Neutrosophic hesitant fuzzy UP (BCC)-filters

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Abstract

In this paper, we introduce the concept of neutrosophic hesitant fuzzy UP (BCC)-filters of UP (BCC)-algebras. The characteristic neutrosophic hesitant fuzzy UP (BCC)-filters have also been studied. The relationship between neutrosophic hesitant fuzzy UP (BCC)-filters and their level subsets is provided. The Cartesian product of neutrosophic hesitant fuzzy UP (BCC)-filters is also supplied. Finally, we also find the property of the homomorphic pre-image of neutrosophic hesitant fuzzy UP (BCC)-filters.
Keywords: UP-algebra; neutrosophic hesitant fuzzy filter; level subset; Cartesian product; homomorphic pre-image.

1 Introduction

The concept of fuzzy sets was proposed by Zadeh.[17] The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches such as soft sets and rough sets has been discussed in.[1–3] In 2009-2010, Torra and Narukawa[15,16] introduced the notion of hesitant fuzzy sets (HFSs), that is a function from a reference set to a power set of the unit interval. The notion of HFSs is the other generalization of the notion of fuzzy sets. The HFS theories developed by Torra and others have found many applications in the domain of mathematics and elsewhere. After the introduction of the notion of HFSs by Torra and Narukawa,[15,16] several pieces of research were conducted on the generalizations of the notion of HFSs and application to many logical algebras such as: in 2012, Zhu et al.[18] introduced the notion of dual HFSs, which is a new extension of fuzzy sets. In 2014, Jun, Ahn and Muhiuddin[8] introduced the notions of hesitant fuzzy soft subalgebras and (closed) hesitant fuzzy soft ideals in BCK/BCI-algebras. Jun and Song[10] introduced the notions of (Boolean, prime, ultra, good) hesitant fuzzy filters and hesitant fuzzy MV-filters of MTL-algebras. Iampan[6] introduced a new algebraic structure called a UP-algebra, and Mosrijai et al.[13] introduced the notion of HFSs on UP-algebras. Iampan et al.[7] introduced the concepts of neutrosophic hesitant fuzzy BCC-subalgebras, neutrosophic hesitant fuzzy BCC-ideals, and neutrosophic hesitant fuzzy strong BCC-ideals of BCC-algebras. The notions of hesitant fuzzy subalgebras, hesitant fuzzy filters and hesitant fuzzy UP-ideals play an important role in studying the many logical algebras. The notion of UP-algebras (see[6]) and the concept of BCC-algebras

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(see[12]) are the same concept, as shown by Jun et al.[9] in 2022. We shall refer to it as BCC rather than UP in this article out of respect for Komori, who initially described it in 1984.

The following research aims are included in this article:

(1) to describe the ideas behind neutrosophic hesitant fuzzy BCC-filters of BCC-algebras.

(2) to examine the properties of the characteristic neutrosophic hesitant fuzzy BCC-filters.

(3) to establish a connection between the level subsets of the neutrosophic hesitant fuzzy BCC filters.

(4) to determine the results of the Cartesian product of neutrosophic hesitant fuzzy BCC-filters.

(5) to find the properties of the homomorphic pre-image of neutrosophic hesitant fuzzy BCC-filters.

2 Preliminaries

The concept of BCC-algebras (see[12]) can be redefined without the condition (6) as follows:

An algebra $X = (X, \cdot, 0)$ of type $(2, 0)$ is called a *BCC-algebra* (see[5]) if it satisfies the following conditions:

$$(\forall x, y, z \in X)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0) \tag{1}$$
$$(\forall x \in X)(0 \cdot x = x) \tag{2}$$
$$(\forall x \in X)(x \cdot 0 = 0) \tag{3}$$
$$(\forall x, y \in X)(x \cdot y = 0 = y \cdot x \Rightarrow x = y) \tag{4}$$

After this, we assign X instead of a BCC-algebra $(X, \cdot, 0)$ until otherwise specified.

We define a binary relation $\leq$ on X as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 0) \tag{5}$$

In X, the following assertions are valid (see[6]).

$$(\forall x \in X)(x \leq x) \tag{6}$$
$$(\forall x, y, z \in X)(x \leq y, y \leq z \Rightarrow x \leq z) \tag{7}$$
$$(\forall x, y, z \in X)(x \leq y \Rightarrow z \cdot x \leq z \cdot y) \tag{8}$$
$$(\forall x, y, z \in X)(x \leq y \Rightarrow y \cdot z \leq x \cdot z) \tag{9}$$
$$(\forall x, y, z \in X)(x \leq y \cdot x, \text{ in particular, } y \cdot z \leq x \cdot (y \cdot z)) \tag{10}$$
$$(\forall x, y \in X)(y \cdot x \leq x \Leftrightarrow x = y \cdot x) \tag{11}$$
$$(\forall x, y \in X)(x \leq y \cdot y) \tag{12}$$
$$(\forall a, x, y, z \in X)(x \cdot (y \cdot z) \leq x \cdot ((a \cdot y) \cdot (a \cdot z))) \tag{13}$$
$$(\forall a, x, y, z \in X)(((a \cdot x) \cdot (a \cdot y)) \cdot z \leq (x \cdot y) \cdot z) \tag{14}$$
$$(\forall x, y, z \in X)((x \cdot y) \cdot z \leq y \cdot z) \tag{15}$$
$$(\forall x, y, z \in X)(x \leq y \Rightarrow x \leq z \cdot y) \tag{16}$$
$$(\forall x, y, z \in X)((x \cdot y) \cdot z \leq x \cdot (y \cdot z)) \tag{17}$$
$$(\forall a, x, y, z \in X)((x \cdot y) \cdot z \leq y \cdot (a \cdot z)) \tag{18}$$

Definition 2.1. [4,6,14] A nonempty subset S of X is called

(1) a *BCC-subalgebra* of X if

$$(\forall x, y \in S)(x \cdot y \in S), \tag{19}$$

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(2) a *BCC-ideal* of X if

$$0 \in S, \tag{20}$$
$$(\forall x, y, z \in X)(x \cdot (y \cdot z), y \in S \Rightarrow x \cdot z \in S), \tag{21}$$

(3) a *BCC-filter* of X if (20) and

$$(\forall x, y, z \in X)(x \cdot y, x \in S \Rightarrow y \in S), \tag{22}$$

(4) a *strong BCC-ideal* of X if (20) and

$$(\forall x, y, z \in X)((z \cdot y) \cdot (z \cdot x), y \in S \Rightarrow x \in S). \tag{23}$$

Definition 2.2. [15] A *hesitant fuzzy set* (HFS) on a reference set X is defined in term of a function h that when applied to X return a subset of $[0, 1]$, that is, $h : X \to \mathcal{P}([0, 1])$.

Definition 2.3. [15] An *neutrosophic hesitant fuzzy set* (NHFS) on a reference set X is defined in the form $\mathcal{N} = (h, k, n)$, where h, k, and n are functions that when applied to X return a subset of $[0, 1]$, that is, $h, k, n : X \to \mathcal{P}([0, 1])$.

