Solving the Inverse Initial Value Problem for the Heat Conductivity Equation by Using the Picard Method

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Abstract

In this work, the inverse initial value problem IVP for the heat equation is formulated and solved. Initial temperature (initial condition) distribution is unknown in this problem, and instead, the temperature spreading at period t= T> 0 is assumed. Among mathematical problems, a class of problems is singled out, the solutions of which are unstable to minor variations in the initial information. It is well identified that this problem is ill-posed. In order to solve the direct problem, we has used the separation of variables way. Note that the method of separation of variables is completely inapplicable for solving IVP, since it principals to rather errors, also divergent series. Ivanov V.K. noticed that if the inverse problem IP is solved by the method separation of variables, and then the resulting series is changed by a incomplete sum of the series, in which the term number is depending on δ, N=N(δ), then as a result we obtain a stable approximate solution. The Picard method customs a regularizing family of operators $\{R_N\}$ that map space $L_2[0,1]$ to same space.

Keywords: Inverse problem; Picard; Ill-posed; Initial value problem

1. Introduction

Numerous applied problems expressed as IP of mathematical physics have its place to the class of ill-posed problems. The inverse IVP in heat conductivty problem is demarcated as an ill-posed [1] in the wisdom that a “small arbitrary change in the data can lead to large errors in the solution”. Note that the IP for the heat that we are investigating can also be resolved by other methods. For example, “Tikhonov A.N.” [2], by the method of “Lavrentiev M.M.” [3] , “Ivanov V.K. “[4], [5] and many others. Thus, many of the methods described for solving inverse problems of mathematical physics also described the solution of the classical problem of direct heat conduction. The foundations of optimal methods were obtained for solving ill-posed problems, as well as methods for estimating the exact solution and the accuracy of order error estimates for these methods [6].

Many works studied the inverse problem [7]–[10].The estimated error between approximation solution and real are studed in [11][12] [13]. The finite-dimensional approximation domain with residual methods for regularization solutions as studied in [7], [8],[16]–[19]. There are numerous papers which explained and used the regularization methods for solving IPs , especially when the IPs in the integral equation for the first kind form [20], [21]. The achievement of these regularization

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methods are based on thoughtful and investigating the mathematical IPs related to the statements of the belongings for problems and classifying explicit problems in solving them [22]–[27].

The key idea of paper is to recover initial function of the heat equation using the Picard method as a regularization method. The method of separation of variables was applied in order to solve the direct problem for the partial differential equation PDE system for the heat equation. All these steps will be executed over the sections in article. In section 1, the mathematical formulation of this problem was considered, and the method of separation of variables was used to solve it as a direct problem. Section 2 is devoted to the IVP for the heat and describes the known data and operator. Section 3 demonstrates the use of the Picard method with the help of a regularizing family of operators $\{R_N\}$ that map the space $L_2[0,1]$ into itself. An example to test the accuracy of our estimation solution is presented in Section 4. Finally, the explanation of the proposed method has been summarized in the final section with proposed future work.

2. Statement IVP for the heat Equation

The idea of this part is recovering the initial function of the diffusion formula by using Picard's method. Separation of variables method was applied to resolve the direct problem for PDE heat equation.

The direct problem is the heat passing over a bar with certain conditions for boundary and initial temperature. The reverse time IVP is expressed as follows: we need to find the function $u(x,t)$,

$$u_t = u_{xx} \quad 0 < x < l, \quad t > 0, \tag{1}$$

with conditions for boundary

$$u(0,t) = u(l,t) = 0, \quad t > 0, \tag{2}$$

at initial moment of time

$$u(x,0) = u_0(x), \quad 0 \le x \le l, \tag{3}$$

Let's assume that $u_0(x) \in H_0^2[0,1]$.

It is essential define a function $u(x,t) \in C\,([0,1] \times [0,T]) \cap C^{2,1}(0,1) \times (0,T])$, satisfying the equation for $x \in [0,1]$ and $t \in (0,T]$, also the initial condition (3) and boundary conditions (2).

