Applications in KU-algebras based on $BMBJ$-neutrosophic Structures

S. Manivasan[1,*], P. Kalidass[2]
[1,2]Department of Mathematics, Annamalai University, Annamalai Nagar - 608 002, India.
Emails: smanivasan63@gmail.com[1]; kalidassp1971@gmail.com[2]

Abstract

We introduce $BMBJ$-neutrosophic sets and subalgebras as a generalisation of neutrosophic sets, and examine their application and related features to KU-algebras in this paper. We give various $BMBJ$-neutrosophic subalgebra characterizations, and we suggest a new $BMBJ$-neutrosophic subalgebra by utilizing a $BMBJ$-neutrosophic subalgebra of a KU-algebra. We look at the homomorphic inverse image of $BMBJ$-neutrosophic subalgebra and $BMBJ$-neutrosophic subalgebra translation.

Keywords: $BMBJ$-N set; $BMBJ$-NSA; $BMBJ$-neutrosophic S-extension.

1 Introduction

Smarandache[6] pioneered the use of neutrosophic sets as a mathematical technique for dealing with uncertain and inconsistent data. Zadeh[9] created the fuzzy set in 1965 to handle uncertainties in numerous real-world applications, and Atanassov introduced the intuitionistic fuzzy set in a universe X in 1983 as a generalisation of the fuzzy set. Another algebraic structure termed KU algebras was presented by Prabpayak and Leerawat.[4,5] In KU algebras, they looked at ideals and congruences. They also introduced the idea of the KU algebra homomorphism and looked at some of its aspects. They also deduced some straightforward results from the links between quotients KU -algebras and isomorphisms. Smarandache[6–8] introduced the notion of the neutrosophic set, which is a more comprehensive platform on which the concepts of the classical set, the fuzzy set (intuitionistic), and the valued interval set (intuitionistic) are expanded. The neutrosophic ideal of the KU-neutrosophic algebras was examined by Bijan Davvaz et al. in 2017. The truth, false, and indeterminate membership functions are fuzzy sets in the neutrosophic set. We selected the interval valued fuzzy set as the indeterminate membership function since interval valued fuzzy set is a generalisation of a fuzzy set, and we dubbed it MBJ-neutrosophic set, where MBJ is the initial of the authors' surnames, Mohseni, Borzooei, and Jun, respectively. As the truth membership function, indeterminate membership function, and false membership function, we employ M_K, $\tilde{B}_K$ and J_K, respectively. We know that Smarandache's neutrosophic sets have several generalisations.

2 Preliminaries

We let $L(\tau)$ be the class of all algebras with type $\tau = (2, 0)$. A KU-algebra[4,5] on a system $P = (P, \diamond, 0) \in L(\tau)$ satisfies

(KU1) $(k_{01} \diamond k_{02}) \diamond ((k_{02} \diamond k_{03}) \diamond (k_{01} \diamond k_{03})) = 0,$

https://doi.org/10.54216/IJNS.200420
Received: December 15, 2022 Accepted: April 12, 2023

(KU2) $k_{01} \diamond 0 = 0$,

(KU3) $0 \diamond k_{01} = k_{01}$,

(KU4) $k_{01} \diamond k_{02} = 0$ & $k_{02} \diamond k_{01} = 0$ implies $k_{01} = k_{01}$,

(KU5) $k_{01} \diamond k_{01} = 0, \forall\, k_{01}, k_{02}, k_{03} \in P$.

Also a binary relation $\leq$ by putting $k_{01} \leq k_{02} \Leftrightarrow k_{02} \diamond k_{01} = 0, \forall\, k_{01}, k_{02} \in P$.

In a KU-algebra P, the following hold:

(KU1′) $(k_{02} \diamond k_{03}) \diamond (k_{01} \diamond k_{03}) \leq (k_{01} \diamond k_{02})$,

(KU2′) $0 \leq k_{01}$,

(KU3′) $k_{01} \leq k_{02}, k_{02} \leq k_{01}$ implies $k_{01} = k_{02}$,

(KU4′) $k_{02} \diamond k_{01} \leq k_{01}$.

Theorem 2.1. [3] The following axioms must be true in a KU-algebra P: $\forall\, k_{01}, k_{02}, k_{03} \in P$,

(i) $k_{01} \leq k_{02}$ imply $k_{02} \diamond k_{03} \leq k_{01} \diamond k_{03}$,

(ii) $k_{01} \diamond (k_{02} \diamond k_{03}) = k_{02} \diamond (k_{01} \diamond k_{03}), \forall\, k_{01}, k_{02}, k_{03} \in P$,

(iii) $((k_{02} \diamond k_{01}) \diamond k_{01}) \leq k_{02}$,

(iv) $(((k_{02} \diamond k_{01}) \diamond k_{01}) \diamond k_{01}) = (k_{02} \diamond k_{01})$.

Definition 2.2. [4,5] A non-empty subset S of a KU-algebra P is called a KU-subalgebra of P if $l_{11} \diamond l_{22} \in S$ $\forall\, l_{11}, l_{22} \in S$.

