frac{\alpha^nt}{|k|^n}\right),\end{array}\right\}\quad(3.1.16)$$

for all $x,y\in X,t>0$ and for large n. Since

$$\left\{\begin{array}{c}\lim_{n\to\infty}\mu'^{\left(\varphi(x,y),\frac{\alpha^nt}{|k|^n}\right)}=1,\ \lim_{n\to\infty}\vartheta'^{\left(\varphi(x,y),\frac{\alpha^nt}{|k|^n}\right)}=0\text{ and}\\ \lim_{n\to\infty}\omega'\left(\varphi(x,y),\frac{\alpha^nt}{|k|^n}\right)=0\end{array}\right\}\quad(3.1.17)$$

which shows that C is additive. Now if $C':X\to Y$ is another additive mapping such that

$$\left\{\begin{array}{c}\mu(C'(x)-f(x),t)\geq M(x,t),\quad\vartheta(C'(x)-f(x),t)\leq N(x,t)\\ \text{and }\omega(C'(x)-f(x),t)\leq P(x,t)\end{array}\right\}\quad(3.1.18)$$

for all $x\in X$ and $t>0$. The, for all $x\in X,t>0$ and $n\in\mathbb{N}$, we have

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$$\left\{\begin{array}{c} \mu(C(x)-C'(x)\geq \mu(C(x)-k^nf(k^{-n}x),t)*\mu(k^nf(k^{-n}x)-C'(x),t) \\ \geq \mu\left(C(k^{-n}x)-f(k^{-n}x),\frac{t}{|k|^n}\right)*\mu\left(f(k^{-n}x)-C'(k^{-n}x),\frac{t}{|k|^n}\right) \\ \geq M\left(k^{-n}x,\frac{\alpha t}{|k|^n}\right)*M\left(x,\frac{\alpha^{n+1}t}{|k|^n}\right), \\ \vartheta(C(x)-C'(x)\leq \vartheta(C(x)-k^nf(k^{-n}x),t)\diamond\vartheta(k^nf(k^{-n}x)-C'(x),t) \\ \leq \vartheta\left(C(k^{-n}x)-f(k^{-n}x),\frac{t}{|k|^n}\right)\diamond\vartheta\left(f(k^{-n}x)-C'(k^{-n}x),\frac{t}{|k|^n}\right) \\ \leq N\left(k^{-n}x,\frac{\alpha t}{|k|^n}\right)\diamond N\left(x,\frac{\alpha^{n+1}t}{|k|^n}\right) \text{ and} \\ \omega(C(x)-C'(x)\leq \omega(C(x)-k^nf(k^{-n}x),t)\odot\omega(k^nf(k^{-n}x)-C'(x),t) \\ \leq \omega\left(C(k^{-n}x)-f(k^{-n}x),\frac{t}{|k|^n}\right)\odot\omega\left(f(k^{-n}x)-C'(k^{-n}x),\frac{t}{|k|^n}\right) \\ \leq P\left(k^{-n}x,\frac{\alpha t}{|k|^n}\right)\odot P\left(x,\frac{\alpha^{n+1}t}{|k|^n}\right). \end{array}\right\} \quad (3.1.19)$$

Therefore

$$\left\{\lim_{n\to\infty} M\left(x,\frac{\alpha^{n+1}t}{|k|^n}\right)=1, \lim_{n\to\infty} N\left(x,\frac{\alpha^{n+1}t}{|k|^n}\right)=0 \text{ and } \lim_{n\to\infty} P\left(x,\frac{\alpha^{n+1}t}{|k|^n}\right)=0\right\} \quad (3.1.20)$$

Hence $C(x)=C'(x)$ for all $x\in X$.

Corollary 3.2:

Let X be a Linear Space ($\mathcal{LS}$) over $\mathcal{NA}$ field $\mathbb{K}$ and let$(Y,\|\cdot\|)$ be a $\mathcal{NA}-\mathcal{NS}$. Suppose that a function $\varphi: X\times X\to\mathbb{R}^+$ satisfies

$$\varphi(k^{-1}x,k^{-1}y)\leq \alpha^{-1}\varphi(x,y), \quad (3.2.1)$$

for all $x,y\in X$, where $\alpha>0$ and k is an integer with $|k|<\alpha$. If a map $f: X\to Y$ satisfies

$$\|f(x+y)-f(x)-f(y)\|\leq \varphi(x,y), \quad (3.2.2)$$

for all $x,y\in X$, then there exists a unique additive mapping $C: X\to Y$ satisfies

$$\|f(x)-C(x)\|\leq \frac{1}{\alpha}\max\{\varphi(x,x)*\varphi(x,2x)*\dots*\varphi(x,(k-1)x)\}. \quad (3.2.3)$$

Proof:

Consider the $\mathcal{NA}-\mathcal{NN}$

$$\left\{\begin{array}{c} \mu(y,t)=\left\{\begin{array}{cc}\frac{t}{t+\|y\|} & \text{if } t>0 \\ 0 & \text{if } t\leq 0\end{array}\right\}, \vartheta(y,t)=\left\{\begin{array}{cc}\frac{\|y\|}{t+\|y\|} & \text{if } t>0 \\ 1 & \text{if } t\leq 0\end{array}\right\} \text{ and} \\ \omega(y,t)=\left\{\begin{array}{cc}\frac{\|y\|}{t} & \text{if } t>0 \\ 1 & \text{if } t\leq 0\end{array}\right\}, \end{array}\right\} \quad (3.2.4)$$

on Y. Let $Z=\mathbb{R}$ and let the function $\mu',\vartheta',\omega': \mathbb{R}\times\mathbb{R}\to[0,1]$ be defined by

$$\left\{\begin{array}{c} \mu'(z,t)=\left\{\begin{array}{cc}\frac{t}{t+|z|} & \text{if } t>0 \\ 0 & \text{if } t\leq 0\end{array}\right\}, \vartheta'(z,t)=\left\{\begin{array}{cc}\frac{|z|}{t+|z|} & \text{if } t>0 \\ 1 & \text{if } t\leq 0\end{array}\right\} \text{ and} \\ \omega'(z,t)=\left\{\begin{array}{cc}\frac{|z|}{t} & \text{if } t>0 \\ 1 & \text{if } t\leq 0\end{array}\right\}. \end{array}\right\} \quad (3.2.5)$$

Then $(\mu',\vartheta',\omega')$ is a $\mathcal{NA}-\mathcal{NN}$ on $\mathbb{R}$. The result follows from the fact that (3.2.1), (3.2.2) and (3.2.3) are equivalent to (3.1.1), (3.1.2) and (3.1.3) respectively.

Example 3.3:

Let X be a $\mathcal{LS}$over $\mathcal{NA}$ field $\mathbb{K}$ and let $(Y,\|\cdot\|)$ be a $\mathcal{NA}-\mathcal{NS}$. Suppose that a function $f: X\to Y$ satisfies $\|f(x+y)-f(x)-f(y)\|\leq\|x\|^p+\|y\|^p$, (3.3.1)

for all $x,y\in X$ and $p\in[0,1]$. Suppose that there exists an integer k such that $|k<1|$. Since $p<1$, by applying Corollary (3.2) for $\varphi(x,y)=\|x\|^p+\|y\|^p$, we observe that (3.2.1) holds for $\alpha=|k|^p$. Inequality (3.2.3) assures the existence of a unique additive mapping $C: X\to Y$ such that

$$\|f(x)-C(x)\|\leq\frac{1+(k-1)^p}{|k|^p}\|x\|^p, \quad \text{for all } x\in X. \quad (3.3.2)$$

4. Stability of Pexiderized Cauchy Functional Equation

The functional equation $f(x+y)=g(x)+h(y)$ is said to be Pexiderized Cauchy, where f,g and h are mapping between $\mathcal{LS}$s. In the case $f=g=h$, it is called Cauchy functional equation.

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Theorem 4.1:

Let X be a $\mathcal{LS}$ over a $\mathcal{NA}$ field $\mathbb{K}$ and let (Y,μ,ϑ,ω) be a $\mathcal{NA}-\mathcal{NBS}$. Suppose that f,g and h are mappings from X to Y with $f(0)=g(0)=h(0)=0$. Suppose that φ is a function from $X\times X$ to a $\mathcal{NA}-\mathcal{NNS}$ $(Z,\mu',\vartheta',\omega')$ such that

$$\begin{Bmatrix} \mu(f(x+y)-g(x)-h(y),t)\geq\mu'(\varphi(x,y),t), \\ \vartheta(f(x+y)-g(x)-h(y),t)\leq\vartheta'(\varphi(x,y),t)\text{and} \\ \omega(f(x+y)-g(x)-h(y),t)\leq\omega'(\varphi(x,y),t), \end{Bmatrix} \quad (4.1.1)$$

for all $x,y\in X$ and $t>0$. If

$$\begin{Bmatrix} \mu'(\varphi(k^{-1}x,k^{-1}y),t)\geq\mu'(\varphi(x,y),\alpha t),\ \vartheta'(\varphi(k^{-1}x,k^{-1}y),t)\leq\vartheta'(\varphi(x,y),\alpha t) \\ \text{and } \omega'(\varphi(k^{-1}x,k^{-1}y),t)\leq\omega'(\varphi(x,y),\alpha t), \end{Bmatrix} (4.1.2)$$

for some positive real number $\alpha>0$ and some positive integer k with $|k|<\alpha$, then there exists a unique additive mapping $C:X\to Y$ such that