Solving the problem (1) – (3) by separation of variables, we obtain

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n\pi)^2 t} \sin(n\pi x), \tag{4}$$

where $x \in [0,1], t \in [0,T],$ a

$$u_0(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x), \tag{5}$$

where

$$a_n = 2\int_0^1 u_0(x)\sin(n\pi x)dx. \tag{6}$$

In (1) – (3) one can consider the inverse IVP for the heat equation, i.e. the temperature distribution at time $t = T > 0, u(x,T),$ is known and it is required to find the initial distribution $u_0(x),$ assuming that $u_0(x) \in H_0^2[0,1].$

Assume that we know the solution to the direct problem for $t = T,$

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$$u(x,T)=f_0(x), \tag{7}$$

The precise value of the $f_0(x)$ is unidentified, and instead of the pair $(f_\delta(x),\delta),$ is given where $f_\delta(x)\in C[0,1],\delta>0,$

$$\|f_0(x)-f_\delta(x)\|^2\le\delta^2, \tag{8}$$

need to define a function $u_0(x)\in M_r,$

$$M_r=\left\{u(x):u(x)\in H_0^2[0,1],\|u(x)\|_{H_0^2}^2\le r^2,u_0'(0)=u_0'(1)=0\right\}, \tag{9}$$

such that, substituting it into complaint (3) we obtain a solution $u(x,t)$ of problem (1) – (3), that satisfies the condition. By using the initial data of the problem $f_\delta(x)$ and δ, it is required to regulate the estimated $u_\delta(x)$, as well as to obtain an approximation of the error $\|u_\delta(x)-u_0(x)\|_{L_2}$.

3. Solving the inverse IVP using the Picard method

The method of Picard uses a regularizing family of $\{R_N\}$, operators that map the space $L_2[0,1]$ into itself and are defined by a formula. Under uniform regularization, $\forall N\Rightarrow\|R_N\|\le1$ and $\forall u\Rightarrow R_NAu\to u$ for $N\to\infty$, this family of operators is called regularizing.

$$R_Nf(x)=\sum_{n=1}^{N}f_ne^{(n\pi)^2T}\sin(n\pi x), \tag{10}$$

where

$$f_n=2\int_0^1 f(x)\sin(n\pi x)dx. \tag{11}$$

Let us present a number of properties of the family $\{R_N\}$, formulated in the form of lemmas.

Lemma 1. For any N the operator R_N, defined by formula (10), is linear bounded with the norm $\|R_N\|=e^{(\pi N)^2T}$.

Proof. The linearity of the operator R_N follows from formula (10).

Now let's prove that

$$\|R_N\|=e^{(\pi N)^2T}, \tag{12}$$

the boundedness of the operator R_N will follow from formula (12).

Since

$$\|R_N\|^2-\sup\left\{\|R_Nf\|^2:f\in L_2[0,1],\|f\|^2\le1\right\}, \tag{13}$$

and

$$\|f\|^2 = \sum_{n=1}^{\infty} f_n^2, \tag{14}$$

then from (13) and (14) it will follow that for $f \in L_2[0,1]$ и $\|f\|^2 \le 1$

$$\|R_N\| \le e^{2(N\pi)^2 T}. \tag{15}$$

Consider the element $f_N(x) = 2\sin(\pi N x)$.

since $f_N \in L_2[0,1]$ and $\|f_N\| = 1$, then we act on it with the operator R_N, then

$$\|R_N f\|^2 = e^{2(\pi N)^2 T} \sin(\pi N x), \tag{16}$$

from (16) follows that

$$\|R_N f\|^2 = e^{2(\pi N)^2 T}, \tag{17}$$

and from (17), that

$$\|R_N f\|^2 \ge e^{2(\pi N)^2 T}, \tag{18}$$

from (15) and (18) follows that $\|R_N\| = e^{2(N\pi)^2 T}$.

Thus, the lemma is proved.

Lemma 2. The family of operators $\{R_N\}$ regularizes the inverse problem(1), (2) и (7) in M_r.

Proof. It follows from the definition of a regularizing family of operators [28], [29] that for any element of $u(x) \in M_r$ the relation, $R_N = \sum_{n=1}^{N} \to u(x)$ for $N \to \infty$ metric $L_2[0,1]$,

$$R_N\left[\sum_{n=1}^{\infty} u_n e^{(n\pi)^2 T} \sin(n\pi x)\right] \to u(x), \tag{19}$$

for $N \to \infty$ metric $L_2[0,1]$, where

$$u_n = 2\int_0^1 u(x) \sin d(n\pi x) dx. \tag{20}$$

Since

$$R_N\left[\sum_{n=1}^{\infty} u_n e^{(n\pi)^2 T} \sin(n\pi x)\right] - u(x) = \sum_{n=N+1}^{\infty} u_n \sin(n\pi x), \tag{21}$$

then

$$\left\| R_N\left[\sum_{n=1}^{\infty} u_n e^{(n\pi)^2 T} \sin(n\pi x)\right] - u(x)\right\|^2 = \sum_{n=N+1}^{\infty} u_n^2, \tag{22}$$

in view of the fact that $u(x) \in L_2[0,1]$ follows the convergence of the series

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$$\sum_{n=N+1}^{\infty} u_n^2. \tag{23}$$

In this way,

$$\sum_{n=N+1}^{\infty} u_n^2 \to 0 \ \ npu \ \ N \to \infty. \tag{24}$$

Thus, the lemma is proved.