A closed subinterval $\tilde{l} = [l^-, l^+]$ of I, where $0 \leq l^- \leq l^+ \leq 1$ and $[I]$ denotes the set of all interval numbers. Consider two interval numbers $\tilde{l}_{01} := [l_{01}^-, l_{01}^+]$ and $\tilde{l}_{02} := [l_{02}^-, l_{02}^+]$. Let us define as refined minimum and refined maximum (briefly, $re\min$ and $re\max$) of two elements in $[I]$, then

$$re\min\{\tilde{l}_{01}, \tilde{l}_{02}\} = [\min\{l_{01}^-, l_{02}^-\}, \min\{l_{01}^+, l_{02}^+\}],$$
$$re\max\{\tilde{l}_{01}, \tilde{l}_{02}\} = [\max\{l_{01}^-, l_{02}^-\}, \max\{l_{01}^+, l_{02}^+\}],$$

Let P be a nonempty set. A function $L : P \to [I]$ is called an interval-valued fuzzy (in short, IVF) set in P. Let $[I]^P$ denotes the set of all IVF sets in P. For every $L \in [I]^P$ and $k \in P, L(k) = [L^-(k), L^+(k)]$ is called the degree of membership of an element k to L, where $L^- : P \to I$ is a a lower fuzzy set in P and $L^+ : P \to I$ an upper fuzzy set in P, respectively. We denote $L = [L^-, L^+]$.

Definition 2.3. [7] Let P be a non-empty set. A neutrosophic set (NS) in P is of the form:

$$L := \{\langle k; L_T(k), L_I(k), L_F(k)\rangle | k \in P\}$$

where $L_T, L_I, L_F : P \to [0,1]$ is a truth, an indeterminate and a false membership function. We use the symbol $L = (L_T, L_I, L_F)$.

Definition 2.4. [2] "Let P be a non-empty set. By an MBJ-neutrosophic (briefly, MBJ-N) set in P is of the form

$$\mathcal{L} := \{\langle k; M_L(n), \tilde{B}_L(n), J_L(n)\rangle | n \in P\}$$

where M_L and J_L are fuzzy sets in P called a truth and a false membership functions and $\tilde{B}_L$ is an IVF set in P called an indeterminate interval-valued membership function." We use the symbol $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$.

https://doi.org/10.54216/IJNS.200420
Received: December 15, 2022 Accepted: April 12, 2023

3 $BMBJ$-neutrosophic Structures with Applications in KU-algebras

Definition 3.1. Let P be a KU-algebra. An MBJ-N set $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ in P is called an $BMBJ$-neutrosophic subalgebra (briefly, $BMBJ$-NSA) of P if it satisfies:

$$(\forall\, l_{01}, l_{02} \in P) \left(\begin{array}{l} M_L(l_{01} \diamond l_{02}) \geq \min\{M_L(l_{01}), M_L(l_{02})\}, \\ B_L^-(l_{01} \diamond l_{02}) \leq \max\{B_L^-(l_{01}), B_L^-(l_{02})\}, \\ B_L^+(l_{01} \diamond l_{02}) \geq \min\{B_L^+(l_{01}), B_L^+(l_{02})\}, \\ J_L(l_{01} \diamond l_{02}) \leq \max\{J_L(l_{01}), J_L(l_{02})\}, \\ M_L(l_{01}) + B_L^-(l_{01}) \leq 1,\ B_L^+(l_{01}) + J_L(l_{01}) \geq 1. \end{array} \right) \tag{1}$$

Example 3.2. *Consider a set $P = \{0_6, a_6, b_6, c_6\}$ with the binary operator $\diamond$ which is given by*

Table 1:

$\diamond$	0_6	a_6	b_6	c_6
0_6	0_6	a_6	b_6	c_6
a_6	0_6	0_6	a_6	c_6
b_6	0_6	0_6	0_6	c_6
c_6	0_6	a_6	b_6	0_6

Then $(P;\ \diamond, 0)$ is a KU-algebra.

Table 2:

P	$M_L(l)$	$\tilde{B}_L(l)$	$J_L(l)$
0_6	0.7	$[0.3, 0.8]$	0.2
a_6	0.3	$[0.4, 0.5]$	0.6
b_6	0.1	$[0.3, 0.8]$	0.4
c_6	0.5	$[0.4, 0.5]$	0.7

Let $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ be an MBJ-N set in P as in Table 2. It is a $BMBJ$-NSA of P.

Proposition 3.3. If $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ is an $BMBJ$-NSA of P, then $M_L(0) \geq M_L(k_0)$, $\tilde{B}_L^-(0) \leq \tilde{B}_L^-(k_0)$, $\tilde{B}_L^+(0) \geq \tilde{B}_L^+(k_0)$ and $J_L(0) \leq J(k_0)$ for all $k_0 \in P$.

Proof. For any $k_0 \in P$, we have

$$M_L(0) = M_L(k_0 \diamond k_0) \geq \min\{M_L(k_0), M_L(k_0)\} = M_L(k_0)\ ,$$

$$\tilde{B}_L^-(0) = \tilde{B}_L^-(k_0 \diamond k_0) \leq \max\{\tilde{B}_L^-(k_0), \tilde{B}_L^-(k_0)\} = \tilde{B}_L^-(k_0)\ ,$$

$$\tilde{B}_L^+(0) = \tilde{B}_L^-(k_0 \diamond k_0) \geq \min\{\tilde{B}_L^+(k_0), \tilde{B}_L^-(k_0)\} = \tilde{B}_L^+(k_0)$$

and

$$J_L(0) = J_L(k_0 \diamond k_0) \leq \max\{J_L(k_0), J_L(k_0)\} = J_L(k_0).$$

□

Proposition 3.4. Let $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ be an $BMBJ$-NSA of P. If $\exists$ a sequence $\{b_l\}$ in P such that

$$\lim_{l\to\infty} M_L(b_l) = 1, \lim_{l\to\infty} \tilde{B}_L^-(b_l) = 0, \lim_{l\to\infty} \tilde{B}_L^+(b_l) = 1 \text{ and } \lim_{l\to\infty} J_L(b_l) = 0, \tag{2}$$

then $M_L(0) = 1$, $\tilde{B}_L^-(0) = 0$, $\tilde{B}_L^+(0) = 1$ & $J_L(0) = 0$.

