Algebraic Operations on Pythagorean neutrosophic sets (PNS): Extending Applicability and Decision-Making Capabilities

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Abstract

Pythagorean neutrosophic sets (PNS) have been recognized as a highly effective mechanism for managing situations characterized by indeterminacy and inconsistency within decision-making procedures. This paper delves into the examination of algebraic operations performed on PNS, thereby expanding their scope of application, and enhancing their utility. In this study, we put forth a set of algebraic operations that can be applied to PNS. These operations encompass addition, multiplication, scalar multiplication, and power. These operations facilitate the efficient manipulation and combination of PNS, thereby enhancing decision-making in scenarios characterized by uncertainty and vagueness. To demonstrate the efficacy of these operations, we will present several illustrative examples accompanied by corroborating proofs. The introduction of algebraic operations enhances the capabilities of PNS, thereby creating opportunities for their practical application.

Keywords: Pythagorean Neutrosophic Set; Algebraic Operations; decision making; addition; multiplication; scalar multiplication.

1. Introduction

The presence of uncertainty is a prevalent characteristic of the tangible world, exerting a substantial impact on the process of making decisions in a wide range of fields such as engineering, finance, medicine, and artificial intelligence [1]. While classical set theory offers a robust framework for managing precise data, it is limited in its ability to capture the inherent vagueness and ambiguity that is commonly encountered in real-world situations [2]. To overcome these constraints, notable advancements such as fuzzy sets and neutrosophic sets have emerged. These developments enable elements to exhibit diverse levels of membership, effectively conveying the degree of uncertainty associated with each membership degree [3], [4]. Furthermore, there have been numerous additional developments in the field of fuzzy sets and neutrosophic sets that have found extensive application in Multiple Criteria Decision Making (MCDM). These advancements have provided decision-makers with the ability to effectively handle intricate and uncertain decision scenarios [5]–[12].

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In recent times, there has been a development in the field of Pythagorean neutrosophic set, which aims to integrate the fundamental concepts of neutrosophic sets with the robust principles of Pythagorean fuzzy sets [13], [14]. The introduction of Pythagorean fuzzy sets allowed for a more adaptable approach to handling uncertainty through the incorporation of three membership degrees: the degree of truth membership, the degree of indeterminacy membership, and the degree of falsity membership. By utilizing this novel concept, PNS provide a more comprehensive framework for addressing intricate decision-making problems with increased granularity. This framework also effectively deals with the uncertainties encountered in real-world data analysis, decision-making processes, and knowledge representation [15-19].

The main aim of this study is to conduct a comprehensive examination of PNS and acquire a deep comprehension of their operational principles. Our objective is to establish the foundation for the utilization of PNS across various fields, thereby facilitating improved decision-making procedures in situations characterized by uncertainty. Furthermore, the primary objective of this research is to make a valuable contribution to the expanding field of neutrosophic set theory. By doing so, it aims to provide a foundation for future scholars to expand upon these findings and broaden the scope of this novel methodology.

2. Preliminaries

This section presented the fundamental theories that are useful in the development of operational rules for PNS.

Definition 2.1 [20] Let X be a non-empty set (universe). A Pythagorean fuzzy set A is an object having the form

$$A=\left\{\left\langle x,\psi_A(x),\kappa_A(x)\right\rangle \mid x\in X\right\}, \tag{1}$$

where $\psi,\kappa\in[0,1]$ denote respectively the truth membership and false membership of each element $x \in X$ to the set A, and $0\le\psi^2+\kappa^2\le 1$ for each $x \in X$. The degree of indeterminacy of x to A is $\varsigma_A(x)=\sqrt{1-\psi_A{}^2(x)-\kappa_A{}^2(x)}$.