Definition 2.4. [13] A HFS h on X is said to be

(1) a *hesitant fuzzy BCC-subalgebra* of X if the following condition holds:

$$(\forall x, y \in X)(h(x \cdot y) \supseteq h(x) \cap h(y)) \tag{24}$$

(2) a *hesitant fuzzy BCC-ideal* of X if the following conditions hold:

$$(\forall x \in X)(h(0) \supseteq h(x)) \tag{25}$$
$$(\forall x, y, z \in X)(h(x \cdot y) \supseteq h(x \cdot (y \cdot z)) \cap h(y)) \tag{26}$$

(3) a *hesitant fuzzy BCC-filter* of X if (25) and the following condition hold:

$$(\forall x, y, z \in X)(h(y) \supseteq h(x \cdot y) \cap h(x)) \tag{27}$$

(4) a *hesitant fuzzy strong BCC-ideal* of X if (25) and the following condition hold:

$$(\forall x, y, z \in X)(h(x) \supseteq h((z \cdot y) \cdot (z \cdot x)) \cap h(y)) \tag{28}$$

Definition 2.5. [15] The *complement* of a HFS h in a reference set X is the HFS $\overline{h}$ defined by $\overline{h}(x) = [0, 1] - h(x)$ for all $x \in X$.

Definition 2.6. [15] The *complement* of an NHFS $\mathcal{N} = (h, k, n)$ on a reference set X is the NHFS $\overline{\mathcal{N}} = (\overline{k}, \overline{h}, n)$.

3 Neutrosophic hesitant fuzzy BCC-filters

In this section, we introduce the concept of neutrosophic hesitant fuzzy BCC-filters of BCC-algebras and study their properties.

Definition 3.1. [7] An NHFS $\mathcal{N} = (h, k, n)$ on X is called a *neutrosophic hesitant fuzzy BCC-subalgebra* of X if it satisfies the following property:

$$(\forall x, y \in X)\begin{pmatrix} h(x \cdot y) \supseteq h(x) \cap h(y) \\ k(x \cdot y) \subseteq k(x) \cup k(y) \\ n(x \cdot y) \supseteq n(x) \cap n(y) \end{pmatrix} \tag{29}$$

Definition 3.2. [7] The *characteristic neutrosophic hesitant fuzzy set* (characteristic NHFS) of a subset A of a set X is defined to be the structure $\chi_A = (h_{\chi_A}, k_{\chi_A}, n_{\chi_A})$, where

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$$h_{\chi_A}(x) = \begin{cases} [0,1] & \text{if } x \in A \\ \emptyset & \text{otherwise} \end{cases}, k_{\chi_A}(x) = \begin{cases} \emptyset & \text{if } x \in A \\ [0,1] & \text{otherwise} \end{cases}, \text{ and } n_{\chi_A}(x) = \begin{cases} [0,1] & \text{if } x \in A \\ \emptyset & \text{otherwise} \end{cases}.$$

Lemma 3.3. [7] *The constant 0 of X is in a nonempty subset B of X if and only if $h_{\chi_B}(0) \supseteq h_{\chi_B}(x), k_{\chi_B}(0) \subseteq k_{\chi_B}(x)$, and $n_{\chi_B}(0) \supseteq n_{\chi_B}(x)$ for all $x \in X$.*

For the purposes of this article, the following definition is used.

Definition 3.4. An NHFS $\mathcal{N} = (h, k, n)$ on X is called a *neutrosophic hesitant fuzzy BCC-filter* of X if the following conditions hold:

$$(\forall x \in X) \begin{pmatrix} h(0) \supseteq h(x) \\ k(0) \subseteq k(x) \\ n(0) \supseteq n(x) \end{pmatrix} \tag{30}$$

$$(\forall x, y \in X) \begin{pmatrix} h(y) \supseteq h(x \cdot y) \cap h(x) \\ k(y) \subseteq k(x \cdot y) \cup k(x) \\ n(y) \supseteq n(x \cdot y) \cap n(x) \end{pmatrix} \tag{31}$$

Example 3.5. [11] Let $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	3
3	0	1	2	0	3
4	0	1	2	0	0

Then X is a BCC-algebra. We define an NHFS $\mathcal{N} = (h, k, n)$ on X as follows:

$$h = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.1, 0.5] & [0.1, 0.3] & [0.1, 0.2] & \{0.1\} & \{0.1\} \end{pmatrix}$$

$$k = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ \{0\} & [0, 0.3] & [0, 0.5] & [0, 0.6] & [0, 0.6] \end{pmatrix}$$

$$n = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0, 0.4] & [0, 0.4] & [0, 0.3] & \{0\} & \{0\} \end{pmatrix}$$

Then $\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-filter of X.

Definition 3.6. [7] An NHFS $\mathcal{N} = (h, k, n)$ on X is called a *neutrosophic hesitant fuzzy BCC-ideal* of X if (30) and the following condition hold:

$$(\forall x, y, z \in X) \begin{pmatrix} h(x \cdot z) \supseteq h(x \cdot (y \cdot z)) \cap h(y) \\ k(x \cdot z) \subseteq k(x \cdot (y \cdot z)) \cup k(y) \\ n(x \cdot z) \supseteq n(x \cdot (y \cdot z)) \cap n(y) \end{pmatrix} \tag{32}$$

Theorem 3.7. *Every neutrosophic hesitant fuzzy BCC-ideal of X is a neutrosophic hesitant fuzzy BCC-filter of X.*

Proof. Let $\mathcal{N} = (h, k, n)$ be a neutrosophic hesitant fuzzy BCC-ideal of X. Then (30) holds. Let $x, y \in X$. Then

$$\begin{array}{rcl} h(y) & = & h(0 \cdot y) \\ & \supseteq & h(0 \cdot (x \cdot y)) \cap h(x) \\ & = & h(x \cdot y) \cap h(x), \\ k(y) & = & k(0 \cdot y) \\ & \subseteq & k(0 \cdot (x \cdot y)) \cup k(x) \\ & = & k(x \cdot y) \cup k(x), \\ n(y) & = & n(0 \cdot y) \\ & \supseteq & n(0 \cdot (x \cdot y)) \cap n(x) \\ & = & n(x \cdot y) \cap n(x). \end{array}$$

Hence, $\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-filter of X. □

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Remark 3.8. The converse of Theorem 3.7 is not true in general. From Example 3.5, we have $\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-filter of X. Since $k(3 \cdot 4) = [0, 0.6] \nsubseteq [0, 0.5] = k(3 \cdot (2 \cdot 4)) \cup k(2)$, we have $\mathcal{N}$ is not a neutrosophic hesitant fuzzy BCC-ideal of X.

Theorem 3.9. *Every neutrosophic hesitant fuzzy BCC-filter of X is a neutrosophic hesitant fuzzy BCC-subalgebra of X.*

Proof. Let $\mathcal{N} = (h, k, n)$ be a neutrosophic hesitant fuzzy BCC-filter of X. Let $x, y \in X$. Then

$$\begin{aligned} h(x \cdot y) &\supseteq h(y \cdot (x \cdot y)) \cap h(y) \\ &= h(0) \cap h(y) \\ &= h(y) \\ &\supseteq h(x) \cap h(y), \end{aligned}$$

$$\begin{aligned} k(x \cdot y) &\subseteq k(y \cdot (x \cdot y)) \cup k(y) \\ &= k(0) \cup k(y) \\ &= k(y) \\ &\subseteq k(x) \cup k(y), \end{aligned}$$

$$\begin{aligned} n(x \cdot y) &\supseteq n(y \cdot (x \cdot y)) \cap n(y) \\ &= n(0) \cap n(y) \\ &= n(y) \\ &\supseteq n(x) \cap n(y). \end{aligned}$$

Hence, $\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X. □

The following example shows that the converse of Theorem 3.9 is not true in general.