Received: December 15, 2022 Accepted: April 12, 2023

Proof. Using Proposition 3.3, $M_L(0) \geq M_L(b)$, $\tilde{B}_L^-(0) \leq \tilde{B}_L^-(b)$, $\tilde{B}_L^+(0) \geq \tilde{B}_L^+(b)$ and $J_L(0) \leq J(b)$ $\forall$ $b \in P$. for every $+ve$ integer l. Note that

$$1 \geq M_L(0) \geq \lim_{l\to\infty} M_L(b_l) = 1,$$

$$0 \leq \tilde{B}_L^-(0) \leq \lim_{l\to\infty} \tilde{B}_L^-(b_l) = 0,$$

$$1 \geq \tilde{B}_L^+(0) \geq \lim_{l\to\infty} \tilde{B}_L^+(b_l) = 1,$$

$$0 \leq J_L(0) \leq \lim_{l\to\infty} J_L(b_l) = 0.$$

Therefore $M_L(0) = 1$, $\tilde{B}_L^-(0) = 0$, $\tilde{B}_L^+(0) = 1$ and $J_L(0) = 0$. □

Theorem 3.5. Given an $BMBJ$-N set $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ in P, if (M_L, J_L) is an intuitionistic fuzzy subalgebra (IFSA) of P, and B_L^- and B_L^+ are fuzzy subalgebras of P, then $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ is an $BMBJ$-NSA of P.

Proof. It suffices to demonstrate that $\tilde{B}_L$ satisfies

$$(\forall\, l_{01}, l_{02} \in P)(\tilde{B}_L^-(l_{01} \diamond l_{02}) \leq \max\{\tilde{B}_L^-(l_{01}), \tilde{B}_L^-(l_{02})\}), \tag{3}$$

$$(\forall\, l_{01}, l_{02} \in P)(\tilde{B}_L^+(l_{01} \diamond l_{02}) \geq \min\{\tilde{B}_L^+(l_{01}), \tilde{B}_L^+(l_{02})\}). \tag{4}$$

For any $l_{01}, l_{02} \in P$, we get

$$\begin{aligned}\tilde{B}_L(l_{01} \diamond l_{02}) =&[\tilde{B}_L^-(l_{01} \diamond l_{02}), \tilde{B}_L^+(l_{01} \diamond l_{02})]\\ \geq&[\{\max \tilde{B}_L^-(l_{01}), \tilde{B}_L^-(l_{02})\}, \min\{\tilde{B}_L^+(l_{01}), \tilde{B}_L^+(l_{02})\}].\end{aligned}$$

Thus $\tilde{B}_L$ satisfies (3), and so $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ is an $BMBJ$-NSA of P. □

If $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ is an $BMBJ$-NSA of P, then

$$\begin{aligned}[B_L^-(l_{01} \diamond l_{02}), B_L^+(l_{01} \diamond l_{02})] =&\tilde{B}_L(l_{01} \diamond l_{02}) \succeq re\min\{\tilde{B}_L(l_{01}), \tilde{B}_L(l_{02})\}\\ =&re\min\{[B_L^-(l_{01}), B_L^+(l_{01})], [B_L^-(l_{02}), B_L^+(l_{02})]\}\\ =&[\min\{B_L^-(l_{01}), B_L^-(l_{02})\}, \min\{B_L^+(l_{01}), B_L^+(l_{02})\}]\end{aligned}$$

for all $l_{01}, l_{02} \in P$.

It follows that $B_L^-(l_{01} \diamond l_{02}) \geq \min\{B_L^-(l_{01}), B_L^-(l_{02})\}$ and $B_L^+(l_{01} \diamond l_{02}) \geq \min\{B_L^+(l_{01}), B_L^+(l_{02})\}$. Thus B_L^- and B_L^+ are fuzzy subalgebras of P. But (M_L, J_L) is not an IFSA of P as seen in Example 3.2 Therefore the converse of Theorem 3.5 is not true.

Given an $BMBJ$-N set $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ in P, then

$$U(M_L; t_1) := \{x \in P | M_L(x) \geq t_1\},$$

$$L(\tilde{B}_L^-; \eta_1) := \{x \in P | \tilde{B}_L^-(x) \leq \eta_1\},$$

$$U(\tilde{B}_L^+; \eta_2) := \{x \in P | \tilde{B}_L^+(x) \geq \eta_2\},$$

$$L(J_L; t_2) := \{x \in P | J_L(x) \leq t_2\}$$

where $t_1, t_2 \in [0, 1]$ & $[\eta_1, \eta_2] \in [I]$.