Definition 2.2 [21] Given two PFSs $x_1=\left(\psi_{x_1},\kappa_{x_1}\right)$ and $x_2=\left(\psi_{x_2},\kappa_{x_2}\right)$, then the operational rules for PFS are defined as follows:

(i) $$x_1\oplus x_2=\left(\sqrt{\psi_{x_1}{}^2+\psi_{x_2}{}^2-\psi_{x_1}{}^2\psi_{x_2}{}^2},\kappa_{x_1}\kappa_{x_2}\right) \tag{2}$$

(ii) $$x_1\otimes x_2=\left(\psi_{x_1}\psi_{x_2},\sqrt{\kappa_{x_1}{}^2+\kappa_{x_2}{}^2-\kappa_{x_1}{}^2\kappa_{x_2}{}^2}\right) \tag{3}$$

(iii) $$\mu x=\left(\sqrt{1-\left(1-\psi_x{}^2\right)^{\mu}},\kappa_x{}^{\mu}\right) \text{ where } \mu\in\Re \text{ and } \lambda\ge 0, \tag{4}$$

(iv) $$x^{\mu}=\left(\psi_x{}^{\mu},\sqrt{1-\left(1-\kappa_x{}^2\right)^{\mu}}\right) \text{ where } \mu\in\Re \text{ and } \lambda\ge 0. \tag{5}$$

Definition 2.3 (Wang et. al., 2010) Let X be a non-empty set (universe). A single valued neutrosophic set X in β is defined as:

$$\beta=\left\{\left\langle x,\psi_\beta(x),\varsigma_\beta(x),\kappa_\beta(x)\right\rangle \mid x\in X\right\}, \cdots\cdots(6)$$

Where $\psi_\beta(x),\varsigma_\beta(x),\kappa_\beta(x)\in[0,1]$, and no restriction on the sum of the components so $0\le\psi_\beta(x)+\varsigma_\beta(x)+\kappa_\beta(x)\le 3$.

Definition 2.4 (Wang et. al., 2010) Let $x_1=\left(\psi_{x_1},\varsigma_{x_1},\kappa_{x_1}\right)$, $x_2=\left(\psi_{x_2},\varsigma_{x_2},\kappa_{x_2}\right)$ are any two SVNNs and $x=\left(\psi_x,\varsigma_x,\kappa_x\right)$, then the operational rules for SVNSs are defined as follows:

(i) $$x_1\oplus x_2=\left(\psi_{x_1}+\psi_{x_2}-\psi_{x_1}\psi_{x_2},\varsigma_{x_1}\varsigma_{x_2},\kappa_{x_1}\kappa_{x_2}\right) \tag{7}$$

(ii) $$x_1\otimes x_2=\left(\psi_{x_1}\psi_{x_2},\varsigma_{x_1}+\varsigma_{x_2}-\varsigma_{x_1}\varsigma_{x_2},\kappa_{x_1}+\kappa_{x_2}-\kappa_{x_1}\kappa_{x_2}\right) \tag{8}$$

(iii) $$\mu x=\left(1-\left(1-\psi_x\right)^{\mu},\varsigma_x{}^{\mu},\kappa_x{}^{\mu}\right) \text{ where } \mu\in\Re \text{ and } \mu\ge 0, \tag{9}$$

(iv) $$x^{\mu}=\left(\psi_x{}^{\mu},1-\left(1-\varsigma_x\right)^{\mu},1-\left(1-\kappa_x\right)^{\mu}\right) \text{ where } \mu\in\Re \text{ and } \mu\ge 0. \tag{10}$$

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Definition 2.5 [22] Let $x=(\psi_x,\varsigma_x,\kappa_x)$ be a SVNN, then the cosine similarity degree

$$\vartheta(x)=\frac{\psi_x}{\sqrt{\psi_x^{\ 2}+\varsigma_x^{\ 2}+\kappa_x^{\ 2}}} \tag{11}$$

It measures the cosine similarity between $x=(\psi_x,\varsigma_x,\kappa_x)$. ideal solution (1, 0, 0) for the comparison of SVNNs.

Definition 2.6 [13] Let X be a non-empty set (universe). A Pythagorean neutrosophic set with T and F as dependent neutrosophic components A on X is an object of the form

$$A=\{\langle x,\psi_A(x),\varsigma_A(x),\kappa_A(x)\rangle \mid x\in X\} \tag{12}$$

where $\psi_A(x)$, $\varsigma_A(x)$ and $\kappa_A(x)$ are the truth, indeterminacy and false membership respectively such that $\psi,\varsigma,\kappa\in[0,1]$. Here when ψ and κ are dependent components, then for all x in X;

$$\psi+\kappa\le 1 \tag{13}$$

$$0\le\psi^2+\kappa^2\le 1 \tag{14}$$

$$0\le\psi^2+\varsigma^2+\kappa^2\le 2 \tag{15}$$

3. Operations for Pythagorean Neutrosophic Set

In this study, we propose basic operations for PNS, offering enhanced capabilities for handling uncertainty and inconsistency in decision-making. These operations pave the way for more effective and comprehensive data analysis and knowledge representation in complex real-world scenarios.