Example 3.10. [11] Let $X = \{0, 1, 2, 3\}$ with the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

Then X is a BCC-algebra. We define an NHFS $\mathcal{N} = (h, k, n)$ on X as follows:

$$h = \begin{pmatrix} 0 & 1 & 2 & 3 \\ [0, 0.5] & [0, 0.3] & [0, 0.4] & [0, 0.3] \end{pmatrix}$$

$$k = \begin{pmatrix} 0 & 1 & 2 & 3 \\ [0, 0.1] & [0, 0.2] & [0, 0.2] & [0, 0.3] \end{pmatrix}$$

$$n = \begin{pmatrix} 0 & 1 & 2 & 3 \\ [0, 0.3] & [0, 0.3] & [0, 0.3] & [0, 0.2] \end{pmatrix}$$

Then $\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-subalgebra of X. But $n(3) = [0, 0.2] \nsupseteq [0, 0.3] = n(2 \cdot 3) \cap n(2)$, we have $\mathcal{N}$ is not a neutrosophic hesitant fuzzy BCC-filter of X.

Theorem 3.11. *A nonempty subset S of X is a BCC-filter of X if and only if the characteristic NHFS χ_S is a neutrosophic hesitant fuzzy BCC-filter of X.*

Proof. Assume that S is a BCC-filter of X. Since $0 \in S$, it follows from Lemma 3.3 that $h_{\chi_S}(0) \supseteq h_{\chi_S}(x)$ for all $x \in X$. Next, let $x, y \in X$.

Case 1 : If $x, y \in S$, then $h_{\chi_S}(x) = [0, 1]$ and $h_{\chi_S}(y) = [0, 1]$. Hence, $h_{\chi_S}(y) = [0, 1] \supseteq h_{\chi_S}(x \cdot y) = h_{\chi_S}(x \cdot y) \cap h_{\chi_S}(x)$. Also, $k_{\chi_S}(x) = \emptyset$ and $k_{\chi_S}(y) = \emptyset$. Hence, $k_{\chi_S}(y) = \emptyset \subseteq k_{\chi_S}(x \cdot y) = k_{\chi_S}(x \cdot y) \cup k_{\chi_S}(x)$. Also, $n_{\chi_S}(x) = [0, 1]$ and $n_{\chi_S}(y) = [0, 1]$. Hence, $n_{\chi_S}(y) = [0, 1] \supseteq n_{\chi_S}(x \cdot y) = n_{\chi_S}(x \cdot y) \cap n_{\chi_S}(x)$.

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Case 2 : If $x \notin S$ and $y \in S$, then $h_{\chi_S}(x) = \emptyset$ and $h_{\chi_S}(y) = [0, 1]$. Thus $h_{\chi_S}(y) = [0, 1] \supseteq \emptyset = h_{\chi_S}(x \cdot y) \cap h_{\chi_S}(x)$. Also, $k_{\chi_S}(x) = [0, 1]$ and $k_{\chi_S}(y) = \emptyset$. Thus $k_{\chi_S}(y) = \emptyset \subseteq [0, 1] = k_{\chi_S}(x \cdot y) \cup k_{\chi_S}(x)$. Also, $n_{\chi_S}(x) = \emptyset$ and $n_{\chi_S}(y) = [0, 1]$. Thus $n_{\chi_S}(y) = [0, 1] \supseteq \emptyset = n_{\chi_S}(x \cdot y) \cap n_{\chi_S}(x)$.

Case 3 : If $x \in S$ and $y \notin S$, then $h_{\chi_S}(x) = [0, 1]$ and $h_{\chi_S}(y) = \emptyset$. Since S is a BCC-filter of X, we have $x \cdot y \notin S$ or $x \notin S$. But $x \in S$, so $x \cdot y \notin S$. Then $h_{\chi_S}(x \cdot y) = \emptyset$. Thus $h_{\chi_S}(y) = \emptyset \supseteq \emptyset = h_{\chi_S}(x \cdot y) \cap h_{\chi_S}(x)$. Also, $k_{\chi_S}(x) = \emptyset$, $k_{\chi_S}(y) = [0, 1]$ and $k_{\chi_S}(x \cdot y) = [0, 1]$. Thus $k_{\chi_S}(y) = [0, 1] \subseteq [0, 1] = k_{\chi_S}(x \cdot y) \cup k_{\chi_S}(x)$. Also, $n_{\chi_S}(x) = [0, 1]$ and $n_{\chi_S}(y) = \emptyset$. Since S is a BCC-filter of X, we have $x \cdot y \notin S$ or $x \notin S$. But $x \in S$, so $x \cdot y \notin S$. Then $n_{\chi_S}(x \cdot y) = \emptyset$. Thus $n_{\chi_S}(y) = \emptyset \supseteq \emptyset = n_{\chi_S}(x \cdot y) \cap n_{\chi_S}(x)$.

Case 4 : If $x \notin S$ and $y \notin S$, then $h_{\chi_S}(x) = \emptyset$ and $h_{\chi_S}(y) = \emptyset$. Thus $h_{\chi_S}(y) = \emptyset \subseteq \emptyset = h_{\chi_S}(x \cdot y) \cap h_{\chi_S}(x)$. Also, $k_{\chi_S}(x) = [0, 1]$ and $k_{\chi_S}(y) = [0, 1]$. Thus $k_{\chi_S}(y) = [0, 1] \subseteq [0, 1] = k_{\chi_S}(x \cdot y) \cup k_{\chi_S}(x)$. Also, $n_{\chi_S}(x) = \emptyset$ and $n_{\chi_S}(y) = \emptyset$. Thus $n_{\chi_S}(y) = \emptyset \subseteq \emptyset = n_{\chi_S}(x \cdot y) \cap n_{\chi_S}(x)$.

Hence, χ_S is a neutrosophic hesitant fuzzy BCC-filter of X.

Conversely, assume that χ_S is a neutrosophic hesitant fuzzy BCC-filter of X. Since $h_{\chi_S}(0) \supseteq h_{\chi_S}(x)$ for all $x \in X$, it follows from Lemma 3.3 that $0 \in S$. Next, let $x, y \in X$ be such that $x \cdot y \in S$ and $x \in S$. Then $h_{\chi_S}(x \cdot y) = [0, 1]$ and $h_{\chi_S}(x) = [0, 1]$. Thus $h_{\chi_S}(y) \supseteq h_{\chi_S}(x \cdot y) \cap h_{\chi_S}(x) = [0, 1]$, so $h_{\chi_S}(y) = [0, 1]$. Therefore, $y \in S$ and so S is a BCC-filter of X. □

Definition 3.12. [7] An NHFS $\mathcal{N} = (h, k, n)$ on X is called a *prime NHFS* on X if it satisfies the following property:

$$(\forall x, y \in X) \begin{pmatrix} h(x \cdot y) \subseteq h(x) \cup h(y) \\ k(x \cdot y) \supseteq k(x) \cap k(y) \\ n(x \cdot y) \subseteq n(x) \cup n(y) \end{pmatrix} \tag{33}$$

Theorem 3.13. [7] *A nonempty subset S of X is a prime subset of X if and only if the characteristic NHFS χ_S is a prime NHFS on X.*

Theorem 3.14. *An NHFS $\mathcal{N} = (h, k, n)$ on X is a prime neutrosophic hesitant fuzzy BCC-filter of X if and only if h, k, and n are constant HFSs on X.*

Proof. Assume that $\mathcal{N} = (h, k, n)$ is a prime neutrosophic hesitant fuzzy BCC-filter of X. Then $h(0) \supseteq h(x)$, $k(0) \subseteq k(x)$, and $n(0) \supseteq n(x)$ for all $x \in X$. By (6), we have $h(0) = h(x \cdot x) \subseteq h(x) \cup h(x) = h(x)$, $k(0) = k(x \cdot x) \supseteq k(x) \cup k(x) = k(x)$, and $n(0) = n(x \cdot x) \subseteq n(x) \cup n(x) = n(x)$ for all $x \in X$ and so $h(x) = h(0)$, $k(x) = k(0)$, and $n(x) = n(0)$ for all $x \in X$. Hence, h, k, and n are constant HFSs on X.