Theorem 3.6. An $BMBJ$-N set $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ in P is an $BMBJ$-NSA of P iff the non-empty sets $U(M_L; t_1)$, $L(\tilde{B}_L^-; \zeta_1)$, $U(\tilde{B}_L^+; \zeta_2)$ & $L(J_L; t_2)$ are subalgebras of P for all $t_1, t_2 \in [0, 1]$ & $[\eta_1, \eta_2] \in [I]$.

https://doi.org/10.54216/IJNS.200420
Received: December 15, 2022 Accepted: April 12, 2023

Proof. Suppose that $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ is an $BMBJ\text{-}NSA$ of P. Let $t_1, t_2 \in [0,1]$ and $[\zeta_1, \zeta_2] \in [I]$ be $\ni U(M_L; t_1)$, $L(\tilde{B}_L^-; \zeta_1), U(\tilde{B}_L^+; \zeta_2)$ & $L(J_L; t_2)$ are non-empty. For any $\iota, \kappa, p, q, r, s, m, n \in P$, if $\iota, \kappa \in U(M_L; t_1), p, q \in L(\tilde{B}_L^-; \zeta_1), r, s \in U(\tilde{B}_L^+; \zeta_2)$ and $m, n \in L(J_L; t_2)$, then

$$M_L(\iota \diamond \kappa) \geq \min\{M_L(\iota), M_L(\kappa)\} \geq \min\{t_1, t_1\} = t_1,$$

$$\tilde{B}_L^-(p \diamond q) \leq \max\{\tilde{B}_L^-(p), \tilde{B}_L^-(q)\} \leq \min\{\zeta_1, \zeta_1\} = \zeta_1,$$

$$\tilde{B}_L^+(r \diamond s) \geq \min\{\tilde{B}_L^+(r), \tilde{B}_L^+(s)\} =\geq \min\{\zeta_2, \zeta_2\} = \zeta_1,$$

$$J_L(m \diamond n) \leq \max\{J_L(m), J_L(n)\} \leq \min\{t_2, t_2\} = t_2,$$

and so $\iota \diamond \kappa \in U(M_L; t_1), p \diamond q \in L(\tilde{B}_L^-; \zeta_1), r \diamond s \in U(\tilde{B}_L^+; \zeta_2)$ and $m \diamond n \in L(J_L; t_2)$. Therefore $U(M_L; t_1)$, $L(\tilde{B}_L^-; \zeta_1), U(\tilde{B}_L^+; \zeta_2)$ and $L(J_L; t_2)$ are subalgebras of P.

Conversely, assume that the non-empty sets $U(M_L; t_1), L(\tilde{B}_L^-; \zeta_1), U(\tilde{B}_L^+; \zeta_2)$ & $L(J_L; t_2)$ are subalgebras of P for all $t_1, t_2, \zeta_1, \zeta_2 \in [0,1]$. If $M_L(p_0 \diamond q_0) < \min\{M_L(p_0), M_L(q_0)\}$ for some $p_0, q_0 \in P$, then $p_0, q_0 \in U(M_L; t_0)$ but $p_0 \diamond q_0 \notin U(M_L; t_0)$ for $t_0 := \min\{M_L(p_0), M_L(q_0)\}$. This is a contradiction, and thus $M_L(p \diamond q) \geq \min\{M_L(p), M_L(q)\}$ for all $p, q \in P$. Similarly, we can show that $\tilde{B}_L^-(p \diamond q) \leq \max\{\tilde{B}_L^-(p), \tilde{B}_L^-(q)\}, \tilde{B}_L^+(r \diamond s) \geq \min\{\tilde{B}_L^+(r), \tilde{B}_L^+(s)\}$ and $J_L(p \diamond q) \leq \max\{J_L(p), J_L(q)\}\ \forall\ p, q \in P$. □

We may derive the following corollary using Proposition 3.3 and Theorem 3.6.

Corollary 3.7. If $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ is an $BMBJ\text{-}NSA$ of P, then the sets $P_{M_L} := \{l \in P | M_L(l) = M_L(0)\}$, $P_{\tilde{B}_L^-}- := \{l \in P | \tilde{B}_L^-(l) = \tilde{B}_L^-(0)\}$, $P_{\tilde{B}_L^+} := \{l \in P | \tilde{B}_L^+(l) = \tilde{B}_L^+(0)\}$, and $P_{J_L} := \{l \in P | J_L(l) = J_L(0)\}$ are subalgebras of P.

Thus $U(M_L; t), L(\tilde{B}_L^-; \delta_1), U(\tilde{B}_L^+; \delta_2)$ & $L(J_L; s)$ are $BMBJ$-subalgebras of $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$.

Theorem 3.8. Every subalgebra of P can be realized as $BMBJ$-subalgebras of an $BMBJ\text{-}NSA$ of P.

Proof. Let H be a subalgebra of P and let $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ be an $BMBJ\text{-}N$ set in P,

$$M_L(\iota) = \begin{cases} t_1 & \text{if } \iota \in H, \\ 0 & \text{otherwise} \end{cases}, \quad \tilde{B}_L^-(\iota) = \begin{cases} \rho_1 & \text{if } \iota \in H, \\ 1 & \text{otherwise} \end{cases}, \quad \tilde{B}_L^+(\iota) = \begin{cases} \rho_2 & \text{if } \iota \in H, \\ 0 & \text{otherwise} \end{cases}, \quad J_L(\iota) = \begin{cases} t_2 & \text{if } \iota \in H, \\ 1 & \text{otherwise} \end{cases}, \tag{5}$$

