A General Study of Fuzzy Co-Pre-Hilbert Spaces

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Abstract
This paper introduces the concepts fuzzy pre-Hilbert spaces and fuzzy co-pre-Hilbert spaces and proves some theorems in this subject. Also, it illustrates many examples to clarify the validity of the new concepts.

Keywords: Fuzzy Co-Pre-Hilbert Spaces; Fuzzy Set; Neutrosophic Set

1. Introduction

The notion of fuzzy pre-Hilbert spaces can be considered as the generalization of that notion of pre-Hilbert spaces. The definition of these spaces has been introduced in [3, 4]. The study mentions the definition of fuzzy pre-Hilbert space and then it introduces a definition for fuzzy co-pre Hilbert space and discusses some properties on this subject.

2. Preliminaries

Definition(2.1):[1] A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a t- norm if $*$ is satisfies the following conditions:

(i) $*$ Is commutative and associative;
(ii) $a * 1 = a \text{ for all } a \in [0,1]$;
(iii) $a * b \le c * d \text{ whenever } a \le c$ and $a \le d$ and $a, b, c, d \in [0,1]$.

Definition(2.2):[2,5] A binary operation $\circ: [0,1] \times [0,1] \to [0,1]$ is a t-conorm if $\circ$ is satisfies the following conditions:

(i) $\circ$ Is commutative and associative;
(ii) $a \circ 0 = a \text{ for all } a \in [0,1]$;
(iii) $a \circ b \le c \circ d \text{ whenever } a \le c$ and $a \le d$ and $a, b, c, d \in [0,1]$.

Definition (2.3): [4] Let X be a real vector space, $*$ be a continuous t-norm on $I = [0,1]$. A function $F: : X \times X \times R \to [0,1]$ is called a fuzzy pre-Hilbert function if satisfies the following axioms for every and $x, y, z \in X$ and $s, t, r \in R$

(1) $F(x, x, 0) = 0$ and $F(x, x, t) > 0$ for all $t > 0$;
(2) $F(x, x, t) \ne H(t)$ for some $t \in R \Leftrightarrow x \ne 0$;
(3) $F(x, y, t) = F(y, x, t)$;
(4) For any real number a

$$F(a, x, y, t) = \begin{cases} F(x, y, \frac{t}{a}) & , a > 0 \\ H(t) & , a = 0 \\ 1 - F(x, y, \frac{t}{-a}) & , a < 0 \end{cases}$$

(5) $F(x, x, t) * F(y, y, s) \le F(x + y, x + y, t + s)$;
(6) $sup_{s+r=t}(F(x, z, s) * F(y, z, r)) = F(x + y, z, t)$;
(7) $F(x, y, .): R \to [0,1]$ is continuous on $R/\{0\}$;
(8) $\lim_{t \to \infty} F(x, y, t) = 1$.

$(X, F, *)$ is a fuzzy pre-Hilbert space.

Note: $H(t) = \begin{cases} 1 & , t > 0 \\ 0 & , t \le 0 \end{cases}$

Lemma (2.4): [3] If $(X, F, *)$ is a fuzzy pre-Hilbert space then $F(X, F, t)$ is non-decreasing with respect to t, for each $x, y \in X$.

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Proof: Let $t < s$ for all $s, t \in R$, $F(x,y,t) = F(x,y,t) * 1 = F(x,y,t) * F(0,0,s-t) \le F(x,y,s)$.
Lemma (2.5): If $(X, F, *)$ is a fuzzy pre-Hilbert space then:
(1) $F(ax, ay, t) = F(x, y, \frac{t}{a^2})$for all $a \in R/\{0\}$.
(2) $F(x-y, x-y, t) = F(y-x, y-x, t)$
Proof: (1) and **(2)** are directly from axiom (4).
Theorem (2.6): [3] (Schwartz inequality) If $(X, F, *)$ is a fuzzy pre-Hilbert space then: $F(x,y,ts) \ge F(x,x,t^2) * F(y,y,s^2)$

