A Study of the Eigenvalues of the Matrix Of Distance Reciprocals in K[r, n-r] And The Cycle C_n

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Abstract

This paper Deals with the complete bipartite graph **K(r, n-r)** and the cycle $\mathbf{C_n}$. The matrix of concern is the matrix **B** which is the (n, n) matrix and whose non zero entries are the reciprocals of the non zero entries of the distance matrix **D**. A complete characterization of the spectrum of **B** and a set of n independent eigenvectors of **B** will be presented. Two special cases will be mentioned, namely the star **K(1, n-1)** and the graph **K(2, n-2)**. We will also look at the case of infinite graph, i. e if the size n grows big while r stays finite. Finally, some numerical data will be presented. As for the cycle, we present the complete set of eigenvalues of the matrix **B**.

Keywords: Infinite Graph; Matrix; cycle.

1. Introduction

For the main concepts, definitions and terminology of the subject, one may consult any standard book on graph theory. One may look this in Behzad M. et al. (1979), Graphs and Digraphs. A graph **G** is an ordered pair (**V, E**) where **V** is the vertex set containing the entries (vertices) $V_1, V_2, \dots, V_n$. The set E is the edge set. For an undirected graph **G** the elements of **E** are unordered pairs {vi, vj} where the vertices vi and vj are adjacent. I. e there is an edge joining the two vertices. The order of the graph is the size of the vertex set while the size of the graph is the size of the edge set. The adjacency matrix **A** of the graph **G** is a square nxn symmetric (0, 1) matrix whose (i, j) entry equals to one only if vertices vi, vj are adjacent. The distance matrix **D** of a connected graph **G** is the nxn integral nonnegative matrix whose (i, j) entry is the shortest length of a path joining the two vertices vi and vj.

In the graph **K(r, n-r)** the vertex set is the union of two sets (classes) V_1 and V_2 where every vertex in the first class is adjacent to every vertex in the second class and there are no adjacencies between vertices in the same class. The adjacency matrix of this graph has the following block structure:

$$\mathbf{A} = \left(\begin{array}{c|c} \mathbf{O}_{rxr} & \mathbf{J}_{rx(n-r)} \\ \hline \mathbf{J}_{nx(r,r)} & \mathbf{O}_{(n-r)x(n-r)} \end{array}\right)$$

The symbol **O** denotes the zero matrix and the symbol **J** denotes the matrix of all ones. The symbol **I** will be used to denote the identity matrix. Concerning the matrices to come in this work if appear in block form, then the dimensions of the blocks are the same as those of the corresponding blocks in the previous matrix **A**.

The distance matrix of the graph **K(r, n-r)** is the following matrix:

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$$D = \left(\begin{array}{c|c} 2J - 2I & J \\ \hline J & 2J - 2I \end{array}\right)$$

The matrix **B** is the nxn matrix whose non zero entries are the reciprocals of the corresponding non zero entries of the matrix **D**. The matrix B thus has the following block structure:

$$B = \left(\begin{array}{c|c} .5(J - I) & J \\ \hline J & .5(J - I) \end{array}\right)$$

$$B = \left(\begin{array}{ccccc|cccc} 0 & .5 & .5 & \cdots & .5 & 1 & 1 & \cdots & 1 \\ .5 & 0 & .5 & \cdots & .5 & 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ .5 & .5 & \cdots & & 0 & 1 & 1 & \cdots & 1 \\ \hline 1 & 1 & \cdots & & 1 & 0 & .5 & \cdots & .5 \\ 1 & 1 & \cdots & & 1 & .5 & 0 & \cdots & .5 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & & 1 & .5 & .5 & \cdots & 0 \end{array}\right)$$

For the discussion to be made easier, a new matrix C will be introduced and defined as **C = 2B + I.** This matrix will thus have the following block structure:

$$C = \left(\begin{array}{c|c} J & 2J \\ \hline 2J & J \end{array}\right)$$

2.THE DISTANCE MATRIX D

The distance matrix **D** of the graph **K(m, n)** was treated in Trinajstic N. et al. (1983), On the Distance Polynomial of a Graph. There it was shown that the distance matrix **D** has (n +m) eigenvalues out of which (n+m-2) eigenvalues are equal to (–2) and the remaining two, if denoted by x and y, satisfy the following two relations

$x^2 + y^2 = 2(2m^2 + 2n^2 - 4m - 4m + mn + 4)$ and $x + y = 2(m + n - 2)$

If we go back to the notation used here, the graph **K(r, n-r)** will have a distance matrix **D** whose spectrum is explicitly stated in the following theorem.