where $t_1 \in (0,1]$, $t_2 \in [0,1)$ and $\rho_1, \rho_2 \in (0,1]$ with $\rho_1 < \rho_2$. Clearly, $U(M_L; t_1) = H, L(\tilde{B}_L^-; \rho_1) = H$, $U(\tilde{B}_L^+; \rho_2) = H$ and $L(J_L; t_2) = H$. Let $l, n \in P$. If $l_{01}, l_{02} \in H$, then $l \diamond n \in H$ and so

$$M_L(l_{01}) \diamond l_{02}) = t_1 = \min\{M_L(l_{01})), M_L(l_{02})\}$$

$$\tilde{B}_L^-(l_{01}) \diamond l_{02}) = \rho_1 = \max\{\tilde{B}_L^-(l_{01})), \tilde{B}_L^-(l_{02})\},$$

$$\tilde{B}_L^+(l_{01}) \diamond l_{02}) = \rho_2 = \max\{\tilde{B}_L^+(l_{01})), \tilde{B}_L^+(l_{02})\},$$

$$J_L(l_{01}) \diamond l_{02}) = t_2 = \max\{J_L(l_{01})), J_L(l_{02})\}.$$

If any one of l_{01} & $l_{02} \subset H$, say $l_{01} \in H$, then $M_L(l_{01})) = t_1$, $\tilde{B}_L^-(l_{01})) = \rho_1$, $\tilde{B}_L^+(l_{01})) = \rho_2$, $J_L(l_{01})) = t_2$, $M_L(l_{02}) = 0, \tilde{B}_L^-(l_{02}) = 0$, $\tilde{B}_L^+(l_{02}) = 0$ and $J_L(l_{02}) = 1$. Hence

$$M_L(l_{01}) \diamond l_{02}) \geq 0 = \min\{t_1, 0\} = \min\{M_L(l_{01})), M_L(l_{02})\}$$

$$\tilde{B}_L^-(l_{01}) \diamond l_{02}) \leq 1 = \max\{\rho_1, 1\} = \max\{\tilde{B}_L^-(l_{01})), \tilde{B}_L^-(l_{02})\},$$

$$\tilde{B}_L^+(l_{01}) \diamond l_{02}) \geq 0 = \min\{\rho_2, 0\} = \min\{\tilde{B}_L^+(l_{01})), \tilde{B}_L^+(l_{02})\},$$

$$J_L(l_{01}) \diamond l_{02}) \leq 1 = \max\{t_2, 1\} = \max\{J_L(l_{01})), J_L(l_{02})\}.$$

If $l_{01}, l_{02} \notin H$, then $M_L(l_{01})) = 0 = M_L(l_{02}), \tilde{B}_L^-(l_{01})) = 1 = \tilde{B}_L^-(l_{02}), \tilde{B}_L^+(l_{01})) = 0 = \tilde{B}_L^+(l_{02})$ and $J_L(l_{01})) = 1 = J_L(l_{02})$.

Received: December 15, 2022 Accepted: April 12, 2023

It follows that

$$\begin{aligned} M_L(l_{01}) \diamond l_{02}) &\geq 0 = \min\{0,0\} = \min\{M_L(l_{01})), M_L(l_{02})\} \\ \tilde{B}^-_L(l_{01}) \diamond l_{02}) &\leq 1 = \max\{1,1\} = \max\{\tilde{B}^-_L(l_{01})), \tilde{B}^-_L(l_{02})\}, \\ \tilde{B}^+_L(l_{01}) \diamond l_{02}) &\geq 0 = \min\{0,0\} = \min\{\tilde{B}^+_L(l_{01})), \tilde{B}^+_L(l_{02})\}, \\ J_L(l_{01}) \diamond l_{02}) &\leq 1 = \max\{1,1\} = \max\{J_L(l_{01})), J_L(l_{02})\}. \end{aligned}$$

Therefore $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ is an $BMBJ$-NSA of P. □

Theorem 3.9. For any non-empty subset H of P, let $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ be an $BMBJ$-N set in P which is given in (5). If $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ is an $BMBJ$-NSA of P, then H is a subalgebra of P.

Proof. Let $l_{01}, l_{02} \in H$. Then $M_L(l_{01})) = t_1 = M_L(l_{02}), \tilde{B}^-_L(l_{01})) = \rho_1 = \tilde{B}^-_L(l_{02}), \tilde{B}^+_L(l_{01})) = \rho_2 = \tilde{B}^+_L(l_{02})$ and $J_L(l_{01})) = t_2 = J_L(l_{02})$. Thus

$$\begin{aligned} M_L(l_{01}) \diamond l_{02}) &\geq \min\{M_L(l_{01})), M_L(l_{02})\} = t_1, \\ \tilde{B}^-_L(l_{01}) \diamond l_{02}) &\leq \max\{\tilde{B}^-_L(l_{01})), \tilde{B}^-_L(l_{02})\} = \rho_1, \\ \tilde{B}^+_L(l_{01}) \diamond l_{02}) &\geq \min\{\tilde{B}^+_L(l_{01})), \tilde{B}^+_L(l_{02})\} = \rho_2, \\ J_L(l_{01}) \diamond l_{02}) &\leq \max\{J_L(l_{01})), J_L(l_{02})\} = t_2, \end{aligned}$$

and therefore $l_{01} \diamond l_{02} \in H$. Hence H is a subalgebra of P. □

Theorem 3.10. Given an $BMBJ$-NSA $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ of a KU-algebra P, let $\mathcal{L}^* = (M^*_L, \tilde{B}^*_L, J^*_L)$ be an $BMBJ$-N set in P by $M^*_L(l_{01}) = M_L(0 \diamond l_{01})$, $\tilde{B}^*_L(l_{01}) = \tilde{B}_L(0 \diamond l_{01})$ and $J^*_L(l_{01}) = J_L(0 \diamond l_{01})$ for all $l_{01} \in P$. Then $\mathcal{L}^* = (M^*_L, \tilde{B}^*_L, J^*_L)$ is an $BMBJ$-NSA of P.