Definition 3.1 [12] defined these basic operations on PNS which can be described as follows:

Let X be a non-empty set (universe). A Pythagorean Neutrosophic Set with ψ and κ as dependent neutrosophic components A and B of the form

$A=\{\langle x,\psi_A(x),\varsigma_A(x),\kappa_A(x)\rangle \mid x\in X\}$ and $B=\{\langle x,\psi_B(x),\varsigma_B(x),\kappa_B(x)\rangle \mid x\in X\}$.

The complement of A is $A^C=\{\langle x,\kappa_A(x),1-\varsigma_A(x),\psi_A(x)\rangle : x\in X\}$.

The union and intersection of A and B are

$$A\cup B=\{\max(\psi_A,\psi_B),\min(\varsigma_A,\varsigma_B),\min(\kappa_A,\kappa_B)\}$$

$$A\cap B=\{\min(\psi_A,\psi_B),\max(\varsigma_A,\varsigma_B),\max(\kappa_A,\kappa_B)\}$$

On the basis of relationship between PFS and SVNS in Definition 2.1 – 2.4 , we further define some novel operations on PNS as below :

Definition 3.2 Given $x_1=(\psi_{x_1},\varsigma_{x_1},\kappa_{x_1})$, $x_2=(\psi_{x_2},\varsigma_{x_2},\kappa_{x_2})$ and $x=(\psi_x,\varsigma_x,\kappa_x)$ are any two PNSs, then the operational rules for PNSs such as addition, multiplication, scalar multiplication and power operations are defined as follows:

3.2.1 The PNS addition denoted $x_1\oplus x_2$ are defined as :

$$x_1\oplus x_2=(\psi_{x_1},\varsigma_{x_1},\kappa_{x_1})+(\psi_{x_2},\varsigma_{x_2},\kappa_{x_2})$$
$$=\left(\sqrt{\psi_{x_1}^{\ 2}+\psi_{x_2}^{\ 2}-\psi_{x_1}^{\ 2}\psi_{x_2}^{\ 2}},\varsigma_{x_1}\varsigma_{x_2},\kappa_{x_1}\kappa_{x_2}\right)$$

3.2.2 The PNS multiplication denoted $x_1\otimes x_2$ are defined as :

$$x_1\otimes x_2=(\psi_{x_1},\varsigma_{x_1},\kappa_{x_1})\otimes(\psi_{x_2},\varsigma_{x_2},\kappa_{x_2})$$
$$=\left(\psi_{x_1}\psi_{x_2},\varsigma_{x_1}+\varsigma_{x_2}-\varsigma_{x_1}\varsigma_{x_2},\sqrt{\kappa_{x_1}^{\ 2}+\kappa_{x_2}^{\ 2}-\kappa_{x_1}^{\ 2}\kappa_{x_2}^{\ 2}}\right)$$

3.2.3 The PNS scalar multiplication denoted μx where $\mu\in R$ and $\mu\ge 0$ are defined as :

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$$\mu x = \mu\left(\psi_x, \varsigma_x, \kappa_x\right) = \left(\sqrt{1-\left(1-\psi_x^{\,2}\right)^{\mu}}, \varsigma_x^{\,\mu}, \kappa_x^{\,\mu}\right)$$

3.2.3 The PNS power operation denoted μx where $\mu \in R$ and $\mu \geq 0$ are defined as :

$$x^{\mu} = \left(\psi_x, \varsigma_x, \kappa_x\right)^{\mu} = \left(\psi_x^{\,\mu}, 1-\left(1-\varsigma_x\right)^{\mu}, \sqrt{1-\left(1-\kappa_x^{\,2}\right)^{\mu}}\right)$$

Theorem 3.1 For three PNSs $x_1 = \left(\psi_{x_1}, \varsigma_{x_1}, \kappa_{x_1}\right)$, $x_2 = \left(\psi_{x_2}, \varsigma_{x_2}, \kappa_{x_2}\right)$ and $x = \left(\psi_x, \varsigma_x, \kappa_x\right)$, the following are valid:

(i) $x_1 \oplus x_2 = x_2 \oplus x_1$

(ii) $x_1 \otimes x_2 = x_2 \otimes x_1$

(iii) $\mu\left(x_1 \oplus x_2\right) = \mu x_2 \oplus \mu x_1$ where $\mu > 0$

(iv) $\mu_1 x \oplus \mu_2 x = \left(\mu_1 + \mu_2\right) x$ where $\mu_1, \mu_2 > 0$

(v) $x_1^{\mu} \otimes x_2^{\mu} = \left(x_1 \otimes x_2\right)^{\mu}$ where $\mu > 0$

(vi) $x^{\mu_1} \otimes x^{\mu_2} = x^{(\mu_1+\mu_2)}$ where $\mu_1, \mu_2 > 0$

Proof. For three PNSs x, x_1 and x_2 and $\mu, \mu_1, \mu_2 > 0$, we can obtain :

(i)
$$\begin{aligned}
x_1 \oplus x_2 &= \left(\psi_{x_1}, \varsigma_{x_1}, \kappa_{x_1}\right) + \left(\psi_{x_2}, \varsigma_{x_2}, \kappa_{x_2}\right) \\
&= \left(\sqrt{\psi_{x_1}^{\,2} + \psi_{x_2}^{\,2} - \psi_{x_1}^{\,2}\psi_{x_2}^{\,2}}, \varsigma_{x_1}\varsigma_{x_2}, \kappa_{x_1}\kappa_{x_2}\right) \\
&= \left(\sqrt{\psi_{x_2}^{\,2} + \psi_{x_1}^{\,2} - \psi_{x_2}^{\,2}\psi_{x_1}^{\,2}}, \varsigma_{x_2}\varsigma_{x_1}, \kappa_{x_2}\kappa_{x_1}\right) \\
&= \left(\psi_{x_2}, \varsigma_{x_2}, \kappa_{x_2}\right) + \left(\psi_{x_1}, \varsigma_{x_1}, \kappa_{x_1}\right) \\
&= x_2 \oplus x_1
\end{aligned}$$

(ii)
$$\begin{aligned}
x_1 \otimes x_2 &= \left(\psi_{x_1}, \varsigma_{x_1}, \kappa_{x_1}\right) \otimes \left(\psi_{x_2}, \varsigma_{x_2}, \kappa_{x_2}\right) \\
&= \left(\psi_{x_1}\psi_{x_2}, \varsigma_{x_1} + \varsigma_{x_2} - \varsigma_{x_1}\varsigma_{x_2}, \sqrt{\kappa_{x_1}^{\,2} + \kappa_{x_2}^{\,2} - \kappa_{x_1}^{\,2}\kappa_{x_2}^{\,2}}\right) \\
&= \left(\psi_{x_2}\psi_{x_1}, \varsigma_{x_2} + \varsigma_{x_1} - \varsigma_{x_2}\varsigma_{x_1}, \sqrt{\kappa_{x_2}^{\,2} + \kappa_{x_1}^{\,2} - \kappa_{x_2}^{\,2}\kappa_{x_1}^{\,2}}\right) \\
&= \left(\psi_{x_2}, \varsigma_{x_2}, \kappa_{x_2}\right) \otimes \left(\psi_{x_1}, \varsigma_{x_1}, \kappa_{x_1}\right) \\
&= x_2 \otimes x_1
\end{aligned}$$

(iii) $\mu x_1 \oplus \mu x_2 = \mu\left(\psi_{x_1}, \varsigma_{x_1}, \kappa_{x_1}\right) + \mu\left(\psi_{x_2}, \varsigma_{x_2}, \kappa_{x_2}\right)$

$$= \left(\sqrt{1-\left(1-\psi_{x_1}^{\,2}\right)^{\mu}}, \varsigma_{x_1}^{\,\mu}, \kappa_{x_1}^{\,\mu}\right) + \left(\sqrt{1-\left(1-\psi_{x_2}^{\,2}\right)^{\mu}}, \varsigma_{x_2}^{\,\mu}, \kappa_{x_2}^{\,\mu}\right)$$