THEOREM 2.1

The eigenvalues of the distance matrix **D** of the graph **K(r, n-r)** are given as

$$\lambda_1 = \frac{2n - 4 + \sqrt{4n^2 - 12r(n-r)}}{2}$$

$\lambda_i = -2$ for i=2,3,…,n-1 and

$$\lambda_n = \frac{2n - 4 - \sqrt{4n^2 - 12r(n-r)}}{2}$$

PROOF

We consider the matrix **F = D + 2I**. This matrix has the following structure

$$F = \left(\begin{array}{c|c} 2J & J \\ \hline J & 2J \end{array}\right)$$

This matrix has clearly just two independent rows which makes the matrix of rank equal to two. Thus the number zero is an eigenvalues of **F** with multiplicity (n-2). The characteristic polynomial of F takes the form

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$f(a) = \det(aI - F) = a^n + b_1 a^{n-1} + b_2 a^{n-2}$
where $b_1 = \mathrm{trace}(\mathbf{F})$ and b_2 is equal to the sum of all leading principal (2,2) determinants (see Lancaster M. et al. (1985), Theory of Matrices on p 256). Thus b2 = 3r(n-r) and the characteristic polynomial of the matrix **F** is explicitly the following one
$f(a) = a^n - 2n\, a^{n-1} + 3r(n-r)a^{n-2}$
This shows that the number zero is an eigenvalue of **F** with multiplicity (n-2). The remaining two eigenvalues are

$$\lambda_1, \lambda_n = \frac{2n \mp \sqrt{4n^2 - 12r(n-r)}}{2}$$

If λ is an eigenvalue of **F** then the corresponding eigenvalue of the matrix D is (λ – 2). Thus the spectrum of the matrix **D** contains the number (-2) as an eigenvalue and the remaining two eigenvalues are given as

$$\lambda_1, \lambda_n = \frac{2n - 4 \mp \sqrt{4n^2 - 12r(n-r)}}{2}$$

We mention now two special cases namely when r = 1 and r=2
Case(1) ; The star **K(1, n-1)**
The spectrum of the matrix **D** contains (–2) as an eigenvalue with multiplicity (n-2). The remaining two eigenvalues are given by

$$\lambda_1, \lambda_n = \frac{2n - 4 \mp \sqrt{4n^2 - 12n + 12}}{2}$$

Case(2): The graph **K(2, n-2)**
The spectrum of the matrix **D** contains (–2) as an eigenvalue with multiplicity (n-2). The remaining two eigenvalues are given by

$$\lambda_1, \lambda_n = \frac{2n - 4 \mp \sqrt{4n^2 - 24n + 48}}{2}$$

3.THE SPECTRUM OF THEMATRIX B
We consider in this section the matrix **B** defined before. The spectrum of this matrix will be stated explicitly. For this purpose another matrix will be treated.
This will be the matrix **C = 2B + I.** The matrix **C** is the following matrix:

$$\mathbf{C} = \left(\begin{array}{c|c} \mathbf{J} & \mathbf{2J} \\ \hline \mathbf{2J} & \mathbf{J} \end{array}\right)$$

We first state a theorem that describes fully the spectrum of the matrix **C**. The eigenvalues of the matrix **C** will be denoted by α and the eigenvalues of **B** will be denoted by λ.
THEOREM 3.1
Consider the graph **K(r, n-r).** Let **D** be its distance matrix and let **B** be the nxn matrix whose non zero entries are the reciprocals of the non zero entries of the distance matrix. If the matrix **C** is defined as **C= 2B + I**, then the matrix **C** has the following entries in its spectrum:

$$\alpha_1 = \frac{n + \sqrt{n^2 + 12r(n-r)}}{2}$$

αk = zero for k=2,3,4,…,n-1 and

$$\alpha_n = \frac{n - \sqrt{n^2 + 12r(n-r)}}{2}$$

proof:
The matrix **C** has clearly just two independent rows. It thus has rank equal to (2). This makes the number zero an eigenvalue with multiplicity (n-2). The characteristic polynomial of this matrix will have the form