Proof. Note that $0 \diamond (l_{01} \diamond l_{02}) = (0 \diamond l_{01}) \diamond (0 \diamond l_{02})\ \forall\, l_{01}, l_{02} \in P$. We have

$$\begin{aligned} M^*_L(l_{01} \diamond l_{02}) =& M_L(0 \diamond (l_{01} \diamond l_{02})) = M_L((0 \diamond l_{01}) \diamond (0 \diamond l_{02})) \\ &\geq \min\{M_L(0 \diamond l_{01}), M_L(0 \diamond l_{02})\} \\ &= \min\{M^*_L(l_{01}), M^*_L(l_{02})\}, \\ (\tilde{B}^-_L)^*(l_{01} \diamond l_{02}) =& \tilde{B}^-_L(0 \diamond (l_{01} \diamond l_{02})) = \tilde{B}^-_L((0 \diamond l_{01}) \diamond (0 \diamond l_{02})) \\ &\leq \max\{\tilde{B}^-_L(0 \diamond l_{01}), \tilde{B}^-_L(0 \diamond l_{02})\} \\ &= \max\{(\tilde{B}^-_L)^*(l_{01}), (\tilde{B}^-_L)^*(l_{02})\}, \\ (\tilde{B}^+_L)^*(l_{01} \diamond l_{02}) =& \tilde{B}^+_L(0 \diamond (l_{01} \diamond l_{02})) = \tilde{B}^+_L((0 \diamond l_{01}) \diamond (0 \diamond l_{02})) \\ &\geq \min\{\tilde{B}^+_L(0 \diamond l_{01}), \tilde{B}^+_L(0 \diamond l_{02})\} \\ &= \min\{(\tilde{B}^+_L)^*(l_{01}), (\tilde{B}^+_L)^*(l_{02})\}, \end{aligned}$$

and

$$\begin{aligned} J^*_L(l_{01} \diamond l_{02}) =& J_L(0 \diamond (l_{01} \diamond l_{02})) = J_L((0 \diamond l_{01}) \diamond (0 \diamond l_{02})) \\ &\leq \max\{J_L(0 \diamond l_{01}), J_L(0 \diamond l_{02})\} \\ &= \max\{J^*_L(l_{01}), J^*_L(l_{02})\}, \end{aligned}$$

for all $l_{01}, l_{02} \in P$. Therefore $\mathcal{L}^* = (M^*_L, \tilde{B}^*_L, J^*_L)$ is an $BMBJ$-NSA of P. □

Theorem 3.11. Let $h : P \to Q$ be a homomorphism of KU-algebras. If $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ is an MBJ-NSA of Q, then $h^{-1}(\mathcal{L}) = (h^{-1}(M_L), h^{-1}(\tilde{B}_L), h^{-1}(J))$ is an $BMBJ$-NSA of P, where $h^{-1}(M_L)(l_{01}) = M_L(h(l_{01})), h^{-1}(\tilde{B}_L)(l_{01}) = \tilde{B}_L(h(l_{01}))$ and $h^{-1}(J_L)(l_{01}) = J_L(h(l_{01}))$ for all $l_{01} \in P$.

https://doi.org/10.54216/IJNS.200420

Received: December 15, 2022 Accepted: April 12, 2023

Proof. Let $l_{01}, l_{02} \in P$. Then

$$\begin{aligned} h^{-1}(M_L)(l_{01} \diamond l_{02}) =& M_L(h(l_{01} \diamond l_{02})) = M_L(h(l_{01}) \diamond h(l_{02})) \\ &\geq \min\{M_L(h(l_{01})), M_L(h(l_{02}))\} \\ &= \min\{h^{-1}(M_L)(l_{01}), h^{-1}(M_L)(l_{02})\}, \\ h^{-1}(\tilde{B}_L^-)(l_{01} \diamond l_{02}) =& \tilde{B}_L^-(h(l_{01} \diamond l_{02})) = \tilde{B}_L^-(h(l_{01}) \diamond h(l_{02})) \\ &\leq \max\{\tilde{B}_L^-(h(l_{01})), \tilde{B}_L^-(h(l_{02}))\} \\ &= \max\{h^{-1}(\tilde{B}_L^-)(l_{01}), h^{-1}(\tilde{B}_L^-)(l_{02})\}, \\ h^{-1}(\tilde{B}_L^+)(l_{01} \diamond l_{02}) =& \tilde{B}_L^+(h(l_{01} \diamond l_{02})) = \tilde{B}_L^+(h(l_{01}) \diamond h(l_{02})) \\ &\geq \min\{\tilde{B}_L^+(h(l_{01})), \tilde{B}_L^+(h(l_{02}))\} \\ &= \min\{h^{-1}(\tilde{B}_L^+)(l_{01}), h^{-1}(\tilde{B}_L^+)(l_{02})\}, \end{aligned}$$

and

$$\begin{aligned} h^{-1}(J_L)(l_{01} \diamond l_{02}) =& J_L(h(l_{01} \diamond l_{02})) = J_L(h(l_{01}) \diamond h(l_{02})) \\ &\leq \max\{J_L(h(l_{01})), J_L(h(l_{02}))\} \\ &= \max\{h^{-1}(J_L)(l_{01}), h^{-1}(J_L)(l_{02})\}. \end{aligned}$$