$$\begin{Bmatrix} \mu(f(x)-C(x),t)\geq M(x,\alpha t),\vartheta(f(x)-C(x),t)\leq N(x,\alpha t) \\ \text{and } \omega(f(x)-C(x),t)\leq P(x,\alpha t), \end{Bmatrix} \quad (4.1.3)$$

$$\begin{Bmatrix} \mu(g(x)-C(x),t)\geq M(x,\min\{1,\alpha\}t),\vartheta(g(x)-C(x),t)\leq N(x,\min\{1,\alpha\}t) \\ \text{and } \omega(g(x)-C(x),t)\leq P(x,\min\{1,\alpha\}t), \end{Bmatrix} \quad (4.1.4)$$

$$\begin{Bmatrix} \mu(h(x)-C(x),t)\geq M(x,\min\{1,\alpha\}t),\vartheta(h(x)-C(x),t)\leq N(x,\min\{1,\alpha\}t) \\ \text{and } \omega(h(x)-C(x),t)\leq P(x,\min\{1,\alpha\}t), \end{Bmatrix} \quad (4.1.5)$$

for all $x\in X$ and $t>0$, where

$$\begin{Bmatrix} M(x,t)=[\mu'(\varphi(x,x),t)*\ldots*\mu'(\varphi(x,(k-1)x),t)*\mu'(\varphi(0,x),t)*\ldots \\ *\mu'(\varphi(0,(k-1)x),t)*\mu'(\varphi(x,0),t)*\ldots*\mu'\big(\varphi\big((k-1)x,0\big),t\big)], \\ N(x,t)=[\vartheta'(\varphi(x,x),t)\diamondsuit\ldots\diamondsuit\vartheta'(\varphi(x,(k-1)x),t)\diamondsuit\vartheta'(\varphi(0,x),t)\diamondsuit\ldots \\ \diamondsuit\vartheta'(\varphi(0,(k-1)x),t)\diamondsuit\vartheta'(\varphi(x,0),t)\diamondsuit\ldots\diamondsuit\vartheta'\big(\varphi\big((k-1)x,0\big),t\big)]\text{ and} \\ P(x,t)=[\omega'(\varphi(x,x),t)\odot\ldots\odot\omega'(\varphi(x,(k-1)x),t)\odot\omega'(\varphi(0,x),t)\odot\ldots \\ \odot\omega'(\varphi(0,(k-1)x),t)\odot\omega'(\varphi(x,0),t)\odot\ldots\odot\omega'\big(\varphi\big((k-1)x,0\big),t\big)]. \end{Bmatrix} \quad (4.1.6)$$

Proof:

Put $y=0$ in (4.1.1). Then, for all $x\in X$ and $t>0$

$$\begin{Bmatrix} \mu(f(x)-g(x),t)\geq\mu'(\varphi(x,0),t), \\ \vartheta(f(x)-g(x),t)\leq\vartheta'(\varphi(x,0),t)\text{and} \\ \omega(f(x)-g(x),t)\leq\omega'(\varphi(x,0),t). \end{Bmatrix}. \quad (4.1.7)$$

For $x=0$, (4.1.1) becomes

$$\begin{Bmatrix} \mu(f(y)-h(y),t)\geq\mu'(\varphi(0,y),t), \\ \vartheta(f(y)-h(y),t)\leq\vartheta'(\varphi(0,y),t)\text{and} \\ \omega(f(y)-h(y),t)\leq\omega'(\varphi(0,y),t), \end{Bmatrix}, \quad (4.1.8)$$

for all $y\in X$ and $t>0$. Combining (4.1.1), (4.1.7) and (4.1.8), we obtain

$$\begin{Bmatrix} \mu(f(x+y)-f(x)-f(y),t)\geq\mu'(\varphi(x,y),t)*\mu'(\varphi(x,0),t)*\mu'(\varphi(0,y),t), \\ \vartheta(f(x+y)-f(x)-f(y),t)\leq\vartheta'(\varphi(x,y),t)\diamondsuit\vartheta'(\varphi(x,0),t)\diamondsuit\vartheta'(\varphi(0,y),t)\text{ and} \\ \omega(f(x+y)-f(x)-f(y),t)\leq\omega'(\varphi(x,y),t)\odot\omega'(\varphi(x,0),t)\odot\omega'(\varphi(0,y),t) \end{Bmatrix}, (4.1.9)$$

for each $x,y\in X$ and $t>0$. Replacing $\mu'(\varphi(x,y),t),\vartheta'(\varphi(x,y),t)$ and $\omega'(\varphi(x,y),t)$ by $\mu'(\varphi(x,y),t)*\mu'(\varphi(x,0),t)*\mu'(\varphi(0,y),t)$, $\vartheta'(\varphi(x,y),t)\diamondsuit\vartheta'(\varphi(x,0),t)\diamondsuit\vartheta'(\varphi(0,y),t)$ and $\omega'(\varphi(x,y),t)\odot\omega'(\varphi(x,0),t)\odot\omega'(\varphi(0,y),t)$ respectively, in Theorem (3.1), we can find that there exists an unique additive mapping $C:X\to Y$ that satisfies (4.1.3). From (4.1.3) and (4.1.7), we see that