3. Main results

Definition (3.1): Let X be a real vector space, ° be a continuous t-conorm on $I = [0,1]$. A function $\mathcal{F}: : X \times X \times R \to [0,1]$ is called a fuzzy co-pre-Hilbert function if satisfies the following axioms for every $x, y, z \in X$ and s, t, r ∈ R:
(1) $\mathcal{F}(x,x,0) = 1$ and $\mathcal{F}(x,x,t) < 1$ for all $t > 0$;
(2) $\mathcal{F}(x,x,t) \ne \mathcal{H}(t)$ for some $t \in R \Leftrightarrow x \ne 0$.
(3) $\mathcal{F}(x,y,t) = \mathcal{F}(y,x,t)$;
(4) For any real number a

$$F(a,x,y,t) = \begin{cases} \mathcal{F}(x,y,\frac{t}{a}) & ,a > 0 \\ \mathcal{H}(t) & ,a = 0 \\ 1 - \mathcal{F}(x,y,\frac{t}{-a}) & ,a < 0 \end{cases}$$

(5) $\mathcal{F}(x,x,t)°\,\mathcal{F}(y,y,s) \le \mathcal{F}(x+y, x+y, t+s)$
(6) $sup_{s+r=t}(\mathcal{F}(x,z,s)°\,\mathcal{F}(y,z,r)) = \mathcal{F}(x+y,z,t)$;
(7) $\mathcal{F}(x,y,.): R \to [0,1]$ is continuous on $R/\{0\}$;
(8) $\lim\limits_{t\to\infty} \mathcal{F}(x,y,t) = 0$.
$(X, \mathcal{F}, °)$ is a fuzzy pre-Hilbert space.
Note: $\mathcal{H}(t) = \begin{cases} 1 & ,t \le 0 \\ 0 & ,t > 0 \end{cases} = 1 - H(t)$
Lemma (3.2): If $(X, \mathcal{F}, °)$ is a fuzzy co-pre-Hilbert space then $\mathcal{F}(x,y,t)$ is non-increasing with respect to t, for each $x, y \in X$.
Proof: Let $t < s$ for all $s, t \in R$, $\mathcal{F}(x,y,t) = \mathcal{F}(x,y,t)°0 = \mathcal{F}(x,y,t)°\mathcal{F}(0,0,s-t) \ge \mathcal{F}(x,y,s)$.
Lemma (3.3):
(1) $\mathcal{F}(ax, ay, t) = \mathcal{F}(x, y, \frac{t}{a^2})$for all $a \in R/\{0\}$.
(2) $\mathcal{F}(x-y, x-y, t) = \mathcal{F}(y-x, y-x, t)$
Proof: (1) and **(2)** are directly from axiom (4).
Theorem (3.4): Let $(X, F, *)$ is a fuzzy pre-Hilbert space. Define $\mathcal{F} = 1 - F$ then $(X, \mathcal{F}, °)$ is a fuzzy co-pre-Hilbert space.
Proof: For all $x, y, z \in X$ and $t, s, r \in R$
(1) $\mathcal{F}(x,x,0) = 1 - F(x,x,0) = 1 - 0 = 1$
And $\mathcal{F}(x,x,t) = 1 - F(x,x,t) < 1 - 0 = 1$ for all $t > 0$;
(2) Suppose$\mathcal{F}(x,x,t) \ne \mathcal{H}(t)$ for some
$t \in R \Leftrightarrow \mathcal{F}(x,x,t) \ne \mathcal{H}(t) \Leftrightarrow 1 - \mathcal{F}(x,x,t) \ne 1 - \mathcal{H}(t) \Leftrightarrow F(x,x,t) \ne H(t) \Leftrightarrow x \ne 0$;
(3) $\mathcal{F}(x,y,t) = 1 - F(y,x,t) = 1 - F(y,x,t) = \mathcal{F}(x,y,t)$.
(4) For any real number a

$$\mathcal{F}(a\,x,y,t) = 1 - F(a\,x,y,t) = 1 - \begin{cases} F(x,y,\frac{t}{a}) & ,a > 0 \\ H(t) & ,a = 0 \\ 1 - F(x,y,\frac{t}{-a}) & ,a < 0 \end{cases} = \begin{cases} \mathcal{F}(x,y,\frac{t}{a}) & ,a > 0 \\ \mathcal{H}(t) & ,a = 0 \\ 1 - \mathcal{F}(x,y,\frac{t}{-a}) & ,a < 0 \end{cases}$$