Hence $h^{-1}(\mathcal{L}) = (h^{-1}(M_L), h^{-1}(\tilde{B}_L), h^{-1}(J))$ is an $BMBJ$-NSA of P. □

Let $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ be an $BMBJ$-N set in a set P, then

$$\begin{aligned} T :=& 1 - \sup\{M_L(l_{01}) | l_{01} \in P\}, \\ \Pi :=& \inf\{\tilde{B}_L^-(l_{01}) | l_{01} \in P\}, \\ \pi :=& 1 - \sup\{\tilde{B}_L^+(l_{01}) | l_{01} \in P\}, \\ \perp :=& \inf\{J_L(l_{01}) | l_{01} \in P\}. \end{aligned}$$

For any $m \in [0, T]$, $b \in [0, \Pi]$, $c \in [0, \pi]$ and $j \in [0, \perp]$, we define $\mathcal{L}^T = (M_L^m, \tilde{B}_L^b, \tilde{B}_L^c, J_L^j)$ by $M_L^m(l_{01}) = M_L(l_{01}) + m$, $\tilde{B}_L^b(l_{01}) = \tilde{B}_L^-(l_{01}) + b$, $\tilde{B}_L^c(l_{01}) = \tilde{B}_L^+(l_{01}) + c$ and $J_L^j(l_{01}) = J_L(l_{01}) - j$. Then $\mathcal{L}^T = (M_L^m, \tilde{B}_L^b, \tilde{B}_L^c, J_L^j)$ is an $BMBJ$-N set in P, which is called a (m, b, c, j)-translative $BMBJ$-neutrosophic set of $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ (briefly, (m, b, c, j)-$TBMBJNS(L)$).

Theorem 3.12. If $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ is an $BMBJ$-NSA of P, then the (m, b, c, j)-$TBMBJNS(L)$ is also an $BMBJ$-NSA of P.

Proof. For any $l_{01}, l_{02} \in P$, we get

$$\begin{aligned} M_L^p(l_{01} \diamond l_{02}) =& M_L(l_{01} \diamond l_{02}) + p \geq \min_L\{M_L(l_{01}), M_L(l_{02})\} + p \\ &= \min\{M_L(l_{01}) + p, M(l_{02}) + p\} = \min\{M_L^p(l_{01}), M_L^p(l_{02})\}, \\ \tilde{B}_L^a(l_{01} \diamond l_{02}) =& \tilde{B}_L^-(l_{01} \diamond l_{02}) + a \leq \max\{\tilde{B}_L^-(l_{01}), \tilde{B}_L^-(l_{02})\} + a \\ &= \max\{\tilde{B}_L^-(l_{01}) + a, \tilde{B}_L^-(l_{02}) + a\} = \max\{\tilde{B}_L^a(l_{01}), \tilde{B}_L^a(l_{02})\}, \\ \tilde{B}_L^b(l_{01} \diamond l_{02}) =& \tilde{B}_L^+(l_{01} \diamond l_{02}) + b \geq \min\{\tilde{B}_L^+(l_{01}), \tilde{B}_L^+(l_{02})\} + b \\ &= \min\{\tilde{B}_L^+(l_{01}) + b, \tilde{B}_L^+(l_{02}) + b\} = \max\{\tilde{B}_L^b(l_{01}), \tilde{B}_L^b(l_{02})\}, \end{aligned}$$

and

$$\begin{aligned} J_L^q(l_{01} \diamond l_{02}) =& J_L(l_{01} \diamond l_{02}) - q \leq \max\{J_L(l_{01}), J_L(l_{02})\} - q \\ &= \max\{J_L(l_{01}) - q, J_L(l_{02}) - q\} = \max\{J_L^q(l_{01}), J_L^q(l_{02})\}. \end{aligned}$$

Therefore $\mathcal{L}^T = (M_L^p, \tilde{B}_L^a, \tilde{B}_L^b, J_L^q)$ is an $BMBJ$-NSA of P. □

https://doi.org/10.54216/IJNS.200420

Received: December 15, 2022 Accepted: April 12, 2023

Theorem 3.13. Let $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ be an $BMBJ$-N set in $P \ni (p,a,b,q)$-$TBMBJNS(L)$ is an $BMBJ$-NSA of P for $p \in [0,T]$, $a \in [0,\Pi]$, $b \in [0,\pi]$ & $q \in [0,\perp]$. Then $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ is an $BMBJ$-NSA of P.