$$q(\alpha) = \det(\alpha\mathbf{I} - \mathbf{C}) = \alpha^n + a_{n-1}\alpha^{n-1} + a_{n-2}\alpha^{n-2}$$

where $a_{n-1} = -\mathrm{trace}(\mathbf{C})$ and a_{n-2} is equal to the sum of the determinants of all (2,2) principal sub matrices of the matrix **C**.
Thus $a_{n-1} = -n$ and $a_{n-2} = -3r(n-r)$. The characteristic polynomial of the matrix **C** is

$$q(\alpha) = \alpha^n - n\,\alpha^{n-1} - 3(n-r)\alpha^{n-2} = \alpha^{n-2}(\alpha^2 - n\alpha - 3r(n-r)$$

The two non zero eigenvalues and through the quadratic formula come to be
$\alpha_1 = \frac{n+\sqrt{n^2 12r(n-r)}}{2}$ and $\alpha_n = \frac{n-\sqrt{n^2+12r(n-r)}}{2}$

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We now state the theorem that describes fully the spectrum, the determinant and the characteristic polynomial of the matrix **B.**

THEOREM 3.2

For the graph **K(r, n-r)** let **B** as defined before, the matrix **B** has the following properties:

(1) The spectrum of **B** contains n eigenvalues having the values

$$\lambda_1, \lambda_n = \frac{n-2 \mp \sqrt{n^2+12r(n-r)}}{4}$$

And $\lambda_k = -.5$ for k=2,3,4,…,n-1

(2) The determinant **B** is given as

$$\det(\mathbf{B}) = \frac{-1}{2}(1-n-3r(n-r))$$

(3) The characteristic polynomial of **B** is

$$P(\lambda) = 2^{-n}((2\lambda+1)^n - n(2\lambda+1)^{n-1} - 3r(n-r)(2\lambda+1)^{n-2})$$

Proof:

Let α be an eigenvalue of the matrix **C** and let λ be the corresponding eigenvalue of **B**. Since **C = 2B + I** thus α = 2λ + 1 which gives λ = (α – 1)/2. Corresponding to zero as an eigenvalue of **C**, the number (-.5) is an eigenvalue of the matrix **B** and with multiplicity (n-2). The remaining two eigenvalues of **B** are

$$\lambda_1, \lambda_n = \frac{n-2 \mp \sqrt{n^2+12r(n-r)}}{4}$$

As for the determinant of **B**, it is the product of its eigenvalues which comes to be

$$\det(\mathbf{B}) = (-.5)^{n-2}\lambda_1, \lambda_n = (-.5)^{n-2}(1-n-3r(n-r))$$

As for the characteristic polynomial p(λ) of the matrix **B**, we have

$$P(\lambda) = \det(\lambda I - B) = \det(.5(\alpha-1)I - .5(C-I)) = \det(.5I - .5C) = (.5)^n q(\alpha)$$
$$= 2^{-n}((2\lambda+1)^n - n(2\lambda+1)^{n-1} - 3r(n-r)(2\lambda+1)^{n-2})$$

It is seen from this polynomial that

$\det(B) = (-1)^n$multiplied by the constant term

$= (-1)^n 2^{-n}(1-n-3r(n-r))$

$= (-.5)^n(1-n-3r(n-r))$

4. EIGENSPACES OF THE MATRIX B

We look in this section at eigenspaces of **B**. A complete set of orthogonal eigenvectors of **B** will be stated. Before that we note that an eigenvector of the matrix **B** associated with a given eigenvalue is also an eigenvector of the matrix **C** associated with the corresponding eigenvalue. This is since

$Cx = \alpha x$ implies

$(2\mathbf{B} + \mathbf{I})x = (2\lambda + 1)x$ which is equivalent to $\mathbf{B}x = \lambda x$

Thus we deal with the matrix **C** and use it to get a set of orthogonal eigenvectors for the matrix **B**. We note first that each of the matrices, being real symmetric, has a complete set of orthogonal eigenvectors (see theorem 3.3 on p 35 of Richard Bellman (1985), Introduction to Matrix Theory).