(5) $\mathcal{F}(x,x,t)°\,\mathcal{F}(y,y,s) = (1 - F(x,x,t))°(1 - F(y,y,s)) = 1 - (F(x,x,t) * F(y,y,s) \le 1 - F(x + y, x+y, t+s) = \mathcal{F}(x+y, x+y, t+s)$;
(6) $sup_{s+r=t}(\mathcal{F}(x,z,s)°\,\mathcal{F}(y,z,r)) = sup_{s+r=t}((1 - \mathcal{F}(x,z,s))°(1 - \mathcal{F}(y,z,r))) = 1 - sup_{s+r=t}(F(x,z,s) * F(y,z,r)) = 1 - F(x+y, z, s+r) = \mathcal{F}(x+y,z,s+r)$
(7) $\mathcal{F}(x,y,.): R \to [0,1]$ is continuous on $R/\{0\}$;
(8) $\lim\limits_{t\to\infty} \mathcal{F}(x,y,t) = \lim\limits_{t\to\infty}(1 - F(x,y,t)) = 1 - 1 = 0$.
Therefore $(X, \mathcal{F}, °)$ is a fuzzy co-pre-Hilbert space.
Theorem (3.5): (Schwartz inequality) Let $(X, \mathcal{F}, °)$ is a fuzzy co-pre-Hilbert space then:
$\mathcal{F}(x,y,ts) \le \mathcal{F}(x,x,t^2)°\mathcal{F}(y,y,s^2)$.
Proof: $\mathcal{F}(x,y,ts) = 1 - F(x,y,ts) \le 1 - F(x,x,t^2) * F(y,y,s^2) = 1 - (1 - \mathcal{F}(x,x,t^2)) * (1 - \mathcal{F}(y,y,s^2)) = \mathcal{F}(x,x,t^2)°\mathcal{F}(y,y,s^2)$.

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The orthogonality:

Definition (3.6): [3] Let $(X, F, *)$ be a fuzzy pre-Hilbert space. A subset A *of* X is called fuzzy orthogonal if $x \perp y$, for each $x, y \in A (i.e. A^{\perp} = \{x \in X: x \perp y \ \forall\ y \in A\})$

Definition (3.7): [3] Let $(X, F, *)$ be a fuzzy pre-Hilbert space. $x, y \in X$ is said to be fuzzy orthogonal if $F(x, y, t) = H(t) \ \forall\ t \in R$ and it is denoted by $x \perp y$.

Theorem (3.8): Let $(X, F, *)$ be a fuzzy pre-Hilbert space. And $A \subset X$

(1) The relation of orthogonality is symmetric $(\text{i.e. if } x \perp y \text{ then } y \perp x)$
(2) If $x \perp y$ then $ax \perp y \ \forall\ a \in R$.
(3) Let $A \subset B$ then $B^{\perp} \subset A^{\perp}$.
(4) $A \subset A^{\perp^{\perp}}$
(5) Let $A \subset B^{\perp} \leftrightarrow B \subset A^{\perp}$
(6) If $x \perp x \leftrightarrow x = 0 \ \forall\ t \in R$
(7) $X^{\perp} = \{0\} \ \forall\ t \in R$.
(8) $A \cap A^{\perp} = \{0\} \ \forall\ t \in R$.
(9) The zero vector is orthogonal to every vector.