$$\begin{Bmatrix} \mu(g(x)-T(x),t)\geq\mu(g(x)-f(x),t)*\mu(f(x)-T(x),t)\geq M(x,t), \\ \vartheta(g(x)-T(x),t)\leq\vartheta(g(x)-f(x),t)\diamondsuit\vartheta(f(x)-T(x),t)\leq N(x,t)\text{ and} \\ \omega(g(x)-T(x),t)\leq\omega(g(x)-f(x),t)\odot\omega(f(x)-T(x),t)\leq P(x,t), \end{Bmatrix} \quad (4.1.10)$$

for all $x,y\in X$ and $t>0$, which proves (4.1.4). Similarly, we can prove (4.1.5).

Corollary 4.2:

Let X be a $\mathcal{LS}$ over a $\mathcal{NA}$ field $\mathbb{K}$ and let $(Z,\mu',\vartheta',\omega')$ be a $\mathcal{NA}-\mathcal{NNS}$. Let (Y,μ,ϑ,ω) be a $\mathcal{NA}-\mathcal{NBS}$. Suppose that f,g and h are functions from X to Y such that $f(0)=g(0)=h(0)=0$ and there is an integer k with $|k|<1$ and satisfies

$$\begin{Bmatrix} \mu(f(x+y)-g(x)-h(y),t)\geq\mu'(\|x\|^r\|y\|^s z_0,t), \\ \vartheta(f(x+y)-g(x)-h(y),t)\leq\vartheta'(\|x\|^r\|y\|^s z_0,t)\text{and} \\ \omega(f(x+y)-g(x)-h(y),t)\leq\omega'(\|x\|^r\|y\|^s z_0,t), \end{Bmatrix} \quad (4.2.1)$$

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for all $x, y \in X, t > 0$ and for some fixed $z_0 \in Z$ and $r, s \geq 0$ with $r + s < 1$. Then there exists a unique additive mapping $T: X \to Y$ such that

$$\left\{\begin{array}{c} \mu(f(x) - T(x), t) \geq \mu'((k-1)^s \|x\|^{r+s} z_0, |k|^{r+s} t), \\ \vartheta(f(x) - T(x), t) \leq \vartheta'((k-1)^s \|x\|^{r+s} z_0, |k|^{r+s} t), \\ \omega(f(x) - T(x), t) \leq \omega'((k-1)^s \|x\|^{r+s} z_0, |k|^{r+s} t), \\ \mu(g(x) - T(x), t) \geq \mu'((k-1)^s \|x\|^{r+s} z_0, |k|^{r+s} t), \\ \vartheta(g(x) - T(x), t) \leq \vartheta'((k-1)^s \|x\|^{r+s} z_0, |k|^{r+s} t), \\ \omega(g(x) - T(x), t) \leq \omega'((k-1)^s \|x\|^{r+s} z_0, |k|^{r+s} t), \\ \mu(h(x) - T(x), t) \geq \mu'((k-1)^s \|x\|^{r+s} z_0, |k|^{r+s} t), \\ \vartheta(h(x) - T(x), t) \leq \vartheta'((k-1)^s \|x\|^{r+s} z_0, |k|^{r+s} t) \text{ and} \\ \omega(h(x) - T(x), t) \leq \omega'((k-1)^s \|x\|^{r+s} z_0, |k|^{r+s} t), \end{array}\right\} \quad (4.2.2)$$

for all $x \in X$ and $t > 0$.

Proof:

Let the function $\varphi: X \times X \to Z$ be defined by $\varphi(x, y) = \|x\|^r \|y\|^s z_0$ for all $x, y \in X$ and z_0 is a fixed unit vector in Z. Then (4.1.1) holds. Since

$$\left\{\begin{array}{c} \mu'(\varphi(k^{-1}x, k^{-1}y), t) = \mu'\big((\|k^{-1}x\|^r \|k^{-1}y\|^s) z_0, t\big) = \mu'((\|x\|^r \|y\|^s) z_0, |k|^{r+s} t), \\ \vartheta'(\varphi(k^{-1}x, k^{-1}y), t) = \vartheta'\big((\|k^{-1}x\|^r \|k^{-1}y\|^s) z_0, t\big) = \vartheta'((\|x\|^r \|y\|^s) z_0, |k|^{r+s} t) \text{and} \\ \omega'(\varphi(k^{-1}x, k^{-1}y), t) = \omega'\big((\|k^{-1}x\|^r \|k^{-1}y\|^s) z_0, t\big) = \omega'((\|x\|^r \|y\|^s) z_0, |k|^{r+s} t), \end{array}\right\} \quad (4.2.3)$$

for each $x, y \in X$ and $t > 0$. If $\alpha = |k|^{r+s}$ and $r + s < 1$, then $\alpha > |k|$ holds. It follows from Theorem (4.1.1) that there exists an unique additive mapping $C: X \to Y$ such that (4.1.3) − (4.1.5) hold.