Let $x = (x_1, x_2, x_3, \dots, x_n)^t$ be a non zero eigenvector of the matrix **C** associated with the eigenvalue α. Thus we have $\mathbf{C}x = \alpha x$ which gives the following two independent equations

$$x_1 + x_2 + x_3 + \cdots + x_r + 2(x_{r+1} + x_{r+2} + \cdots + x_n) = \alpha x_k$$

For k=1,2,…,r (*)

$$2(x_1 + x_2 + \cdots + x_r) + x_{r+1} + x_{r+2} + \cdots + x_n = \alpha\, xj$$

For j=r+1,r+2,…n (**)

Upon subtracting these two equations, we get

$$x_1 + x_2 + x_3 + \cdots + x_r - (x_{r+1} + x_{r+2} + \cdots + x_n) = \alpha(x_j - x_k)$$

If α = 0 the we obtain the following

$$x_1 + x_2 + x_3 + \cdots + x_r = x_{r+1} + x_{r+2} + \cdots + x_n$$

This gives the following set of (n-2) mutually pair wise orthogonal vectors written as the union of two sets where vectors in one set are given as

$x_1 = 1$, $x_k = -1$ for k=2,3,4,…,r and $x_i = 0$ for i=r+1, r+2, …,n

and vectors in the second set are given as

$x_k = 0$, for k=1,2,3,…,r and $x_{r+1} = 1$ and $x_j = -1$ for j=r+2,r+3,…,n

We now consider the case when α is non zero namely $\alpha = \alpha_1$or α_n

Upon subtracting two equations of the type (*) for two different values of k sy k_1 and k_2 one gets $\alpha(x_{k1} - x_{k2}) = 0$.

Since α is non zero we get

$x_{k1} = x_{k2}$. I. e x_k = a = constant for every k = 1, 2, 3, …, r

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and similarly through equations of the form (**) one gets xj = b = constant for any j = r+1, r+2, …, n. Thus an eigenvector of the matrix **C**, and consequently of the matrix **B**, associated with the first or with the last eigenvalue has the form
$x = (a, a, \dots, a, b, b, \dots, b)^t$ where the constants a and b depend upon the value of α. If α = α1 then all entries of the eigenvector are positive (see theorem 4.1 on p 312 of Robin L. Wilson et al. (1978), Selected Topics in Graph Theory). while if α = αn then a and b will differ in sign. If a = 1 is assumed arbitrarily then $b = \frac{\alpha-1}{2n-2r}$
That defines completely a set of n mutually pair wise orthogonal eigenvectors of the matrix **B**.

5. SPECIAL CASES

We look in this section at special cases. We look at the two cases where r = 1 and r = 2. In the first case the graph **K(1, n-1)** is the star of order (n) with one center. In the second case the graph **K(2, n-2)** is the star with two centers. The spectrum, the determinant and the characteristic polynomial for the matrix **B** will be stated for each of these two cases. The results will be obtained from the general case shown earlier by just replacing r by its proper value. Then we investigate the case where n grows big while r stays finite.

PROPOSITION 5.1

For the star **K(1, n-1)** we have the following:

(1)The spectrum of the matrix **b** contains n eigenvalues out of which (n-2) are equal to (-.5) and the remaining two are given by

$$\lambda_1, \lambda_n = \frac{n - 2 \mp \sqrt{n^2 + 12n - 12}}{4}$$

(2)The determinant is det

$$(\mathbf{B}) = \left(\frac{-1}{2}\right)^n (4 - 4n)$$

(3)The characteristic polynomial of B is given as

$$P(\lambda) = 2^{-n}((2\lambda + 1)^n - n(2\lambda + 1)^{n-1} - 3(n - 1)(2\lambda + 1)^{n-2})$$

PROPOSITION 5.2

For the graph **K(2, n-2)** we have the following:

(1)The spectrum of the matrix **B** contains n eigenvalues out of which (n-2) are all equal to (-.5) and the remaining two are given as

$$\lambda_1, \lambda_n = \frac{n - 2 \mp \sqrt{n^2 + 24n - 48}}{4}$$

(2) The determinant of **B** is det

$$(\mathbf{B}) = (-.5)^n(13 - 7n)$$

(3) The characteristic polynomial of **B** is

$$P(\lambda) = 2^{-n}((2\lambda + 1)^n - n(2\lambda + 1)^{n-1} - 6(n - 2)(2\lambda + 1)^{n-2})$$

INFINITE CASE:

We now consider the graph **K(r, n-r).** The number r will be held constant while n will be free to grow. If we look at the last eigenvalue

$$\lambda_n = \frac{n - 2 - \sqrt{n^2 + 12r(n - r)}}{4}$$

and utilize the basic techniques in limit theory, it becomes very simple to show that as n grows to infinity then the value of λn approaches the value $\frac{-1-3r}{2}$ The sum of all eigenvalues of the matrix **B** is equal to its trace. I. e
$\lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_n = \text{zero}$. Which is also true for infinite graphs. We also note that

$$\lambda_2 + \lambda_3 + \dots + \lambda_{n-1} = \frac{-(n - 2)}{2}$$

which gives the first eigenvalue in the asymptotic case as

$$\lambda_1 = \frac{(n - 2)}{2} + \frac{1 + 3r}{2} = \frac{n + 3r - 1}{2}$$

For the star **K(1, n-1)** we have the following results in the limiting case when n grows big.
The value of λ_n goes to the value of (-2) while λ_1 goes to the value $\frac{n+2}{2}$
For the graph **K(2, n-2)** we have the following results for the first and last eigenvalues: The value of λ_1 goes to $\frac{n+5}{2}$ while λn goes to the value of -3.5 We present here some examples of the graph **K(r, n-r)** for different values of n and r. The results show the first and last eigenvalues of the matrix **B**. It also shows that the last eigenvalue approaches the value (-3.5) if r = 2 and approaches the value (-8) in case r = 5.

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Graph	first eigenvalue	last eigenvalue
K(2, 5)	5.105551	-2.105551
K(2, 7)	5.694933	-2.194933
K(2, 18)	11.7111	-2.711102
K(2, 58)	32.16352	-3.163522
K(2, 78)	42.23763	-3.237634
K(2, 118)	62.31768	-3.317678
K(2, 138)	72.34177	-3.341771
K(2, 158)	82.36024	-3.360238
K(2, 178)	92.37484	-3.374837
K(2, 198)	102.3867	-3.386671
K(2, 398)	202.4417	-3.441731
K(2, 448)	227.448	-3.44804
K(2, 498)	252.4531	-3.453116
K(2, 548)	277.4573	-3.457289
K(2, 598)	302.4608	-3.460779
K(2, 648)	327.4637	-3.463742
K(2, 698)	352.4663	-3.466289
K(2, 748)	377.4685	-3.468501
K(2, 798)	402.4704	-3.470441
K(2, 848)	427.4722	-3.472156
K(2, 898)	452.4737	-3.473683
K(2, 948)	477.475	-3.476284
K(2, 998)	502.4763	-3.476284
K(2, 1998)	1002.488	-3.488071
K(2, 2198)	1102.489	-3.48915
K(2, 2398)	1202.49	-3.49005
K(2, 2598)	1302.491	-3.490812
K(2, 2798)	1402.491	-3.491465
K(2, 2998)	1502.492	-3.492032
K(2, 3198)	1602.493	-3.492528
K(2, 3398)	1702.493	-3.492966
K(2, 3598)	1802.493	-3.493356
K(2, 3798)	1902.294	-3.493704
K(2, 3998)	2002.494	-3.49018

Graph	first eigenvalue	last eigenvalue
K(5, 15)	7	-3
K(5, 15)	13.51388	-4.513878
K(5, 25)	19.24745	-5.247449
K(5, 35)	24.70691	-5.706906
K(5, 45)	30.02776	-6.027756
K(5, 55)	35.26656	-6.452079
K(5, 95)	55.82491	-6.82491
K5, 395)	206.6506	-7.716678
K(5, 995)	506.8543	-7.926103
K(5, 2995)	1506.95	-7.957506
K(5, 4995)	2506.970	-7.970179
K(5, 5495)	2756.973	-7.972875
K(5, 5795)	2906.974	-7.974271
K(5, 5895)	2956.975	-7.974705
K(5, 5995)	3006.975	-7.975124

The following data shows the eigenvalues and a set of normalized eigenvectors for the matrix **B** corresponding to the graph **K(3, 5)**

Spec(**B**) = {5.405124, -.5, -.5, -.5, -.5, -.5, -.5, -2.405125}

A set of normalized eigenvectors contains the vectors

x1= (.381218, .381218, .381218, .335863, .335863, .335863, .335863, .335863)