The approximate solution $u_\delta^N(x)$ for IP is defined by

$$u_\delta^N(x) = R_N f_\delta(x). \tag{25}$$

Now let's proceed to the definition of dependence $N(\delta)$. To do this, let's estimate

$$\left\|u_0(x) - u_\delta^N(x)\right\| \le \left\|u_0(x) - u_0^N(x)\right\| + \left\|u_0^N(x) - u_\delta^N(x)\right\|, \tag{26}$$

where

$$u_0^N(x) = R_N f_0(x). \tag{27}$$

Let us first estimate the second term in the formula (26)

$$\left\|u_0(x) - u_\delta^N(x)\right\| = \left\|R_N f_0(x) - f_\delta(x)\right\| \le \left\|R_N\right\|\delta. \tag{28}$$

From (28) and lemma 2 follow that

$$\left\|u_0^N(x) - u_\delta^N(x)\right\| = e^{(\pi N)^2 T}\delta. \tag{29}$$

Let us pass to estimating the first term in the formula (26)

$$\left\|u_0^N(x) - u_\delta^N(x)\right\|^2 = \sum_{n=N+1}^{\infty} u_n^2, \tag{30}$$

where

$$u_n = 2\int_0^1 u_0(x)\sin(\pi n x)dx. \tag{31}$$

Since $u_0 \in M_r$, that $u_0'' \in L_2[0,1]$ and $\left\|u_0''(x)\right\|_{L_2}^2 \le r^2$. Thus,

$$u_0''(x) = \sum_{n=1}^{\infty} v_n \sin(\pi n x), \tag{32}$$

where

$$v_n = 2\int_0^1 u_0''(x)\sin(\pi x n)dx. \tag{33}$$

By integrating the formula (32), we get that

$$v_n = -(\pi n)^2 u_n, \tag{34}$$

From (33) follow that

$$\sum_{n=N+1}^{\infty} u_n^2 \le \frac{r^2}{(\pi N)^4}, \tag{35}$$

and from (30) and (35), that

$$\left\| u_0(x) - u_0^N(x) \right\| \le \frac{r}{(\pi N)^2}, \tag{36}$$

finally

$$\left\| u_0(x) - u_0^N(x) \right\| \le \frac{r}{(\pi N)^2} + e^{(\pi N)^2 T}\delta. \tag{37}$$

Thus, from (35) и (37) we determine the regularization parameter $N(\delta)$ from the condition

$$\min_N \frac{r}{(\pi N)^2} + e^{(\pi N)^2 T}\delta. \tag{38}$$

4. Numerical Example

Given the problem from (1) – (3), we need to determine the unknown function for the sources temperature $u(x)$ from the known data $u(x,T) = f(x)$.

Example a) Direct solution for temperature where time (T = 0.001 и T = 0.005) to test the guesstimate we have the exact initial temperature$u_0(x) = 4\sin(3\pi x) + 7\sin(8\pi x)$, as shown in Figure 1. We can discovery a function $f(x)$, by (25).

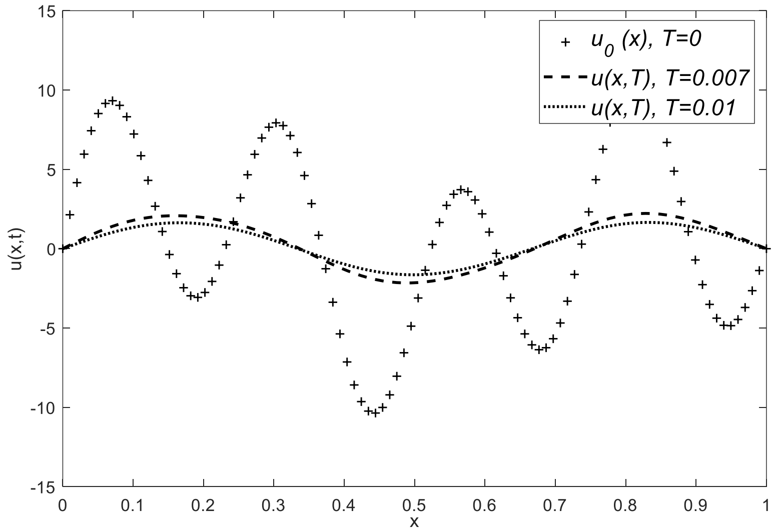

Figure 1: Direct solution for temperature measurement $u_0(x) = 4\sin(3\pi x) + 7\sin(8\pi x)$