Proof:

(1) If $x \perp y$ then by definition (3.7)
$F(x, y, t) = H(t) \ \forall\ t \in R$, but $F(x, y, t) = F(y, x, t)$ then $F(x, y, t) = H(t) \ \forall\ t \in R \Rightarrow y \perp x$.
(2) If $x \perp y$ then $F(x, y, t) = H(t) \ \forall\ t \in R, x, y \in X$.

$$F(a\,x, y, t) = \begin{cases} F(x, y, \frac{t}{a}) & , a > 0 \\ H(t) & , a = 0 \\ 1 - F(x, y, \frac{t}{-a}) & , a < 0 \end{cases} = \begin{cases} H(\frac{t}{a}) \\ H(t) \quad = H(t) \\ 1 - H(\frac{t}{-a}) \end{cases} \Rightarrow ax \perp y$$

(3) Let $x \in B^{\perp} \Rightarrow x \perp y \ \forall\ y \in B$, Since $A \subset B \Rightarrow x \perp y \ \forall\ y \in A \ \ x \subset A^{\perp} \Rightarrow B^{\perp} \subset A^{\perp}$.
(4) Let $x \in A$ then $F(x, y, t) = H(t) \ \forall\ y \in A^{\perp} \Rightarrow x \in A^{\perp^{\perp}} \Rightarrow A \subset A^{\perp^{\perp}}$
(5) Let $A \subset B^{\perp} \Rightarrow B^{\perp^{\perp}} \subset A^{\perp} (by\ 3) \Rightarrow$from (4) $B \subset B^{\perp^{\perp}} \subset A^{\perp} \Rightarrow B \subset A^{\perp}$
(6) Let $x \in X$ and $x \perp x$ then from definition(3.7) $F(x, x, t) = H(t) \ \forall\ t \in R \Rightarrow x = 0 \ \forall\ t \in R$ (by axiom2).
Let $x = 0 \Rightarrow F(x, x, t) = F(0,0, t) = H(t) \ \forall\ t \in R$ (by axiom 4) then $x \perp x \ \forall\ t \in R$ (by definition 3.7)
(7) Let $x \in X^{\perp} \Rightarrow F(x, y, t) = H(t) \ \forall\ t \in R$ and $\forall y \in X$.
Since $x \in X$ then $F(x, x, t) = H(t) \Rightarrow x = 0$ for all $t \in R \Rightarrow X^{\perp} \subseteq \{0\}$ for all $t \in R$
Since $0 \in X^{\perp} \Rightarrow \{0\} \subseteq X^{\perp} \Rightarrow X^{\perp} = \{0\}$ for all $t \in R$.
(8) Let $x \in A \cap A^{\perp} \Rightarrow x \in A$ and $x \in A^{\perp} \Rightarrow F(x, x, t) = H(t) \ \forall\ t \in R \Rightarrow x \perp x \ \forall x \in A \Rightarrow x = 0 \Rightarrow x \in \{0\} \Rightarrow A \cap A^{\perp} \subseteq \{0\}$ for all $t \in R$.
$0 \in A$ and $0 \perp 0 \Rightarrow 0 \in A^{\perp} \Rightarrow 0 \in A \cap A^{\perp}$ then $\{0\} \subseteq A \cap A^{\perp} \Rightarrow A \cap A^{\perp} = \{0\}$ for all $t \in R$.
(9) For every vector $x \in X$, we have $F(0, x, t) = H(t) \ \forall\ t \in R$
Therefore $0 \perp x \forall\ x \in X$.

Theorem (3.9): (parallelogram law) Let $(X, F, *)$is a fuzzy pre-Hilbert space then, $F(x + y, x + y, 4t^2) * F(x - y, x - y, 4s^2) \le F(x, x, (t + s)^2)$.

Proof: $F(x + y, x + y, 4t^2) * F(x - y, x - y, 4s^2) \le F(2x, 2x, 4t^2 + 4s^2) = F(2x, 2x, 4t^2 + 4s^2) * 1 = F(x, x, t^2 + s^2) * F(0,0,2ts) \le F(x, x, (t + s)^2)$

Theorem :(3.10) Let A be a non-empty subset of a fuzzy pre-Hilbert space X. Then $A^{\perp}$is closed fuzzy subspace of X.