Assume that h, k, and n are constant HFSs on X. Hence, we can easily show that $\mathcal{N}$ is a prime neutrosophic hesitant fuzzy BCC-filter of X. □

Definition 3.15. [4] A nonempty subset S of X is called a *weakly prime subset* of X if it satisfies the following property:

$$(\forall x, y \in X, x \neq y)(x \cdot y \in S \Rightarrow x \in S \text{ or } y \in S)$$

Definition 3.16. [4] A BCC-filter S of X is called a *weakly prime BCC-filter* of X if S is a weakly prime subset of X.

Definition 3.17. An NHFS $\mathcal{N} = (h, k, n)$ on X is called a *weakly prime NHFS* on X if it satisfies the following property:

$$(\forall x, y \in X, x \neq y) \begin{pmatrix} h(x \cdot y) \subseteq h(x) \cup h(y) \\ k(x \cdot y) \supseteq k(x) \cap k(y) \\ n(x \cdot y) \subseteq n(x) \cup n(y) \end{pmatrix} \tag{34}$$

Definition 3.18. A neutrosophic hesitant fuzzy BCC-filter $\mathcal{N} = (h, k, n)$ of X is called a *weak prime neutrosophic hesitant fuzzy BCC-filter* of X if $\mathcal{N}$ is a weakly prime NHFS on X.

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Theorem 3.19. [7] *A nonempty subset S of X is a weakly prime subset of X if and only if the characteristic NHFS χ_S is a weakly prime NHFS on X.*

Theorem 3.20. *A nonempty subset S of X is a weakly prime BCC-filter of X if and only if the characteristic NHFS χ_S is a weakly prime neutrosophic hesitant fuzzy BCC-filter on X.*

Proof. It is straightforward by Theorems 3.11 and 3.19. □

Theorem 3.21. *An NHFS $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-filter of X if and only if the HFSs $h, \overline{k}$, and n are hesitant fuzzy BCC-filters of X.*

Proof. Assume that $\mathcal{N} = (h, k, n)$ is a neutrosophic fuzzy BCC-filter of X. Then for any $x, y \in X$, we have $h(0) \supseteq h(x)$, $h(y) \supseteq h(x \cdot y) \cap h(x)$, $n(0) \supseteq n(x)$, and $n(y) \supseteq n(x \cdot y) \cap n(x)$. Hence, h and n are hesitant fuzzy BCC-filters of X. Now for any $x, y \in X$, we have $k(0) \subseteq k(x)$ and $k(y) \subseteq k(x \cdot y) \cup k(x)$. Then $\overline{k}(0) = [0, 1] - k(0) \supseteq [0, 1] - k(x) = \overline{k}(x)$ and

$$\begin{aligned} \overline{k}(y) &= [0,1] - k(y) \\ &\supseteq [0,1] - (k(x \cdot y) \cup k(x)) \\ &= [0,1] - k(x \cdot y) \cap [0,1] - k(x) \\ &= \overline{k}(x \cdot y) \cap \overline{k}(x). \end{aligned}$$

Hence, $\overline{k}$ is a hesitant fuzzy BCC-filter of X.

Conversely, assume that the HFSs $h, \overline{k}$, and n are hesitant fuzzy BCC-filters of X. Then for any $x, y \in X$, we have $h(0) \supseteq h(x)$, $h(y) \supseteq h(x \cdot y) \cap h(x)$, $n(0) \supseteq n(x)$, and $n(y) \supseteq n(x \cdot y) \cap n(x)$. Now for any $x, y \in X$, we have $\overline{k}(0) \supseteq \overline{k}(x)$ and $\overline{k}(y) \supseteq \overline{k}(x \cdot y) \cap \overline{k}(x)$. Then $[0, 1] - k(0) \supseteq [0, 1] - k(x)$ and so $k(0) \subseteq k(x)$. Now, $[0, 1] - k(y) \supseteq [0, 1] - k(x \cdot y) \cap [0, 1] - k(x) = [0, 1] - (k(x \cdot y) \cup k(x))$, so $k(y) \subseteq k(x \cdot y) \cup k(x)$. Hence, $\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-filter of X. □

Theorem 3.22. *An NHFS $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-filter of X if and only if the NHFS $\overline{\mathcal{N}} = (\overline{k}, \overline{h}, n)$ is a neutrosophic hesitant fuzzy BCC-filter of X.*

Proof. Assume that $\mathcal{N} = (h, k, n)$ is a neutrosophic fuzzy BCC-filter of X. Then for any $x, y, z \in X$, we have $h(0) \supseteq h(x)$ and $h(y) \supseteq h(x \cdot y) \cap h(x)$, $n(0) \supseteq n(x)$, and $n(y) \supseteq n(x \cdot y) \cap n(x)$. Hence, for any $x, y, z \in X$, we have $\overline{h}(0) = [0, 1] - h(0) \subseteq [0, 1] - h(x) = \overline{h}(x)$ and

$$\begin{aligned} \overline{h}(y) &= [0,1] - h(y) \\ &\subseteq [0,1] - (h(x \cdot y) \cap h(x)) \\ &= [0,1] - h(x \cdot y) \cup [0,1] - h(x) \\ &= \overline{h}(x \cdot y) \cup \overline{h}(x). \end{aligned}$$

Now for any $x, y, z \in X$, we have $k(0) \subseteq k(x)$ and $k(y) \subseteq k(x \cdot y) \cup k(x)$. Hence, for any $x, y, z \in X$, we have $\overline{k}(0) = [0, 1] - k(0) \supseteq [0, 1] - k(x) = \overline{k}(x)$ and

$$\begin{aligned} \overline{k}(y) &= [0,1] - k(y) \\ &\supseteq [0,1] - (k(x \cdot y) \cup k(x)) \\ &= [0,1] - k(x \cdot y) \cap [0,1] - k(x) \\ &= \overline{k}(x \cdot y) \cap \overline{k}(x). \end{aligned}$$

Hence, $\overline{\mathcal{N}} = (\overline{k}, \overline{h}, n)$ is a neutrosophic hesitant fuzzy BCC-filter of X.

Conversely, assume that the NHFS $\overline{\mathcal{N}} = (\overline{k}, \overline{h}, n)$ is a neutrosophic hesitant fuzzy BCC-filter of X. Then for any $x, y, z \in X$, we have $\overline{k}(0) \supseteq \overline{k}(x)$ and $\overline{k}(y) \supseteq \overline{k}(x \cdot y) \cap \overline{k}(x)$. Then $[0, 1] - k(0) \supseteq [0, 1] - k(x)$ and $[0, 1] - k(y) \supseteq [0, 1] - (k(x \cdot y) \cup k(x))$, so $k(0) \subseteq k(x)$ and $k(y) \subseteq k(x \cdot y) \cup k(x)$. Now for any $x, y, z \in X$, we have $\overline{h}(0) \subseteq \overline{h}(x)$ and $\overline{h}(y) \subseteq \overline{h}(x \cdot y) \cup \overline{h}(x)$. Then $[0, 1] - h(0) \subseteq [0, 1] - h(x)$ and $[0, 1] - h(y) \supseteq [0, 1] - (h(x \cdot y) \cup h(x))$, so $h(0) \supseteq h(x)$ and $h(y) \supseteq h(x \cdot y) \cap h(x)$. Hence, $\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-filter of X. □

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Definition 3.23. Let $\mathcal{N} = (h, k, n)$ be an NHFS on a set X. The NHFSs $\oplus\mathcal{N}, \otimes\mathcal{N}$, and $\odot\mathcal{N}$ are defined as $\oplus\mathcal{N} = (h, \overline{h}, n), \otimes\mathcal{N} = (\overline{k}, k, n)$, and $\otimes\mathcal{N} = (\overline{k}, k, \overline{k})$.