We can add a noise signal to the given $f_0(x)$ data for use in $\delta = 0.05$ problem analysis. Using the inverse problem algorithm, which is defined above we obtain solutions, see figure 2 and figure 3

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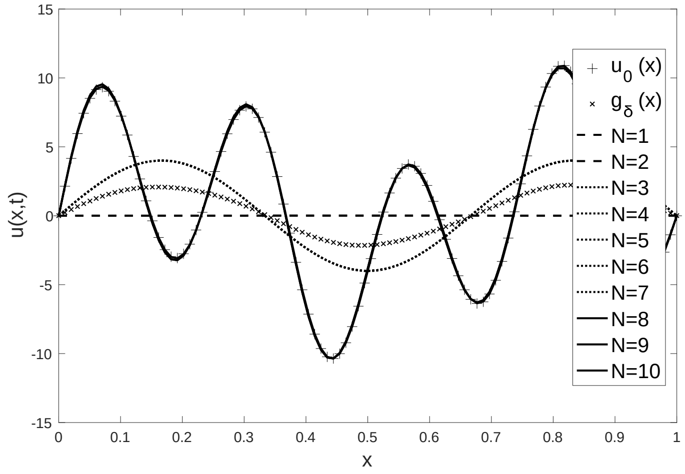

Figure 2: Inverse solution $u_0(x), u_\delta^N(x)$, where T=0,007

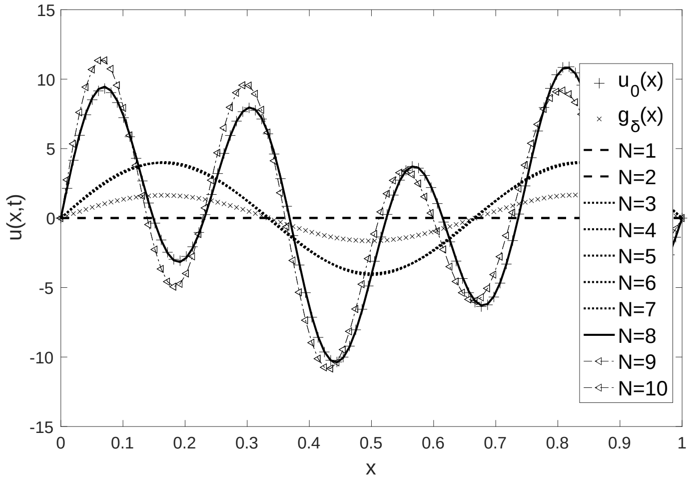

Figure 3: Inverse solution $u_0(x), u_\delta^N(x)$, where T=0,01

Example b): Direct solution for temperature where time (T = 0,05 and T = 0,01) to test the approximation we have the exact starting temperature $u_0(x) = sin(\pi x)$, as shown in Fig. 4. We can find the function $f(x)$, by using (25).

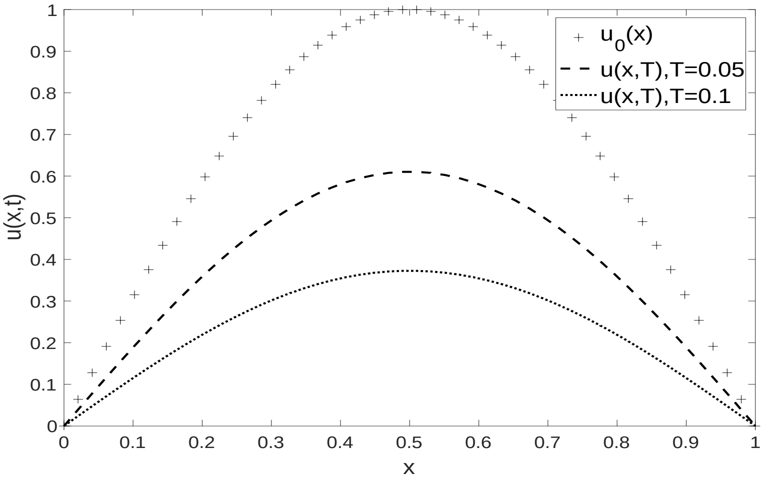

Figure 4: Direct solution for temperature measurement $f_0(x)$ and $u_0(x) = sin\ (\pi x)$

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We can add a noise signal to the given data $f_0(x)$ or its use in$\delta = 0.035$ problem analysis. Using the inverse problem algorithm, which is defined in (10) and (11), we obtain solutions, see figure 5 and figure 6.