Proof: Since $F(0, x, t) = H(t) \ \forall\ x \in A \Rightarrow 0 \in A^{\perp}$ then $A^{\perp} \ne \varphi$ .Let $x, y \in A^{\perp}$and $\alpha, \beta, r \in R$ $F(x, z, r) = H(r) \ \forall\ z \in A$ and $F(y, z, r) = H(r) \ \forall\ z \in A$

For every $z \in A$ we have: If $\alpha > 0, \beta > 0$

$$F(\alpha x + \beta y, z, r) = sup_{t+s=r}(F(x, z, \frac{t}{\alpha}) * (F(y, z, \frac{s}{\beta}) = H(\frac{t}{\alpha}) * H(\frac{s}{\beta}) = H(t) * H(s) = H(r) \forall\ r \in R$$

If $\alpha < 0, \beta < 0$

$$F(\alpha x + \beta y, z, r) = sup_{t+s=r}((1 - (F(x, z, \frac{t}{-\alpha})) * (1 - (F(y, z, \frac{s}{-\beta}))) = H(r)$$

If $\alpha = 0, \beta = 0$
$F(\alpha x + \beta y, z, r) = F(0, z, r) = H(r) \ \forall\ r \in R$
If $\alpha < 0, \beta = 0$ or $\alpha = 0, \beta < 0$ or $\alpha > 0, \beta = 0$ or $\alpha = 0, \beta > 0$
$F(\alpha x + \beta y, z, r) = H(r) \Rightarrow \alpha x + \beta y \in A^{\perp} \Rightarrow A^{\perp}$ is a fuzzy subspace.
Let $x \in \overline{A^{\perp}} \exists \{x_n\}$ in $A^{\perp}$ such that $x_n \to x$
Let $y \in A \Rightarrow F(x_n, y, t) = H(t) \ \forall\ n \in Z^+$and $t \in R (x_n \in A^{\perp} \forall\ n \in Z^+)$
Since $x_n \to x \Rightarrow F(x_n, y, t) \to F(x, y, t) = H(t)$ for all $y \in A$.

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$\Rightarrow x \in A^{\perp} \Rightarrow \overline{A^{\perp}} = A^{\perp} \Rightarrow A^{\perp}$ is closed fuzzy subset of X.

Definition (3.11): Let $(X, \mathcal{F}, \circ)$ be a fuzzy co-pre-Hilbert space. A subset A *of* X is called fuzzy orthogonal if $x \perp y$, for each $x, y \in A$

Definition (3.12): Let $(X, \mathcal{F}, \circ)$ be a fuzzy co-pre-Hilbert space. $x, y \in X$ is said to be fuzzy orthogonal if $\mathcal{F}(x, y, t) = \mathcal{H}(t) \forall\, t \in R$ and it is denoted by $x \perp y$.

Theorem (3.13): Let $(X, \mathcal{F}, \circ)$ be a fuzzy co-pre-Hilbert space. And $A \subset X$

(1) The relation of orthogonally is symmetric (i.e. if $x \perp y$ then $y \perp x$)

(2) If $x \perp y$ then $\alpha x \perp y\ \forall\ \alpha \in R$.

(3) Let $A \subset B$ then $B^{\perp} \subset A^{\perp}$

(4) $A \subset A^{\perp^{\perp}}$.

(5) Let $A \subset B^{\perp} \Leftrightarrow B \subset A^{\perp}$

(6) If $x \perp x \leftrightarrow x = 0\ \forall\ t \in R$

(7) $X^{\perp} = \{0\}\ \forall\ t \in R$

(8) $A \cap A^{\perp} = \{0\}\ \forall\ t \in R$

(9) The zero vector is orthogonal to every vector.

Proof:

(1) If $x \perp y$ then by definition (3.13)
$\mathcal{F}(x, y, t) = \mathcal{H}(t) \forall\, t \in R$ but $\mathcal{F}(x, y, t) = \mathcal{F}(y, x, t)$
then $\mathcal{F}(y, x, t) = \mathcal{H}(t) \forall\, t \in R \Rightarrow y \perp x$.