5. Stability of Jensen Functional Equation

The stability problem for the Jensen functional equation was first proved by Kominek [8] and since then several generalizations and applications of this notion have been investigated by various authors, namely, Jung [7], Mohiuddine [12], Parnami and Vasudeva [16] and many others. The Jensen functional equation is $2f((x + y)/2) = f(x) + f(y)$, where f is a mapping between LSs. It is easy to see that a mapping $f: X \to Y$ between $\mathcal{LS}$ with $f(0) = 0$ satisfies the Jensen equation if and only if it is additive [16].

Theorem 5.1:

Let X be a $\mathcal{LS}$ over a $\mathcal{NA}$ field $\mathbb{K}$ and let $(Z, \mu', \vartheta', \omega')$ be a $\mathcal{NA} - \mathcal{NNS}$. Suppose that $\varphi: X \times X \to Z$ is a function such that for some $\alpha > 0$ and some positive integer k with $|k| < \alpha$ satisfies (3.1.1). Suppose that $(Y, \mu, \vartheta, \omega)$ be a $\mathcal{NA} - \mathcal{NBS}$. If a map $f: X \to Y$ satisfies

$$\left\{\begin{array}{c} \mu\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right) \geq \mu'(\varphi(x, y), t), \\ \vartheta\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right) \leq \vartheta'(\varphi(x, y), t) \text{ and} \\ \omega\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right) \leq \omega'(\varphi(x, y), t), \end{array}\right\} \quad (5.1.1)$$

for all $x, y \in X$ and $t > 0$, then there exists an unique additive mapping $C: X \to Y$ such that

$$\left\{\begin{array}{c} \mu(f(x) - f(0) - C(x), t) \geq M(x, \alpha t), \vartheta(f(x) - f(0) - C(x), t) \leq N(x, \alpha t) \\ \text{and } \omega(f(x) - f(0) - C(x), t) \leq P(x, \alpha t) \end{array}\right\} \quad (5.1.2)$$

for all $x \in X$ and $t > 0$, where

$$\left.\begin{array}{l} M(x, t) = \mu'(\varphi(x, x), t) * \mu'(\varphi(x, 2x), t) * \ldots * \mu'(\varphi(x, (k-1)x), t) \\ \quad * \mu'(\varphi(2x, 0), t) * \mu'(\varphi(3x, 0), t) * \ldots * \mu'(\varphi(kx, 0), t), \end{array}\right\} \quad (5.1.3)$$

$$\left.\begin{array}{l} N(x, t) = \vartheta'(\varphi(x, x), t) \diamond \vartheta'(\varphi(x, 2x), t) \diamond \ldots \diamond \vartheta'(\varphi(x, (k-1)x), t) \\ \quad \diamond \vartheta'(\varphi(2x, 0), t) \diamond \vartheta'(\varphi(3x, 0), t) \diamond \ldots \diamond \vartheta'(\varphi(kx, 0), t) \text{ and} \end{array}\right\} \quad (5.1.4)$$

$$\left.\begin{array}{l} P(x, t) = \omega'(\varphi(x, x), t) \odot \omega'(\varphi(x, 2x), t) \odot \ldots \odot \omega'(\varphi(x, (k-1)x), t) \\ \quad \odot \omega'(\varphi(2x, 0), t) \odot \omega'(\varphi(3x, 0), t) \odot \ldots \odot \omega'(\varphi(kx, 0), t). \end{array}\right\} \quad (5.1.5)$$

Proof:

Suppose that $g(x) = f(x) - f(0)$ for all $x \in X$. Then

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$$\left\{\begin{array}{c}\mu\left(2g\left(\frac{x+y}{2}\right)-g(x)-g(y),t\right)\geq\mu'(\varphi(x,y),t),\\ \vartheta\left(2g\left(\frac{x+y}{2}\right)-g(x)-g(y),t\right)\leq\vartheta'(\varphi(x,y),t)\text{ and}\\ \omega\left(2g\left(\frac{x+y}{2}\right)-g(x)-g(y),t\right)\leq\omega'(\varphi(x,y),t),\end{array}\right\} \tag{5.1.6}$$

for all $x,y\in X$ and $t>0$. Replacing x by $x+y$ and y by 0 in (5.1.6), then, for all $x,y\in X$ and $t>0$, we have

$$\left\{\begin{array}{c}\mu\left(2g\left(\frac{x+y}{2}\right)-g(x+y),t\right)\geq\mu'(\varphi(x+y,0),t),\\ \vartheta\left(2g\left(\frac{x+y}{2}\right)-g(x+y),t\right)\leq\vartheta'(\varphi(x+y,0),t)\text{and}\\ \omega\left(2g\left(\frac{x+y}{2}\right)-g(x+y),t\right)\leq\omega'(\varphi(x+y,0),t).\end{array}\right\} \tag{5.1.7}$$