$$= \left(\begin{array}{l}\sqrt{1-\left(1-\psi_{x_1}^{\,2}\right)^{\mu} + 1-\left(1-\psi_{x_2}^{\,2}\right)^{\mu} - \left(1-\left(1-\psi_{x_1}^{\,2}\right)^{\mu}\right)\left(1-\left(1-\psi_{x_2}^{\,2}\right)^{\mu}\right)}, \\ \varsigma_{x_1}^{\,\mu}\varsigma_{x_2}^{\,\mu}, \kappa_{x_1}^{\,\mu}\kappa_{x_2}^{\,\mu}\end{array}\right)$$

$$= \left(\begin{array}{l}\sqrt{1-\left(1-\psi_{x_1}^{\,2}\right)^{\mu} + 1-\left(1-\psi_{x_2}^{\,2}\right)^{\mu} - \left(1-\left(1-\psi_{x_1}^{\,2}\right)^{\mu} - \left(1-\psi_{x_2}^{\,2}\right)^{\mu} + \left(1-\psi_{x_1}^{\,2}\right)^{\mu}\left(1-\psi_{x_2}^{\,2}\right)^{\mu}\right)}, \\ \varsigma_{x_1}^{\,\mu}\varsigma_{x_2}^{\,\mu}, \kappa_{x_1}^{\,\mu}\kappa_{x_2}^{\,\mu}\end{array}\right)$$

$$= \left(\begin{array}{l}\sqrt{1-\left(1-\psi_{x_1}^{\,2}\right)^{\mu} + 1-\left(1-\psi_{x_2}^{\,2}\right)^{\mu} - 1 + \left(1-\psi_{x_1}^{\,2}\right)^{\mu} + \left(1-\psi_{x_2}^{\,2}\right)^{\mu} - \left(1-\psi_{x_1}^{\,2}\right)^{\mu}\left(1-\psi_{x_2}^{\,2}\right)^{\mu}}, \\ \varsigma_{x_1}^{\,\mu}\varsigma_{x_2}^{\,\mu}, \kappa_{x_1}^{\,\mu}\kappa_{x_2}^{\,\mu}\end{array}\right)$$

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$$=\left(\sqrt{1-\left(1-\psi_{x_1}{}^2\right)^{\mu}\left(1-\psi_{x_2}{}^2\right)^{\mu}},\varsigma_{x_1}{}^{\mu}\varsigma_{x_2}{}^{\mu},\kappa_{x_1}{}^{\mu}\kappa_{x_2}{}^{\mu}\right)$$

$$=\left(\sqrt{1-\left(\left(1-\psi_{x_1}{}^2\right)\left(1-\psi_{x_2}{}^2\right)\right)^{\mu}},\varsigma_{x_1}{}^{\mu}\varsigma_{x_2}{}^{\mu},\kappa_{x_1}{}^{\mu}\kappa_{x_2}{}^{\mu}\right)$$

$$=\left(\sqrt{1-\left(1-\psi_{x_1}{}^2-\psi_{x_2}{}^2+\psi_{x_1}{}^2\psi_{x_2}{}^2\right)^{\mu}},\varsigma_{x_1}{}^{\mu}\varsigma_{x_2}{}^{\mu},\kappa_{x_1}{}^{\mu}\kappa_{x_2}{}^{\mu}\right)$$

$$=\left(\sqrt{1-\left(1-\left(\psi_{x_1}{}^2+\psi_{x_2}{}^2-\psi_{x_1}{}^2\psi_{x_2}{}^2\right)\right)^{\mu}},\left(\varsigma_{x_1}\varsigma_{x_2}\right)^{\mu},\left(\kappa_{x_1}\kappa_{x_2}\right)^{\mu}\right)$$

$$=\mu\left(x_1+x_2\right)$$

(iv) $\mu_1 x\oplus\mu_2 x=\mu_1\left(\psi_x,\varsigma_x,\kappa_x\right)+\mu_2\left(\psi_x,\varsigma_x,\kappa_x\right)$

$$=\left(\sqrt{1-\left(1-\psi_x{}^2\right)^{\mu_1}},\varsigma_x{}^{\mu_1},\kappa_x{}^{\mu_1}\right)+\left(\sqrt{1-\left(1-\psi_x{}^2\right)^{\mu_2}},\varsigma_x{}^{\mu_2},\kappa_x{}^{\mu_2}\right)$$