Proof. Assume that $\mathcal{L}^T = (M_L^p, \tilde{B}_L^a, \tilde{B}_L^b, J_L^q)$ is an $BMBJ$-NSA of P for $p \in [0,T]$, $a \in [0,\Pi]$, $b \in [0,\pi]$ & $q \in [0,\perp]$. Let $l_{01}, l_{02} \in P$. Then

$$\begin{aligned}
M_L(l_{01} \diamond l_{02}) + p =& M_L^p(l_{01} \diamond l_{02}) \geq \min\{M_L^p(l_{01}), M_L^p(l_{02})\}\\
=& \min\{M_L(l_{01}) + p, M_L(l_{02}) + p\}\\
=& \min\{M_L(l_{01}), M_L(l_{02}) + p\}\\
\tilde{B}_L^a(l_{01} \diamond l_{02}) - a =& \tilde{B}_L^-(l_{01} \diamond l_{02}) \leq \max\{\tilde{B}_L^-(l_{01}), \tilde{B}_L^-(l_{02})\}\\
=& \max\{\tilde{B}_L^a(l_{01}) - a, \tilde{B}_L^a(l_{02}) - a\}\\
=& \max\{\tilde{B}_L^-(l_{01}), \tilde{B}_L^-(l_{02})\} - a.\\
\tilde{B}_L^b(l_{01} \diamond l_{02}) - b =& \tilde{B}_L^+(l_{01} \diamond l_{02}) \geq \min\{\tilde{B}_L^+(l_{01}), \tilde{B}_L^+(l_{02})\}\\
=& \min\{\tilde{B}_L^b(l_{01}) - b, \tilde{B}_L^b(l_{02}) - b\}\\
=& \min\{\tilde{B}_L^+(l_{01}), \tilde{B}_L^+(l_{02})\} - b.
\end{aligned}$$

and

$$\begin{aligned}
J_L(l_{01} \diamond l_{02}) - q =& J_L^q(l_{01} \diamond l_{02}) \leq \max\{J_L^q(l_{01}), J_L^q(l_{02})\}\\
=& \max\{J_L(l_{01}) - q, J_L(l_{02}) - q\}\\
=& \max\{J_L(l_{01}), J_L(l_{02})\} - q.
\end{aligned}$$

It follows that $M_L(l_{01} \diamond l_{02}) \geq \min\{M_L(l_{01}), M_L(l_{02})\}$, $\tilde{B}_L^-(l_{01} \diamond l_{02}) \leq \max\{\tilde{B}_L^-(l_{01}), \tilde{B}_L^-(l_{02})\}$, $\tilde{B}_L^+(l_{01} \diamond l_{02}) \geq \min\{\tilde{B}_L^+(l_{01}), \tilde{B}_L^+(l_{02})\}$ and $J_L(l_{01} \diamond l_{02}) \leq \max\{J_L(l_{01}), J_L(l_{02})\}$ for all $l_{01}, l_{02} \in P$. Hence $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ is an $BMBJ$-NSA of P. □

Definition 3.14. Let $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ and $\mathcal{N} = (M_N, \tilde{B}_N, J_N)$ be $BMBJ$-N sets in P. Then $\mathcal{N} = (M_N, \tilde{B}_N, J_N)$ is called an $BMBJ$-neutrosophic S-extension of $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ if

(i) $M_N(l) \geq M_L(l), \tilde{B}_L^-(l) \leq \tilde{B}_L^-(l), \tilde{B}_L^+(l) \geq \tilde{B}_L^+(l)$ and $J_N(l) \leq J(l)$ for all $l \in P$,

(ii) If $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ is an $BMBJ$-NSA of P, then $\mathcal{N} = (M_N, \tilde{B}_N, J_N)$ is an $BMBJ$-NSA of P

are valid.

Theorem 3.15. Given $m \in [0,T], b \in [0,\Pi], c \in [0,\pi]$ and $j \in [0,\perp]$, the (m,b,c,j)-$TBMBJNS(\mathcal{L}^T)$ of an $BMBJ$-NSA $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$ is an $BMBJ$-neutrosophic S-extension of $\mathcal{L} = (M_L, \tilde{B}_L, J_L)$.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

[1] Bijan Davvaz, Samy M. Mostafa and Fatema F. Kareem, *Neutrosophic ideals of neutrosophic KU-algebras*, GU J Sci., **30** (4), (2017), 463-472.

[2] M. Mohseni Takallo, R. A. Borzooei and Young Bae Jun, *MBJ-neutrosophic structures and its applications in BCK/BCI-algebras*, Neutrosophic Sets and Systems, **23**, (2018), 72-84.

[3] S. M. Mostafa, M. A. Abd-Elnaby and M. M. M. Yousef, *Fuzzy ideals of KU-algebras*, International Math Forum., **6** (63) (2011) 3139-3149.

[4] C. Prabpayak and U. Leerawat, *On ideals and congruence in KU-algebras*, Scientia Magna Journal, **5** (1) (2009), 54-57.

[5] C. Prabpayak and U. Leerawat, *On isomorphisms of KU-algebras*, Scientia Magna Journal, **5** (3) (2009), 25-31.

[6] F. Smarandache, *Neutrosophy, Neutrosophic Probability, Set, and Logic, ProQuest Information & Learning*, Ann Arbor, Michigan, USA, **105** (1998). http://fs.gallup.unm.edu/eBook-neutrosophics6.pdf(last edition online).

[7] F. Smarandache, *A unifying field in logics. Neutrosophy: Neutrosophic probabilFy, set and logic*, Rehoboth: American Research Press (1999).

[8] F. Smarandache, *Neutrosophic set, a generalization of intuitionistic fuzzy sets*, International Journal of Pure and Applied Mathematics, **24** (5) (2005), 287-297.

[9] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (3) (1965), 338-353.

Received: December 15, 2022 Accepted: April 12, 2023