(2) If $x \perp y$ then $\mathcal{F}(x, y, t) = \mathcal{H}(t) \forall\, t \in R, x, y \in X$.

$$\mathcal{F}(a\,x, y, t) = \begin{cases} \mathcal{F}(x, y, \frac{t}{a}) & , a > 0 \\ \mathcal{H}(t) & , a = 0 \\ 1 - \mathcal{F}(x, y, \frac{t}{-a}) & , a < 0 \end{cases} = \begin{cases} \mathcal{H}(t) \\ \mathcal{H}(t) \\ 1 - \mathcal{H}(\frac{t}{-a}) \end{cases} = \mathcal{H}(t)$$

(3) Let $x \in B^{\perp} \Rightarrow x \perp y\ \forall\ y \in B$, Since $A \subset B \Rightarrow x \perp y\ \forall\ y \in A x \in A^{\perp} \Longrightarrow B^{\perp} \subset A^{\perp}$

(4) Let $x \in A \Rightarrow \mathcal{F}(x, y, t) = \mathcal{H}(t) \forall\, y \in A^{\perp} \Longrightarrow x \in A^{\perp^{\perp}} \Longrightarrow A \subset A^{\perp^{\perp}}$

(5) Let $A \subset B^{\perp} \Longrightarrow B^{\perp^{\perp}} \subset A^{\perp}(by3) \Longrightarrow$ from(4) $B \subset B^{\perp^{\perp}} \subset A^{\perp} \Longrightarrow B \subset A^{\perp}$

(6) Let $x \in X$ and $x \perp x$ then from definition (3.13) $\mathcal{F}(x, x, t) = \mathcal{H}(t) \forall\, t \in R \Longrightarrow x = 0\ \forall\ t \in R$ (by axiom2).

Let $x = 0 \Longrightarrow \mathcal{F}(x, x, t) = \mathcal{F}(0,0, t) = \mathcal{H}(t) \forall\, t \in R$ (by axiom 4) then $x \perp x\ \forall\ t \in R$ (by definition 3.13)

(7) Let $x \in X^{\perp} \Longrightarrow \mathcal{F}(x, y, t) = \mathcal{H}(t) \forall\, t \in R$ and $\forall\ y \in X$.

Since $x \in X$ then $\mathcal{F}(x, x, t) = \mathcal{H}(t) \Longrightarrow x = 0\ for\ all\ \ t \in R \Longrightarrow X^{\perp} \subseteq \{0\}$. Since $0 \in X^{\perp} \Longrightarrow \{0\} \subseteq X^{\perp} \Longrightarrow X^{\perp} = \{0\}\ for\ all\ \ t \in R$.

(8) Let $x \in A \cap A^{\perp} \Longrightarrow x \in A\ and\ x \in A^{\perp} \Longrightarrow \mathcal{F}(x, x, t) = \mathcal{H}(t)\ \forall\ t \in R\ \Longrightarrow x \perp x\ \forall\ x \in A \Longrightarrow x = 0 \Longrightarrow x \in \{0\} \Longrightarrow A \cap A^{\perp} \subseteq \{0\}\ for\ all\ \ t \in R. 0 \in A\ and\ 0 \perp 0 \Longrightarrow 0 \in A^{\perp} \Longrightarrow 0 \in A \cap A^{\perp}$.

Then $\{0\} \subseteq A \cap A^{\perp} \Longrightarrow A \cap A^{\perp} = \{0\} for\ all\ \ t \in R$.

(9) For every vector $x \in X$ we have $\mathcal{F}(0, x, t) = \mathcal{H}(t)\ \forall\ t \in R$. Therefore $0 \perp x\ \forall\ x \in X$

Theorem (3.14): (parallelogram law) Let $(X, \mathcal{F}, \circ)$ be a fuzzy co-pre-Hilbert space then: $\mathcal{F}(x + y, x + y, 4t^2) \circ \mathcal{F}(x - y, x - y, 4s^2) \geq \mathcal{F}(x, x, (t + s)^2)$.

Proof: $\mathcal{F}(\mathrm{x + y, x + y, 4t^2}) \circ \mathcal{F}(\mathrm{x - y, x - y, 4s^2}) \geq \mathrm{F(2x, 2x, 4t^2 + 4s^2)} = \mathcal{F}(\mathrm{2x, 2x, 4t^2 + 4s^2}) \circ 0 = \mathcal{F}(\mathrm{x, x, t^2 + s^2}) \circ \mathcal{F}(\mathrm{0,0,2ts}) \geq \mathcal{F}(\mathrm{x, x, (t + s)^2})$

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