From (5.1.6) and (5.1.7), we conclude that that

$$\left\{\begin{array}{c}\mu(g(x+y)-g(x)-g(y),t)\geq\mu'(\varphi(x,y),t)*\mu'(\varphi(x+y,0),t)\\ \vartheta(g(x+y)-g(x)-g(y),t)\leq\vartheta'(\varphi(x,y),t)\diamond\vartheta'(\varphi(x+y,0),t)\text{ and}\\ \omega(g(x+y)-g(x)-g(y),t)\leq\omega'(\varphi(x,y),t)\odot\omega'(\varphi(x+y,0),t),\end{array}\right\} \tag{5.1.8}$$

for all $x,y\in X$ and $t>0$. Proceeding the same lines as in the proof of Theorem(3.1), one can show that there exists an unique additive mapping $C:X\to Y$ such that

$$\left\{\begin{array}{c}\mu(f(x)-f(0)-C(x),t)=\mu(g(x)-T(x),\alpha t)\geq M(x,t),\\ \vartheta(f(x)-f(0)-C(x),t)=\vartheta(g(x)-T(x),\alpha t)\leq N(x,t)\text{ and}\\ \omega(f(x)-f(0)-C(x),t)=\omega(g(x)-T(x),\alpha t)\leq P(x,t),\end{array}\right\} \tag{5.1.9}$$

for all $x\in X$ and $t>0$.

6. Stability of Pexiderized Jensen Functional Equation

The functional equation $2f((x+y)/2)=g(x)+h(y)$ is said to be Pexiderized Jensen, where f,g and h are mappings between $\mathcal{LS}$s. In this case $f=g=h$, it is called Jensen Functional Equation.

Theorem 6.1:

Let X be a $\mathcal{LS}$ over a $\mathcal{NA}$ field $\mathbb{K}$ and let (Y,μ,ϑ,ω) be a $\mathcal{NA}-\mathcal{NBS}$. Suppose that f,g and h are mappings from X to Y with $f(0)=g(0)=h(0)=0$. Let $(Z,\mu',\vartheta',\omega')$ be $\mathcal{NA}-\mathcal{NNS}$. Suppose that $\varphi:X\times X\to Z$ is a function such that for some $\alpha>0$, and some positive integer k with $|k|<\alpha$ satisfies (3.1.1) and inequality

$$\left\{\begin{array}{c}\mu\left(2f\left(\frac{x+y}{2}\right)-g(x)-h(y),t\right)\geq\mu'(\varphi(x,y),t),\\ \vartheta\left(2f\left(\frac{x+y}{2}\right)-g(x)-h(y),t\right)\leq\vartheta'(\varphi(x,y),t)\text{ and}\\ \omega\left(2f\left(\frac{x+y}{2}\right)-g(x)-h(y),t\right)\leq\omega'(\varphi(x,y),t),\end{array}\right\} \tag{6.1.1}$$

for all $x,y\in X$ and $t>0$. Then there exists an unique additive mapping $C:X\to Y$ such that

$$\left\{\begin{array}{c}\mu(f(x)-C(x),t)\geq M(x,\alpha t),\vartheta(f(x)-C(x),t)\leq N(x,\alpha t)\\ \text{and }\omega(f(x)-C(x),t)\leq P(x,\alpha t),\end{array}\right\} \tag{6.1.2}$$

$$\left\{\begin{array}{c}\mu(g(x)-C(x),t)\geq M\left(\frac{x}{2},\frac{\alpha t}{2}\right)*\mu'(\varphi(x,0),t),\\ \vartheta(g(x)-C(x),t)\leq N\left(\frac{x}{2},\frac{\alpha t}{2}\right)\diamond\vartheta'(\varphi(x,0),t)\text{ and}\\ \omega(g(x)-C(x),t)\leq P\left(\frac{x}{2},\frac{\alpha t}{2}\right)\odot\omega'(\varphi(x,0),t),\end{array}\right\} \tag{6.1.3}$$

$$\left\{\begin{array}{c}\mu(h(x)-C(x),t)\geq M\left(\frac{x}{2},\frac{\alpha t}{2}\right)*\mu'(\varphi(0,x),t),\\ \vartheta(h(x)-C(x),t)\leq N\left(\frac{x}{2},\frac{\alpha t}{2}\right)\diamond\vartheta'(\varphi(0,x),t)\text{ and}\\ \omega(h(x)-C(x),t)\leq P\left(\frac{x}{2},\frac{\alpha t}{2}\right)\odot\omega'(\varphi(0,x),t)\end{array}\right\} \tag{6.1.4}$$