Theorem 3.24. *If $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-filter of X, then the sets $X_h := \{x \in X \mid h(x) = h(0)\}, X_k := \{x \in X \mid k(x) = k(0)\}$, and $X_n := \{x \in X \mid n(x) = n(0)\}$ are BCC-filters of X.*

Proof. Assume that $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-filter of X. Clearly, $0 \in X_h \cap X_k \cap X_n$. Let $x, y \in X$ be such that $x \cdot y, x \in X_h$. Then $h(x \cdot y) = h(0)$ and $h(x) = h(0)$. By (31), we have $h(y) \supseteq h(x \cdot y) \cap h(x) = h(0)$, whence $h(y) = h(0)$, by (30). This means that $y \in X_h$. Hence, X_h is a BCC-filter of X. Let $x, y \in X$ be such that $x \cdot y, x \in X_k$. Then $k(x \cdot y) = k(0)$ and $k(x) = k(0)$. By (31), we have $k(y) \subseteq k(x \cdot y) \cup k(x) = k(0)$, whence $k(y) = k(0)$, by (30). This means that $y \in X_k$. Hence, X_k is a BCC-filter of X. Let $x, y \in X$ be such that $x \cdot y, x \in X_n$. Then $n(x \cdot y) = n(0)$ and $n(x) = n(0)$. By (31), we have $n(y) \supseteq n(x \cdot y) \cap n(x) = n(0)$, whence $n(y) = n(0)$, by (30). This means that $y \in X_n$. Hence, X_n is a BCC-filter of X. □

Theorem 3.25. *An NHFS $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-filter of X if and only if the NHFSs $\oplus\mathcal{N}, \otimes\mathcal{N}$, and $\otimes\mathcal{N}$ are neutrosophic hesitant fuzzy BCC-filters of X.*

Proof. Assume that $\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-filter of X. Let $x \in X$. Then $\overline{h}(1) = [0, 1] - h(0) \subseteq [0, 1] - h(x) = \overline{h}(x)$. Let $x, y \in X$. Then

$$\begin{array}{rcl} \overline{h}(y) & = & [0,1] - h(y) \\ & \subseteq & [0,1] - (h(x \cdot y) \cap h(x)) \\ & = & ([0,1] - h(x \cdot y)) \cup ([0,1] - h(x)) \\ & = & \overline{h}(x \cdot y) \cup \overline{h}(x). \end{array}$$

Hence, $\oplus\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-filter of X. Let $x \in X$. Then $\overline{k}(1) = [0, 1] - k(0) \supseteq [0, 1] - k(x) = \overline{k}(x)$. Let $x, y \in X$. Then

$$\begin{array}{rcl} \overline{k}(y) & = & [0,1] - k(y) \\ & \supseteq & [0,1] - (k(x \cdot y) \cup k(x)) \\ & = & ([0,1] - k(x \cdot y)) \cap ([0,1] - k(x)) \\ & = & \overline{k}(x \cdot y) \cap \overline{k}(x). \end{array}$$

Hence, $\otimes\mathcal{N}$ and $\odot\mathcal{N}$ are neutrosophic hesitant fuzzy BCC-filters of X.

Conversely, assume that $\oplus\mathcal{N}, \otimes\mathcal{N}$, and $\odot\mathcal{N}$ are neutrosophic hesitant fuzzy BCC-filters of X. Then for any $x, y \in X$, we have $h(0) \supseteq h(x), h(y) \supseteq h(x \cdot y) \cap h(x), k(0) \subseteq k(x), k(y) \subseteq k(x \cdot y) \cup k(x), n(0) \supseteq n(x)$, and $n(y) \supseteq n(x \cdot y) \cap n(x)$. Hence, $\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-filter of X. □

Lemma 3.26. *If $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-filter of X, then*

$$(\forall x, y, z \in X) \left(z \leq x \cdot y \Rightarrow \left\{ \begin{array}{l} h(y) \supseteq h(x) \cap h(z) \\ k(y) \subseteq k(x) \cup k(z) \\ n(y) \supseteq n(x) \cap n(z) \end{array} \right. \right). \tag{35}$$

Proof. Assume that $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-filter of X. Let $x, y, z \in X$ be such that $z \leq x \cdot y$. Then $z \cdot (x \cdot y) = 0$. Thus

$$\begin{array}{rcl} h(y) & \supseteq & h(x \cdot y) \cap h(x) \\ & \supseteq & h(z \cdot (x \cdot y)) \cap h(z) \cap h(x) \\ & = & h(0) \cap h(z) \cap h(x) \\ & = & h(x) \cap h(z), \end{array}$$

$$\begin{array}{rcl} k(y) & \subseteq & k(x \cdot y) \cup k(x) \\ & \subseteq & k(z \cdot (x \cdot y)) \cup k(z) \cup k(x) \\ & = & k(0) \cup k(z) \cup k(x) \\ & = & k(x) \cup k(z), \end{array}$$

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$$\begin{aligned} n(y) &\supseteq n(x \cdot y) \cap n(x) \\ &\supseteq n(z \cdot (x \cdot y)) \cap n(z) \cap n(x) \\ &= n(0) \cap n(z) \cap n(x) \\ &= n(x) \cap n(z). \end{aligned}$$

□

Lemma 3.27. *If $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-filter of X, then*

$$(\forall x, y \in X) \left(x \leq y \Rightarrow \left\{ \begin{array}{l} h(y) \supseteq h(x) \\ k(y) \subseteq k(x) \\ n(y) \supseteq n(x) \end{array} \right. \right). \tag{36}$$

Proof. Assume that $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-filter of X. Let $x, y \in X$ be such that $x \leq y$. Then $x \cdot y = 0$ and so

$$h(y) \supseteq h(x \cdot y) \cap h(x) = h(0) \cap h(x) = h(x),$$

$$k(y) \subseteq k(x \cdot y) \cup k(x) = k(0) \cup k(x) = k(x),$$

$$n(y) \supseteq n(x \cdot y) \cap n(x) = n(0) \cap n(x) = n(x).$$

□

Definition 3.28. Let $h : X \to \mathcal{P}([0, 1])$. For any $\pi \in \mathcal{P}([0, 1])$, the sets $U(h, \pi) = \{x \in X \mid h(x) \supseteq \pi\}$ and $U^+(h, \pi) = \{x \in X \mid h(x) \supset \pi\}$ are called an upper π-level subset and an upper π-strong level subset of h, respectively. The sets $L(h, \pi) = \{x \in X \mid h(x) \subseteq \pi\}$ and $L^-(h, \pi) = \{x \in X \mid h(x) \subset \pi\}$ are called a lower π-level subset and a lower π-strong level subset of h, respectively. The set $E(h, \pi) = \{x \in X \mid h(x) = \pi\}$ is called an equal π-level subset of h. Then $U(h, \pi) = U^+(h, \pi) \cup E(h, \pi)$ and $L(h, \pi) = L^-(h, \pi) \cup E(h, \pi)$.