$$=\left(\begin{array}{l}\sqrt{1-\left(1-\psi_x{}^2\right)^{\mu_1}+1-\left(1-\psi_x{}^2\right)^{\mu_1}-\left(1-\left(1-\psi_x{}^2\right)^{\mu_2}\right)\left(1-\left(1-\psi_x{}^2\right)^{\mu_2}\right)},\\ \varsigma_x{}^{\mu_1}\varsigma_x{}^{\mu_2},\kappa_x{}^{\mu_1}\kappa_x{}^{\mu_2}\end{array}\right)$$

$$=\left(\sqrt{1-\left(1-\psi_x{}^2\right)^{\mu_1}\left(1-\psi_x{}^2\right)^{\mu_2}},\varsigma_x{}^{\mu_1}\varsigma_x{}^{\mu_2},\kappa_x{}^{\mu_1}\kappa_x{}^{\mu_2}\right)$$

$$=\left(\sqrt{1-\left(1-\psi_x{}^2\right)^{\mu_1+\mu_2}},\varsigma_x\varsigma_x{}^{\mu_1+\mu_2},\kappa_x\kappa_x{}^{\mu_1+\mu_2}\right)$$

$$=\left(\mu_1+\mu_2\right)x$$

(v) $\left(x_1\otimes x_2\right)^{\mu}=\left(\left(\psi_{x_1},\varsigma_{x_1},\kappa_{x_1}\right)\otimes\left(\psi_{x_2},\varsigma_{x_2},\kappa_{x_2}\right)\right)^{\mu}$

$$=\left(\psi_{x_1}\psi_{x_2},\varsigma_{x_1}+\varsigma_{x_2}-\varsigma_{x_1}\varsigma_{x_2},\sqrt{\kappa_{x_1}{}^2+\kappa_{x_2}{}^2-\kappa_{x_1}{}^2\kappa_{x_2}{}^2}\right)^{\mu}$$

$$=\left(\left(\psi_{x_1}\psi_{x_2}\right)^{\mu},1-\left(1-\varsigma_{x_1}+\varsigma_{x_2}-\varsigma_{x_1}\varsigma_{x_2}\right)^{\mu},\sqrt{1-\left(1-\kappa_{x_1}{}^2+\kappa_{x_2}{}^2-\kappa_{x_1}{}^2\kappa_{x_2}{}^2\right)^{\mu}}\right)$$

$$=\left(\psi_{x_1}{}^{\mu}\psi_{x_2}{}^{\mu},1-\left(\left(1-\varsigma_{x_1}\right)\left(1+\varsigma_{x_2}\right)\right)^{\mu},\sqrt{1-\left(\left(1-\kappa_{x_1}{}^2\right)\left(1+\kappa_{x_2}{}^2\right)\right)^{\mu}}\right)$$

$$=\left(\psi_{x_1}{}^{\mu}\psi_{x_2}{}^{\mu},1-\left(1-\varsigma_{x_1}\right)^{\mu}\left(1+\varsigma_{x_2}\right)^{\mu},\sqrt{1-\left(1-\kappa_{x_1}{}^2\right)^{\mu}\left(1+\kappa_{x_2}{}^2\right)^{\mu}}\right)$$

$$=x_1{}^{\mu}\otimes x_2{}^{\mu}$$

(vi) $x^{\mu_1}\otimes x^{\mu_2}=\left(\psi_x,\varsigma_x,\kappa_x\right)^{\mu_1}\otimes\left(\psi_x,\varsigma_x,\kappa_x\right)^{\mu_2}$

$$=\left(\psi_x{}^{\mu_1},1-\left(1-\varsigma_x\right)^{\mu_1},\sqrt{1-\left(1-\kappa_x{}^2\right)^{\mu_1}}\right)\otimes\left(\psi_x{}^{\mu_2},1-\left(1-\varsigma_x\right)^{\mu_2},\sqrt{1-\left(1-\kappa_x{}^2\right)^{\mu_2}}\right)$$

$$=\left(\begin{array}{l},1-\left(1-\varsigma_x\right)^{\mu_1}+1-\left(1-\varsigma_x\right)^{\mu_2}-\left(1-\left(1-\varsigma_x\right)^{\mu_1}\right)\left(1-\left(1-\varsigma_x\right)^{\mu_2}\right),\\ \sqrt{1-\left(1-\kappa_x{}^2\right)^{\mu_1}+1-\left(1-\kappa_x{}^2\right)^{\mu_2}-\left(1-\left(1-\kappa_x{}^2\right)^{\mu_1}\right)\left(1-\left(1-\kappa_x{}^2\right)^{\mu_2}\right)}\end{array}\right)$$