for all $x\in X$ and $t>0$, where

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$$\left\{\begin{array}{c} M(x,t)=\prod_{m=1}^{k-1}\{\mu'(\varphi(x,mx),|2|t)*\mu'(\varphi(mx,mx),|2|t)\} \\ *\prod_{m=0}^{k}\{\mu'(\varphi(mx,0),|2|t)*\mu'(\varphi(0,mx),|2|t)\}, \\ N(x,t)=\coprod_{m=1}^{k-1}\{\vartheta'(\varphi(x,mx),|2|t)\diamond\vartheta'(\varphi(mx,mx),|2|t)\} \\ \diamond\coprod_{m=0}^{k}\{\vartheta'(\varphi(mx,0),|2|t)\diamond\vartheta'(\varphi(0,mx),|2|t)\}\text{ and} \\ P(x,t)=\coprod_{m=1}^{k-1}\{\omega'(\varphi(x,mx),|2|t)\odot\omega'(\varphi(mx,mx),|2|t)\} \\ \odot\coprod_{m=0}^{k}\{\omega'(\varphi(mx,0),|2|t)\odot\omega'(\varphi(0,mx),|2|t)\}. \end{array}\right\} \quad (6.1.5)$$

Proof:

Put $y=x$ in (6.1.1). Then for all $x\in X$ and $t>0$

$$\left\{\begin{array}{c} \mu(2f(x)-g(x)-h(x),t)\geq\mu'(\varphi(x,x),t), \\ \vartheta(2f(x)-g(x)-h(x),t)\leq\vartheta'(\varphi(x,x),t)\text{and} \\ \omega(2f(x)-g(x)-h(x),t)\leq\omega'(\varphi(x,x),t). \end{array}\right\} \quad (6.1.6)$$

Replacing x by y in (6.1.1), we get

$$\left\{\begin{array}{c} \mu(2f(y)-g(y)-h(y),t)\geq\mu'(\varphi(y,y),t), \\ \vartheta(2f(y)-g(y)-h(y),t)\leq\vartheta'(\varphi(y,y),t)\text{and} \\ \omega(2f(y)-g(y)-h(y),t)\leq\omega'(\varphi(y,y),t), \end{array}\right\} \quad (6.1.7)$$

for all $y\in X$ and $t>0$. Again replacing x by y as well as y by x in (6.1.1), we get

$$\left\{\begin{array}{c} \mu\left(2f\left(\frac{x+y}{2}\right)-g(y)-h(x),t\right)\geq\mu'(\varphi(y,x),t), \\ \vartheta\left(2f\left(\frac{x+y}{2}\right)-g(y)-h(x),t\right)\leq\vartheta'(\varphi(y,x),t)\text{ and} \\ \omega\left(2f\left(\frac{x+y}{2}\right)-g(y)-h(x),t\right)\leq\omega'(\varphi(y,x),t), \end{array}\right\} \quad (6.1.8)$$

for all $x,y\in X$ and $t>0$. It follows from (6.1.1) and (6.1.6), (6.1.7), (6.1.8) that

$$\left\{\begin{array}{c} \mu\left(4f\left(\frac{x+y}{2}\right)-2f(x)-2f(y),t\right) \\ \geq\mu'(\varphi(x,x),t)*\mu'(\varphi(x,y),t)*\mu'(\varphi(y,y),t)*\mu'(\varphi(y,x),t), \\ \vartheta\left(4f\left(\frac{x+y}{2}\right)-2f(x)-2f(y),t\right) \\ \leq\vartheta'(\varphi(x,x),t)\diamond\vartheta'(\varphi(x,y),t)\diamond\vartheta'(\varphi(y,y),t)\diamond\vartheta'(\varphi(y,x),t)\text{ and} \\ \omega\left(4f\left(\frac{x+y}{2}\right)-2f(x)-2f(y),t\right) \\ \leq\omega'(\varphi(x,x),t)\odot\omega'(\varphi(x,y),t)\odot\omega'(\varphi(y,y),t)\odot\omega'(\varphi(y,x),t). \end{array}\right\} \quad (6.1.9)$$

Thus, for all $x,y\in X$ and $t>0$,

$$\left\{\begin{array}{c} \mu\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y),t\right) \\ \geq\mu'(\varphi(x,x),|2|t)*\mu'(\varphi(x,y),|2|t)*\mu'(\varphi(y,y),|2|t)*\mu'(\varphi(y,x),|2|t), \\ \vartheta\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y),t\right) \\ \leq\vartheta'(\varphi(x,x),|2|t)\diamond\vartheta'(\varphi(x,y),|2|t)\diamond\vartheta'(\varphi(y,y),|2|t)\diamond\vartheta'(\varphi(y,x),|2|t)\text{ and} \\ \omega\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y),t\right) \\ \leq\omega'(\varphi(x,x),|2|t)\odot\omega'(\varphi(x,y),|2|t)\odot\omega'(\varphi(y,y),|2|t)\odot\omega'(\varphi(y,x),|2|t). \end{array}\right\} \quad (6.1.10)$$