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x2 = (0, 0, 0, -.297122, -.284909, -.277918, -.867896, -.0077947)
x3 = (-.408248, -.408248, .816497, 0, 0, 0, 0, 0)
x4 = (0, 0, 0, -.412223, -.394250, -.821292, -.007622, -.0071871)
x5 = (-.707107, -.707107, 0, 0, 0, 0, 0, 0)
x6 = (0, 0, 0, -.697885, .716132, -.006627, -.006004, -.0056151)
x7 = (0, 0, 0, -.233961, -.224831, -.219539, -.216015, -.894345)
x8 = (-.433597, -.433597, -.433597, .295290, .295290, .295290, .295290)

6.THECYClECN

We look in this section at the cycle of order n and study the matrix **B** related to the cycle. The distance matrix of the cycle is a circulant symmetric matrix of the form

$$D=\begin{pmatrix} 0 & 1 & 2 & 3 & . & . & . & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & . & . & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & . & . & 4 & 3 \\ . & . & . & . & . & . & . & . & . \\ 2 & 3 & . & . & . & . & . & 0 & 1 \\ 1 & 2 & 3 & . & . & . & . & 1 & 0 \end{pmatrix}$$

This matrix **B(CN)** is also a circulant matrix with the non zero entries being reciprocated. To study this kind of matrix, we will first have a quick look at the circulant matrix to be used here. The primitive matrix W being a special permutation matrix has the form **W** = [wij]

$$\text{Where Wij} = \begin{cases} 1 \ \text{ if } j = i + 1 \text{ for } i = 1,2,3,\dots,n-1 \\ 1 \ \text{ if } i = n \text{ and } j = 1 \\ 0 \ \text{ other wise} \end{cases}$$

The matrix **W** has ones on the first super diagonal and a one in the position (n, 1) and zeros else where. The matrix **W** is explicitly the following one

$$W=\begin{pmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & . & 0 \end{pmatrix}$$

A circulant matrix M with first row being $(a_0, a_1, a_2, a_3, \dots, a_{n-1})$ will be denoted by $A = circ(a_0, a_1, a_2, a_3, \dots, a_{n-1})$. In the matrix $A = [m_{ij}]$ we have

$$mij = \begin{cases} a_0 \ \text{ if } i = j \\ a_k \ \text{ if } j = i + k \text{ and for } j \geq 2 \\ a_k \ i - j = n - k \ \text{ for } i > j \end{cases}$$

for k ≥ 1

As for powers of the matrix **W**, the matrix $\mathbf{W}^{\mathbf{k}} = \mathbf{V} = [V_{ij}]$ for 1 ≤ k ≤ n-1, we have

$$V_{ij} = \begin{cases} 1 \ \text{ if } i = j + k \ \text{ for } j \geq 2 \\ 1 \ \text{ if } i - j = n - k \ \text{ for } i \geq 2 \\ 0 \text{ else where} \end{cases}$$

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With this notation, The matrix $\mathbf{D}(\mathbf{C_n}) = \mathrm{circ}(0,1,2,3, \dots, 3,2,1)$ and the matrix
$\mathbf{B}(\mathbf{C_n}) = \mathrm{circ}(0,1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots,\frac{2}{n-2},\frac{2}{n},\frac{2}{n-2},\dots,\frac{1}{3},\frac{1}{2},1)$
$\mathrm{B}(\mathrm{C_n}) = (0,1,\frac{1}{2},\frac{1}{3},\dots,\frac{2}{n-1},\frac{2}{n-1},\dots,\frac{1}{3},\frac{1}{2},1)$ for odd n.
So
If $\mathrm{B}(\mathrm{C_n}) = \mathrm{circ}(\mathrm{b_0},\mathrm{b_1},\mathrm{b_2},\dots,\mathrm{b_{n-1}})$ then $\mathrm{b_i} = \mathrm{b_j}$ whenever i+j=n.
Back to the primitive matrix **W** and its characteristic polynomial and eigenvalues, we have $\mathrm{P}(\alpha) = \alpha^{\mathrm{n}} - 1$. The eigenvalues of the matrix **W** are the n^{th} roots of unity. Namely, the n eigenvalues are given by the formula
$\alpha = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ for k=0,1,2,…,n-1
which accounts for the n eigenvalues.
The circulant matrix $\mathbf{M} = \mathrm{circ}(\mathrm{a_0},\mathrm{a_1},\mathrm{a_2},\mathrm{a_3},\dots,\mathrm{a_{n-1}})$ is expressed in terms of the matrix **W** and its powers as $\mathbf{M} = \mathrm{a_0}\mathbf{I} + \mathrm{a_1}\mathbf{W} + \mathrm{a_2}\mathbf{W^2} + \mathrm{a_3}\mathbf{W^3} + \dots + \mathrm{a_{n-1}}\mathbf{W^{n-1}}$.
This gives the eigenvalues of **M** as