Theorem 3.29. *An NHFS $\mathcal{N} = (h, k, n)$ on X is a neutrosophic hesitant fuzzy BCC-filter of X if and only if for all $\pi \in \mathcal{P}([0, 1])$, the nonempty subsets $U(h, \pi), L(k, \pi)$, and $U(n, \pi)$ of X are BCC-filters.*

Proof. Assume that $\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-filter of X. Let $\pi \in \mathcal{P}([0, 1])$ be such that $U(h, \pi) \neq \emptyset$ and let $x \in U(h, \pi)$. Then $h(x) \supseteq \pi$. By (30), we have $h(0) \supseteq h(x) \supseteq \pi$. Thus $0 \in U(h, \pi)$. Next, let $x, y \in X$ be such that $x, x \cdot y \in U(h, \pi)$. Then $h(x) \supseteq \pi$ and $h(x \cdot y) \supseteq \pi$. By (31), we have $h(y) \supseteq h(x \cdot y) \cap h(x) \supseteq \pi$. So $y \in U(h, \pi)$. Let $\pi \in \mathcal{P}([0, 1])$ be such that $L(k, \pi) \neq \emptyset$ and let $x \in L(k, \pi)$. Then $k(x) \subseteq \pi$. By (30), we have $k(0) \subseteq k(x) \subseteq \pi$. Thus $0 \in L(k, \pi)$. Next, let $x, y \in X$ be such that $x, x \cdot y \in L(k, \pi)$. Then $k(x) \subseteq \pi$ and $k(x \cdot y) \subseteq \pi$. By (31), we have $k(y) \subseteq k(x \cdot y) \cup k(x) \subseteq \pi$. So $y \in L(k, \pi)$. Let $\pi \in \mathcal{P}([0, 1])$ be such that $U(n, \pi) \neq \emptyset$ and let $x \in U(n, \pi)$. Then $n(x) \supseteq \pi$. By (30), we have $n(0) \supseteq n(x) \supseteq \pi$. Thus $0 \in U(n, \pi)$. Next, let $x, y \in X$ be such that $x, x \cdot y \in U(n, \pi)$. Then $n(x) \supseteq \pi$ and $n(x \cdot y) \supseteq \pi$. By (31), we have $n(y) \supseteq n(x \cdot y) \cap n(x) \supseteq \pi$. So $y \in U(n, \pi)$.

Conversely, assume that for all $\pi \in \mathcal{P}([0, 1])$, the nonempty subsets $U(h, \pi), L(k, \pi)$, and $U(n, \pi)$ of X are BCC-filters. Let $x \in X$. Then $h(x) \in \mathcal{P}([0, 1])$. Choose $\pi = h(x) \in \mathcal{P}([0, 1])$. Then $h(x) \supseteq \pi$. Thus $x \in U(h, \pi)$. By assumption, we have $U(h, \pi)$ is a BCC-filter of X and thus $0 \in U(h, \pi)$. So $h(0) \supseteq \pi = h(x)$. Let $x, y \in X$. Then $h(x), h(x \cdot y) \in \mathcal{P}([0, 1])$. Choose $\pi = h(x) \cap h(x \cdot y) \in \mathcal{P}([0, 1])$. Then $h(x) \supseteq \pi$ and $h(x \cdot y) \supseteq \pi$. Since $x, x \cdot y \in U(h, \pi) \neq \emptyset$. By assumption, we have $U(h, \pi)$ is a BCC-filter of X and then $y \in U(h, \pi)$. Thus $h(y) \supseteq \pi = h(x) \cap h(x \cdot y)$. Let $x \in X$. Then $k(x) \in \mathcal{P}([0, 1])$. Choose $\pi_1 = k(x) \in \mathcal{P}([0, 1])$. Then $k(x) \subseteq \pi_1$. Thus $x \in L(k, \pi_1)$. By assumption, we have $L(k, \pi_1)$ is a BCC-filter of X and thus $0 \in L(k, \pi_1)$. So $k(0) \subseteq \pi_1 = k(x)$. Let $x, y \in X$. Then $k(x), k(x \cdot y) \in \mathcal{P}([0, 1])$. Choose $\pi_1 = k(x) \cup k(x \cdot y) \in \mathcal{P}([0, 1])$. Then $k(x) \subseteq \pi_1$ and $k(x \cdot y) \subseteq \pi_1$. Since $x, x \cdot y \in L(k, \pi_1) \neq \emptyset$. By assumption, we have $L(k, \pi_1)$ is a BCC-filter of X and then $y \in L(k, \pi_1)$. Thus $k(y) \subseteq \pi_1 = k(x) \cup k(x \cdot y)$. Also let $x \in X$. Then $n(x) \in \mathcal{P}([0, 1])$. Choose $\pi_2 = n(x) \in \mathcal{P}([0, 1])$. Then $n(x) \supseteq \pi_2$. Thus $x \in U(n, \pi_2)$. By assumption, we have $U(n, \pi_2)$ is a BCC-filter of X and thus $0 \in U(n, \pi_2)$. So $n(0) \supseteq \pi_2 = n(x)$. Let $x, y \in X$. Then $n(x), n(x \cdot y) \in \mathcal{P}([0, 1])$. Choose $\pi_3 = n(x) \cap n(x \cdot y) \in \mathcal{P}([0, 1])$. Then $n(x) \supseteq \pi_3$ and $n(x \cdot y) \supseteq \pi_3$. Since $x, x \cdot y \in U(n, \pi_3) \neq \emptyset$. By assumption, we have $U(n, \pi_3)$ is a BCC-filter of X and then $y \in U(n, \pi_3)$. Thus $n(y) \supseteq \pi_3 = n(x) \cap n(x \cdot y)$. Hence, $\mathcal{N}$ is a neutrosophic hesitant fuzzy BCC-filter of X. □

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Definition 3.30. Let $\{\mathcal{N}_\alpha \mid \alpha \in \Delta\}$ be a family of NHFSs on a reference set X. We define the NHFS $\bigcap_{\alpha\in\Delta} \mathcal{N}_\alpha = (\bigcap_{\alpha\in\Delta} h_\alpha, \bigcup_{\alpha\in\Delta} k_\alpha, \bigcap_{\alpha\in\Delta} n_\alpha)$ by $(\bigcap_{\alpha\in\Delta} h_\alpha)(x) = \bigcap_{\alpha\in\Delta} h_\alpha(x), (\bigcup_{\alpha\in\Delta} k_\alpha)(x) = \bigcup_{\alpha\in\Delta} k_\alpha(x)$, and $(\bigcap_{\alpha\in\Delta} n_\alpha)(x) = \bigcap_{\alpha\in\Delta} n_\alpha(x)$ for all $x \in X$, which is called the *neutrosophic hesitant intersection of NHFSs*.