Proceeding the same argument used in Theorem (5.1) shows that there exists an unique additive mapping $C: X\to Y$ such that (6.1.2) holds. Therefore

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$$\left\{\begin{matrix}\mu\left(2f\left(\frac{x}{2}\right)-C(x),t\right)\geq M\left(\frac{x}{2},\frac{\alpha t}{2}\right),\vartheta\left(2f\left(\frac{x}{2}\right)-C(x),t\right)\leq N\left(\frac{x}{2},\frac{\alpha t}{2}\right)\\ \text{and } \omega\left(2f\left(\frac{x}{2}\right)-C(x),t\right)\leq P\left(\frac{x}{2},\frac{\alpha t}{2}\right),\end{matrix}\right\} \quad (6.1.11)$$

for all $x \in X$ and $t > 0$. Put $y = 0$ in (6.1.1), we get

$$\left\{\begin{matrix}\mu\left(2f\left(\frac{x}{2}\right)-g(x),t\right)\geq \mu'(\varphi(x,0),t),\vartheta\left(2f\left(\frac{x}{2}\right)-g(x),t\right)\leq \vartheta'(\varphi(x,0),t)\\ \text{and } \omega\left(2f\left(\frac{x}{2}\right)-g(x),t\right)\leq \omega'(\varphi(x,0),t),\end{matrix}\right\} (6.1.12)$$

for all $x \in X$ and $t > 0$. It follows from (6.1.11) and (6.1.12) that (6.1.3) holds.
Similarly we can show that (6.1.4) holds.

Corollary 6.2:

Let X be a $\mathcal{NA}-\mathcal{NS}$. Suppose that $f,g,h : X \to Y$ such that $f(0) = g(0) = h(0) = 0$ and there is an integer k with $|k| < 1$ and satisfies

$$\left\|2f\left(\frac{x+y}{2}\right)-g(x)-h(y)\right\|\leq \varepsilon, \quad (6.2.1)$$

for all $x, y \in X$. Then there exists an unique additive mapping $C: X \to Y$ such that

$$\{\|f(x)-C(x)\|\leq\varepsilon, \|f(x)-C(x)\|\leq\varepsilon \text{ and } \|f(x)-C(x)\|\leq\varepsilon\}, \quad (6.2.2)$$

for all $x \in X$.

Proof:

Let the function $\mu, \vartheta, \omega : Y \times \mathbb{R} \to [0,1]$ be defined by

$$\left\{\mu(x,t)=\left\{\begin{matrix}\frac{t}{t+\|x\|} & \text{if } t>0\\ 0 & \text{if } t\leq 0\end{matrix}\right\},\vartheta(x,t)=\left\{\begin{matrix}\frac{\|x\|}{t+\|x\|} & \text{if } t>0\\ 1 & \text{if } t\leq 0\end{matrix}\right\} and\ \ \omega(x,t)=\left\{\begin{matrix}\frac{\|x\|}{t} & \text{if } t>0\\ 1 & \text{if } t\leq 0\end{matrix}\right\},\right\} (6.2.3)$$

on Y. It is easy to see that $(Y, \mu, \vartheta, \omega)$ is a $\mathcal{NA}-\mathcal{NBS}$. Consider the $\mathcal{NA}-\mathcal{NN}$.

$$\left\{\mu'(z,t)=\left\{\begin{matrix}\frac{t}{t+|z|} & \text{if } t>0\\ 0 & \text{if } t\leq 0\end{matrix}\right\},\vartheta'(z,t)=\left\{\begin{matrix}\frac{|z|}{t+|z|} & \text{if } t>0\\ 1 & \text{if } t\leq 0\end{matrix}\right\} \text{ and } \omega'(z,t)=\left\{\begin{matrix}\frac{|z|}{t} & \text{if } t>0\\ 1 & \text{if } t\leq 0\end{matrix}\right\}.\right\} (6.2.4)$$

Then $(\mu', \vartheta', \omega')$ is a $\mathcal{NA}-\mathcal{NN}$ on $\mathbb{R}$. It is easy to see that (4.1.1) holds for $\varphi(x,y) = \varepsilon$ and $\alpha = 1$ satisfies (3.1.1). Therefore the condition of Theorem (6.1) is fulfilled. Hence there exists an unique additive mapping $C: X \to Y$ such that (6.2.2) holds.

5. Conclusion

This paper discussed the Hyers - Ulam - Rassias Stability of Various Functional Equations in Non-Archimedean Neutrosophic Normed Spaces. This work indeed presents a relationship between four various disciplines, the theory of neutrosophic normed space, non – Archimedean, Hyers-Ulam-Rassias stability and functional equation.

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