$$\lambda = a_0\alpha + a_1\alpha + a_2\alpha^2 + \dots + a_{n-1}\alpha^{n-1}$$

Back to the matrix $\mathbf{B}(\mathbf{C_n}) = \mathrm{circ}(0,\mathrm{b_1},\mathrm{b_2},\mathrm{b_3},\dots,\mathrm{b_{n-1}})$. For cycles with odd order n we have
$\mathrm{b_k} = \mathrm{b_{n-k}} = \frac{1}{k}$ for $\mathrm{k} = 1,2,\dots,\frac{n-1}{2}$
For cycles with even order, n we have
$\mathrm{b_k} = \mathrm{b_{n-k}} = 1/\mathrm{k}$ for $\mathrm{k} = 1,2,3,\dots,\frac{n-1}{2}$ and $\mathrm{b}_{\frac{n}{2}} = \frac{2}{n}$
since the matrix **B** is also symmetric besides being circulant. This gives the eigenvalues of the matrix $\mathbf{B}(\mathbf{C_n})$ as

$$\lambda = \alpha + \frac{1}{2}\alpha^2 + \frac{1}{3}\alpha^3 + \dots + \frac{1}{3}\alpha^{n-3} + \frac{1}{2}\alpha^{n-2} + \alpha^{n-1}$$

For odd n = 2m + 1, we have

$$\lambda = \sum_{1}^{m}\frac{1}{k}\alpha^k + \sum_{1}^{m}\frac{1}{m-k}\alpha^{m+k} = 2\sum_{1}^{m} real(\frac{1}{k}\alpha^k)$$

where $\alpha = \exp(\frac{2\pi p}{n}i)$ for i=0,1,…,n-1.
And for even n = 2m, we have

$$\lambda = 2\sum_{1}^{m-1} real\left(\frac{1}{k}\alpha^k\right) - 1$$

and where $\alpha = \exp(\frac{2\pi p}{n}i)$ for p=0,1,2,…,n-1.
We consider two special cases as examples, first for an odd n = 11. Here m = 5.
The matrix $\mathbf{B}(\mathbf{C_{11}}) = \mathrm{circ}(0,1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{1}{2},1)$
The eigenvalues of the matrix **B** are given as

$$\lambda = \alpha + \frac{1}{2}\alpha^2 + \frac{1}{3}\alpha^3 + \frac{1}{4}\alpha^4 + \frac{1}{5}\alpha^5 + \frac{1}{5}\alpha^6 + \frac{1}{4}\alpha^7 + \frac{1}{3}\alpha^8 + \frac{1}{2}\alpha^9 + \alpha^{10}$$
$$= 2\sum_{1}^{5} real\left(\frac{1}{k}\alpha^k\right)$$

Where $\alpha = \exp(\frac{2\pi p}{11}i)$ for p=0,1,2,…,10.
While for an even value of n = 10, we have m = 5. The eigenvalues of the matrix **B** are given as

$$\lambda = \alpha + \frac{1}{2}\alpha^2 + \frac{1}{3}\alpha^3 + \frac{1}{4}\alpha^4 + \frac{1}{5}\alpha^5 + \frac{1}{4}\alpha^6 + \frac{1}{3}\alpha^7 + \frac{1}{2}\alpha^8 + \alpha^9$$

where $\alpha = \exp(\frac{2\pi p}{10}i)$ for p=0,1,2,…,9.

$$= \cos\frac{2\pi p}{10} + i\sin\frac{2\pi p}{10}$$

And so

$$\lambda = 2\left(\cos\frac{2\pi p}{10} + \frac{1}{2}\cos\frac{2\pi p}{10} + \frac{1}{3}\cos\frac{2\pi p}{10} + \frac{1}{4}\cos\frac{2\pi p}{10}\right) + \cos\pi p$$

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