Proposition 3.31. *If $\{\mathcal{N}_\alpha \mid \alpha \in \Delta\}$ is a family of neutrosophic hesitant fuzzy BCC-filters of X, then $\bigcap_{\alpha\in\Delta} \mathcal{N}_\alpha$ is a neutrosophic hesitant fuzzy BCC-filter of X.*

Proof. Let $\{\mathcal{N}_\alpha \mid \alpha \in \Delta\}$ be a family of neutrosophic hesitant fuzzy BCC-filters of X. Let $x \in X$. Then

$$(\bigcap_{\alpha\in\Delta} h_\alpha)(0) = \bigcap_{\alpha\in\Delta} h_\alpha(0) \supseteq \bigcap_{\alpha\in\Delta} h_\alpha(x) = (\bigcap_{\alpha\in\Delta} h_\alpha)(x),$$

$$(\bigcup_{\alpha\in\Delta} k_\alpha)(0) = \bigcup_{\alpha\in\Delta} k_\alpha(0) \subseteq \bigcup_{\alpha\in\Delta} k_\alpha(x) = (\bigcup_{\alpha\in\Delta} k_\alpha)(x),$$

$$(\bigcap_{\alpha\in\Delta} n_\alpha)(0) = \bigcap_{\alpha\in\Delta} n_\alpha(0) \supseteq \bigcap_{\alpha\in\Delta} n_\alpha(x) = (\bigcap_{\alpha\in\Delta} n_\alpha)(x).$$

Let $x, y \in X$. Then

$$\begin{aligned}
(\bigcap_{\alpha\in\Delta} h_\alpha)(y) &= \bigcap_{\alpha\in\Delta} h_\alpha(y) \\
&\supseteq \bigcap_{\alpha\in\Delta} (h_\alpha(x\cdot y) \cap h_\alpha(x)) \\
&= (\bigcap_{\alpha\in\Delta} h_\alpha(x\cdot y)) \cap (\bigcap_{\alpha\in\Delta} h_\alpha(x)) \\
&= (\bigcap_{\alpha\in\Delta} h_\alpha)(x\cdot y) \cap (\bigcap_{\alpha\in\Delta} h_\alpha)(x), \\
(\bigcup_{\alpha\in\Delta} k_\alpha)(y) &= \bigcup_{\alpha\in\Delta} k_\alpha(y) \\
&\subseteq \bigcup_{\alpha\in\Delta} (k_\alpha(x\cdot y) \cup k_\alpha(x)) \\
&= \bigcup_{\alpha\in\Delta} k_\alpha(x\cdot y) \cup \bigcup_{\alpha\in\Delta} k_\alpha(x) \\
&= (\bigcup_{\alpha\in\Delta} k_\alpha)(x\cdot y) \cup (\bigcup_{\alpha\in\Delta} k_\alpha)(x), \\
(\bigcap_{\alpha\in\Delta} n_\alpha)(y) &= \bigcap_{\alpha\in\Delta} n_\alpha(y) \\
&\supseteq \bigcap_{\alpha\in\Delta} (n_\alpha(x\cdot y) \cap n_\alpha(x)) \\
&= (\bigcap_{\alpha\in\Delta} n_\alpha(x\cdot y)) \cap (\bigcap_{\alpha\in\Delta} n_\alpha(x)) \\
&= (\bigcap_{\alpha\in\Delta} n_\alpha)(x\cdot y) \cap (\bigcap_{\alpha\in\Delta} n_\alpha)(x).
\end{aligned}$$

Hence, $\bigcap_{\alpha\in\Delta} \mathcal{N}_\alpha$ is a neutrosophic hesitant fuzzy BCC-filter of X. □

Definition 3.32. Let $A = (h_A, k_A, n_A)$ and $B = (h_B, k_B, k_B)$ be NHFSs on sets X and Y, respectively. The Cartesian product $A \times B = (h, k, n)$ defined by $h(x, y) = h_A(x) \cap h_B(y), k(x, y) = k_A(x) \cup k_B(y)$, and $n(x, y) = n_A(x) \cap n_B(y)$, where $h : X \times Y \to \mathcal{P}([0, 1])$ and $k : X \times Y \to \mathcal{P}([0, 1])$ for all $x \in X$ and $y \in Y$.

Let $(X, \cdot, 0_X)$ and $(Y, \star, 0_Y)$ be BCC-algebras. Then $(X \times Y, \diamond, (0_X, 0_Y))$ is a BCC-algebra defined by $(x, y) \diamond (u, v) = (x \cdot u, y \star v)$ for every $x, u \in X$ and $y, v \in Y$.

Proposition 3.33. *If $A = (h_A, k_A, n_A)$ and $B = (h_B, k_B, k_B)$ are neutrosophic hesitant fuzzy BCC-filters of BCC-algebras X and Y, respectively, then the Cartesian product $A \times B$ is a neutrosophic hesitant fuzzy BCC-filter of $X \times Y$.*

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Proof. Assume that $A = (h_A, k_A, n_A)$ and $B = (h_B, k_B, k_B)$ are neutrosophic hesitant fuzzy BCC-filters of BCC-algebras X and Y, respectively. Let $(0_X, 0_Y), (x, y) \in X \times Y$. Then

$$\begin{aligned} h(0_X, 0_Y) &= h_A(0_X) \cap h_B(0_Y) \\ &\supseteq h_A(x) \cap h_B(y) \\ &= h(x, y), \end{aligned}$$

$$\begin{aligned} k(0_X, 0_Y) &= k_A(0_X) \cup k_B(0_Y) \\ &\subseteq k_A(x) \cup k_B(y) \\ &= k(x, y), \end{aligned}$$

$$\begin{aligned} n(0_X, 0_Y) &= n_A(0_X) \cap n_B(0_Y) \\ &\supseteq n_A(x) \cap n_B(y) \\ &= n(x, y). \end{aligned}$$

Let $(x_1, x_2), (y_1, y_2) \in X \times Y$. Then

$$\begin{aligned} h(y_1, y_2) &= h_A(y_1) \cap h_B(y_2) \\ &\supseteq (h_A(x_1 \cdot y_1) \cap h_A(x_1)) \cap (h_B(x_2 \star y_2) \cap h_B(x_2)) \\ &= h_A(x_1 \cdot y_1) \cap h_B(x_2 \star y_2) \cap h_A(x_1) \cap h_B(x_2) \\ &= h(x_1 \cdot y_1, x_2 \star y_2) \cap h(x_1, x_2) \\ &= h((x_1, x_2) \diamond (y_1, y_2)) \cap h(x_1, x_2), \end{aligned}$$

$$\begin{aligned} k(y_1, y_2) &= k_A(y_1) \cup k_B(y_2) \\ &\subseteq (k_A(x_1 \cdot y_1) \cup k_A(x_1)) \cup (k_B(x_2 \star y_2) \cup k_B(x_2)) \\ &= k_A(x_1 \cdot y_1) \cup k_B(x_2 \star y_2) \cup k_A(x_1) \cup k_B(x_2) \\ &= k(x_1 \cdot y_1, x_2 \star y_2) \cup k(x_1, x_2) \\ &= k((x_1, x_2) \diamond (y_1, y_2)) \cup k(x_1, x_2), \end{aligned}$$

$$\begin{aligned} n(y_1, y_2) &= n_A(y_1) \cap n_B(y_2) \\ &\supseteq (n_A(x_1 \cdot y_1) \cap n_A(x_1)) \cap (n_B(x_2 \star y_2) \cap n_B(x_2)) \\ &= n_A(x_1 \cdot y_1) \cap n_B(x_2 \star y_2) \cap n_A(x_1) \cap n_B(x_2) \\ &= n(x_1 \cdot y_1, x_2 \star y_2) \cap n(x_1, x_2) \\ &= n((x_1, x_2) \diamond (y_1, y_2)) \cap n(x_1, x_2). \end{aligned}$$

Hence, $A \times B$ is a neutrosophic hesitant fuzzy BCC-filter of $X \times Y$. □

Theorem 3.34. *Two NHFSs $A = (h_A, k_A, n_A)$ and $B = (h_B, k_B, k_B)$ are neutrosophic hesitant fuzzy BCC-filters of BCC-algebras X and Y, respectively if and only if the NHFSs $\oplus(A \times B), \otimes(A \times B)$, and $\odot(A \times B)$ are neutrosophic hesitant fuzzy BCC-filters of $X \times Y$.*

Proof. It follows from Proposition 3.33 and Theorem 3.25. □

A mapping $f : (X, \cdot, 0_X) \to (Y, \star, 0_Y)$ of BCC-algebras is called a *homomorphism* if $f(x \cdot y) = f(x) \star f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of BCC-algebras, then $f(0_X) = 0_Y$.