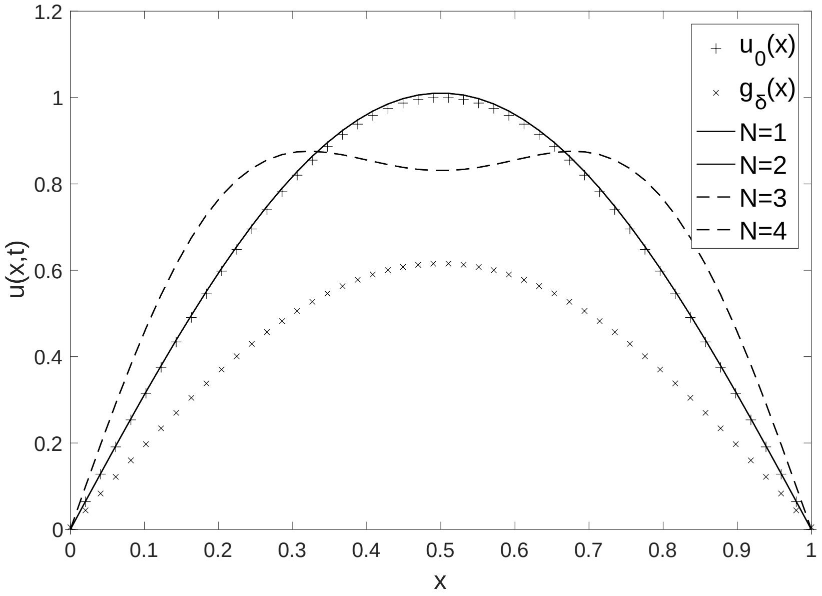

Figure 5: Inverse solution $u_0(x), u_\delta^N(x)$, where T=0,05

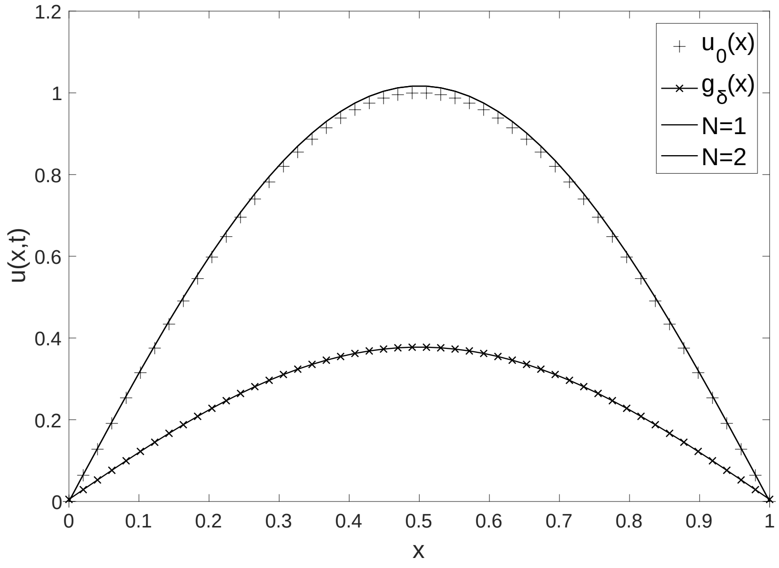

Figure 6. Inverse solution $u_0(x), u_\delta^N(x)$, where T=0,1

5. Conclusion

This article is devoted to an algorithm for solving the problem of inverse IVP for heat conduction. This problem must be solved by an ill-posed Inverse initial value problem and a special method. The separation of variables method is used to solve the equation of PDE heat conduction. Numerical investigates are effectively used to solve IP of heat conduction using Picard's theorem. The algorithm selects the appropriate parameter to regularize the estimated solution. It can be well-known that the algorithm was effective for estimating the initial temperature depending on the given measurement temperature with a known noise level δ. The proposed future work will solve the inverse IVP for the heat PDE in composite materials.

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