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$$=\left(\psi_x^{\mu_1}\psi_x^{\mu_2},1-(1-\varsigma_x)^{\mu_1}(1-\varsigma_x)^{\mu_2},\sqrt{1-(1-\kappa_x^{\ 2})^{\mu_1}(1-\kappa_x^{\ 2})^{\mu_2}}\right)$$

$$=\left((\psi_x\psi_x)^{\mu_1+\mu_2},1-(1-\varsigma_x)^{\mu_1+\mu_2},\sqrt{1-(1-\kappa_x^{\ 2})^{\mu_1+\mu_2}}\right)$$

$$=(\psi_x,\varsigma_x,\kappa_x)^{\mu_1+\mu_2}=x^{\mu_1+\mu_2}$$

which complete the proof of the theorem.

4. Numerical Examples

To illustrate these operations, examples are given and we recall Theorem 1, definition 3.2.1, 3.2.2, 3.2.3 and 3.2.4.

Example 4.1 Let $A_1=\{\langle 0.1,0.8,0.9\rangle \mid x\in X\}$ and $A_2=\{\langle 0.2,0.7,0.8\rangle \mid x\in X\}$ are the Pythagorean neutrosophic numbers, the the PNS addition operation is given as:

By definition 3.2.1, $A_1\oplus A_2=(0.1,0.8,0.9)+(0.2,0.7,0.8)$

$$=\left(\sqrt{(0.1)^2+(0.2)^2-(0.1)^2(0.2)^2},(0.8)(0.7),(0.9)(0.8)\right)$$

$$=(0.2227,\ 0.5600,\ 0.7200)$$

From Definition 2.6, we have $\psi+\kappa=0.9427$, $\psi^2+\kappa^2=0.568$ and $\psi^2+\varsigma^2+\kappa^2=0.8816$ satisfying all the three conditions of PNS.

Example 4.2 Consider Example 4.1, then by Definition 3.2.2, the PNS multiplication between A_1 and A_2 can be calculated as follows:

$$A_1\otimes A_2=(0.1,0.8,0.9)\otimes(0.2,0.7,0.8)$$

$$=\left((0.1)(0.2),(0.8)+(0.7)-(0.8)(0.7),\sqrt{0.9^2+0.8^2-(0.9)^2(0.8)^2}\right)$$

$$=(0.0200,0.9400,0.4658)$$

where $\psi+\kappa=0.4858$, $\psi^2+\kappa^2=0.2174$ and $\psi^2+\varsigma^2+\kappa^2=1.1010$ satisfying all the three conditions of PNS.

Example 4.3 Consider Example 4.1, then by Definition 3.2.3 with $\mu=3$, we have

$$\mu A=\left(\sqrt{1-(1-0.1^2)^{\mu}},0.8^{\mu},0.9^{\mu}\right)$$

$$3A=\left(\sqrt{1-(1-0.1^2)^{3}},0.8^{3},0.9^{3}\right)$$

$$=(0.1723,0.5120,0.7290)$$

where $\psi+\kappa=0.9013$, $\psi^2+\kappa^2=0.5611$ and $\psi^2+\varsigma^2+\kappa^2=0.8233$ satisfying all the three conditions of PNS.

Example 4.4 Consider Example 4.1, then by Definition 3.2.4 with $\mu=3$, we have

$$A^3=(0.1,0.8,0.9)^3$$

$$=\left(0.1^3,1-(1-0.8)^3,\sqrt{1-(1-0.9^2)^3}\right)$$

$$=(0.0010,0.9920,0.9966)$$

where $\psi+\kappa=0.9976$, $\psi^2+\kappa^2=0.9931$ and $\psi^2+\varsigma^2+\kappa^2=1.9772$ satisfying all the three conditions of PNS.

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5. Conclusion

By investigating PNS, this research aims to enrich the domain of set theory and fuzzy logic with an innovative approach to handle uncertainty. The findings of this study will contribute to a deeper understanding of PNS, their operational rules, and potential applications. Ultimately, we hope that this research will inspire further exploration and advancements in the field, leading to practical implementations of this novel concept in real-world decision-making and problem-solving domains.

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