Definition 3.35. [7] Let f be a function from a nonempty set X to a nonempty set Y. If $\mathcal{N} = (h, k, n)$ is an NHFS on Y, then the NHFS $f^{-1}(\mathcal{N}) = (h \circ f, k \circ f, n \circ f)$ on X is called the *pre-image of $\mathcal{N}$ under f*.

Theorem 3.36. *Let $f : (X, \cdot, 0_X) \to (Y, \star, 0_Y)$ be a homomorphism of BCC-algebras. If $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-filter of Y, then $f^{-1}(\mathcal{N}) = (h \circ f, k \circ f, n \circ f)$ is a neutrosophic hesitant fuzzy BCC-filter of X.*

Proof. Assume that $\mathcal{N} = (h, k, n)$ is a neutrosophic hesitant fuzzy BCC-filter of Y. By assumption, we have $h(f(0_X)) = h(0_X) \supseteq h(y)$ for every $y \in Y$. In particular, $(h \circ f)(0_X) = h(f(0_X)) \supseteq h(f(x)) = (h \circ f)(x)$ for all $x \in X$. Also, $k(f(0_X)) = k(0_Y) \subseteq k(y)$ and $n(f(0_X)) = n(0_Y) \supseteq n(y)$ for every $y \in Y$. In particular, $(k \circ f)(0_X) = k(f(0_X)) \subseteq k(f(x)) = (k \circ f)(x)$ and $(n \circ f)(0_X) = n(f(0_X)) \supseteq n(f(x)) = (n \circ f)(x)$ for all $x \in X$. Let $x, y \in X$. Then

$$\begin{aligned} (h \circ f)(y) &= h(f(y)) \\ &\supseteq h(f(x) \star f(y)) \cap h(f(x)) \\ &= h(f(x \cdot y)) \cap h(f(x)) \\ &= (h \circ f)(x \cdot y) \cap (h \circ f)(x), \end{aligned}$$

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$$\begin{aligned}(k \circ f)(y) &= k(f(y)) \\ &\subseteq k(f(x) \star f(y)) \cup k(f(x)) \\ &= k(f(x \cdot y)) \cup k(f(x)) \\ &= (k \circ f)(x \cdot y) \cup (k \circ f)(x), \\ (n \circ f)(y) &= n(f(y)) \\ &\supseteq n(f(x) \star f(y)) \cap n(f(x)) \\ &= n(f(x \cdot y)) \cap n(f(x)) \\ &= (n \circ f)(x \cdot y) \cap (n \circ f)(x).\end{aligned}$$

Hence, $f^{-1}(\mathcal{N})$ is a neutrosophic hesitant fuzzy BCC-filter of X. □

4 Conclusion

In the present paper, we have introduced the concept of neutrosophic hesitant fuzzy BCC-filters of BCC-algebras. The relationship between neutrosophic hesitant fuzzy BCC-filters and their level subsets are described. Moreover, the homomorphic pre-images of neutrosophic hesitant fuzzy BCC-filters in BCC-algebras are also studied, and some related properties are investigated.

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References

[1] B. Ahmad, A. Kharal, On fuzzy soft sets, Adv. Fuzzy Syst., vol. 2009, Article ID: 586507, 6 pages, 2009.

[2] M. Atef, M. I. Ali, T. Al-shami, Fuzzy soft covering based multi-granulation fuzzy rough sets and their applications, Comput. Appl. Math., vol. 40, no. 4, Article number: 115, 2021.

[3] N. Cağman, S. Enginoğlu, F. Citak, Fuzzy soft set theory and its application, Iran. J. Fuzzy Syst., vol. 8, no. 3, pp. 137-147, 2011.

[4] T. Guntasow, S. Sajak, A. Jomkham, A. Iampan, Fuzzy translations of a fuzzy set in UP-algebras, J. Indones. Math. Soc., vol. 23, no. 2, pp. 1-19, 2017.

[5] Y. Huang, BCI-algebra, Science Press, Beijing, China, 2006.

[6] A. Iampan, A new branch of the logical algebra: UP-algebras, J. Algebra Relat. Top., vol. 5, no. 1, pp. 35-54, 2017.

[7] A. Iampan, S. Yamunadevi, P. Maragatha Meenakshi, N. Rajesh, A novel extension of hesitant fuzzy sets on UP (BCC)-algebras: neutrosophic hesitant fuzzy sets, Int. J. Neutrosophic Sci., vol. 20, no. 4, pp. 78-92, 2023.

[8] Y. B. Jun, S. S. Ahn, G. Muhiuddin, Hesitant fuzzy soft subalgebra and ideal in BCI/BCK-algebras, Sci. World J., vol. 2014, Article ID: 763929, 7 pages, 2014.

[9] Y. B. Jun, B. Brundha, N. Rajesh, R. K. Bandaru, $(3, 2)$-Fuzzy UP (BCC)-subalgebras and $(3, 2)$-fuzzy UP (BCC)-filters, J. Mahani Math. Res. Cent., vol. 11, no. 3, pp. 1-14, 2022.

[10] Y. B. Jun, S.-Z. Song, Hesitant fuzzy set theory applied to filters in MTL-algebras, Honam Math. J., vol. 36, no. 4, pp. 813-830, 2014.

[11] P. Kankaew, S. Yuphaphin, N. Lapo, R. Chinram, A. Iampan, Picture fuzzy set theory applied to UP-algebras, Missouri J. Math. Sci., vol. 34, no. 1, pp. 94-120, 2022.

[12] Y. Komori, The class of BCC-algebras is not a variety, Math. Japon., vol. 29, no. 3, pp. 391-394, 1984.

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[13] P. Mosrijai, W. Kamti, A. Satirad, A. Iampan, Hesitant fuzzy sets on UP-algebras, Konuralp J. Math., vol. 5, no. 2, pp. 268-280, 2017.

[14] J. Somjanta, N. Thuekaew, P. Kumpeangkeaw, A. Iampan, Fuzzy sets in UP-algebras, Ann. Fuzzy Math. Inform., vol. 12, no. 6, pp. 739-756, 2016.

[15] V. Torra, Hesitant fuzzy sets, Int. J. Intell. Syst., vol. 25, no. 6, pp. 529-539, 2010.

[16] V. Torra, Y. Narukawa, On hesitant fuzzy sets and decision, 18th IEEE Int. Conf. Fuzzy Syst., Jeju Island, pp. 1378-1382, 2009.

[17] L. A. Zadeh, Fuzzy sets, Inf. Control, vol. 8, no. 3, pp. 338-353, 1965.

[18] B. Zhu, Z. Xu, M. Xia, Dual hesitant fuzzy sets, J. Appl. Math., vol. 2012, Article ID: 879629, 13 pages, 2012.

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