Fixed point theorems for self maps on 4-dimensional ball metric spaces and extension to $\mathrm{n}-\mathrm{dimensional}\ (\mathrm{n} \geq 4)$ ball metric spaces

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Abstract

Metric spaces are generalized to three variables and are termed as S - metric spaces, which in turn are extended to four variables and are termed as B_4 - metric spaces. Now we extend this notion to n - variables $(n \geq 4)$, which we name them as B_n - metric spaces. We study novel contractive mappings on B_n - metric spaces. In this paper, we obtain a fixed point theorem for a self map on a 4-dimensional ball metric spaces and also obtain a fixed point theorem for a self map on a n - dimensional $(n \geq 4)$ ball metric spaces generalize known results in fixed point theorems on metric spaces.

Keywords: B_4 - metric spaces, B_n - metric spaces, $4-dimensional$ ball metric spaces, $n-dimensional$ $(n \geq 4)$ ball metric spaces, Fixed point theorems.

1. Introduction

B_4 - metric space is introduced in [1] as an extension of S - metric space [6] (S. Sedghi, N. Shobe, A. Aliouche). Further B_n - metric space $(n \geq 4)$ is introduced in [2]. Fixed point results are established in [3] and [4]. In this paper a B_4 - metric space is renamed as $4-dimensional$ ball metric space and B_n - metric space ($n \geq 4$) is renamed as $n-dimensional$ ball metric space. This nomenclature is in conjunction with rectangular S - metric spaces [5]. We establish a fixed point theorem for a self-map on a $4-dimensional$ ball metric space and obtain applications. We also obtain a fixed point theorem for a self map on a $n-dimensional$ ball metric space and obtain applications. Examples of the above spaces can be found in [3] and [4].

However, we provide some more examples in this paper.

Extensions of the contraction principles can also be found in ([7] $-$ [20]) to cite a few.

2. Preliminaries

For 4-dimensional ball metric spaces:

2.1 Definition: [1] Let $\Omega \neq \phi$ and $B_4 : \Omega^4 \to \mathbb{R}^+$ satisfy the following axioms: for all $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \alpha \in \Omega$

(i) $B_4(\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4) = 0 \Leftrightarrow \varsigma_1 = \varsigma_2 = \varsigma_3 = \varsigma_4$

(ii) $B_4(\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4) \leqslant \begin{Bmatrix} B_4(\varsigma_1, \varsigma_1, \varsigma_1, \alpha) + B_4(\varsigma_2, \varsigma_2, \varsigma_2, \alpha) \\ +B_4(\varsigma_3, \varsigma_3, \varsigma_3, \alpha) + B_4(\varsigma_4, \varsigma_4, \varsigma_4, \alpha) \end{Bmatrix}$

Then we say that B_4 is a $4-dimensional$ ball metric on Ω and the pair (Ω, B_4) is a $4-dimensional$ ball metric space.

Various examples of $4-dimensional$ ball metric spaces can be found in [1].

The following are some more examples.

2.2 Example: Suppose $\Omega = N \cup \{0\}$ and define $B_4 : \Omega^4 \to \mathbb{R}^+ \cup \{0\}$ by

$$B_4(\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4)=\begin{cases}0, & \text{if } \varsigma_1=\varsigma_2=\varsigma_3=\varsigma_4\\ \varsigma_1^2+\varsigma_2^2+\varsigma_1^3+\varsigma_4^2, & \text{otherwise}\end{cases}$$

Then (Ω, B_4) is a $4-\mathit{dimensional}$ ball metric space.

2.3 Example: Let $\Omega = N \cup \{0\}$ and define $B_4\colon \Omega^4 \to \mathbb{R}^+ \cup \{0\}$ by

$$B_4(\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4)=\begin{cases}0, & \text{if } \varsigma_1=\varsigma_2=\varsigma_3=\varsigma_4\\ \varsigma_1+\varsigma_2+\varsigma_3+\varsigma_4, & \text{otherwise}\end{cases}$$

Then (Ω, B_4) is a $4-$dimensional ball metric space

2.4 Example: Let $\Omega = N \cup \{0\}, \lambda > 0$. Define $B_4\colon \Omega^4 \to \mathbb{R}^+ \cup \{0\}$

by $B_4(\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4)=\begin{cases}0, & \text{if } \varsigma_1=\varsigma_2=\varsigma_3=\varsigma_4\\ \lambda, & \text{otherwise}\end{cases}$

where $\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4 \in \Omega$.

Then (Ω, B_4) is a $4-\mathit{dimensional}$ ball metric space.

The definition of Convergence, Cauchy sequence, and completeness in a 4 -dimensional ball metric space can be found in [1].

We now state a few lemmas, which we use in our further development.

$\mathbf{2.5}$ **Lemma**: [1] *Let* (Ω, B_4) *be a* $4-\mathit{dimensional\ ball\ metric\ space.}$

Then $B_4(\varsigma_1,\varsigma_2,\varsigma_2,\varsigma_2)=B_4(\varsigma_2,\varsigma_2,\varsigma_2,\varsigma_1), \mathit{for\ all} \varsigma_1,\varsigma_2 \in \Omega.$

$\mathbf{2.6}$ **Lemma**: [1] $\varsigma_m \to \varsigma \Leftrightarrow B_4(\varsigma,\varsigma,\varsigma,\varsigma_m) \to 0 \ \mathit{as}\ m \to \infty$

$\mathbf{2.7}$ **Lemma:** [1] $\varsigma_m \to \eta_1, \varsigma_m \to \eta_2 \Rightarrow \eta_1 = \eta_2$.

$\mathbf{2.8}$ **Lemma:** [1] $\varsigma_m \to \varsigma \Rightarrow \{\varsigma_m\}$ *is a Cauchy Sequence.*

<u>For n -dimensional ball metric spaces:</u>

Suppose $n(\geq 4)$ is a positive integer.

$\mathbf{2.9}$ **Definition** [2] Let $\Omega \neq \emptyset$ and $B_n\colon \Omega^n \to \mathbb{R}$ satisfy the following axioms: for all$\varsigma_1,\varsigma_2,\dots,\varsigma_n,\alpha \in \Omega$

(i) $B_n(\varsigma_1,\varsigma_2,\dots,\varsigma_n)=0 \Leftrightarrow \varsigma_1=\varsigma_2=\cdots=\varsigma_n,$

(ii) $B_n(\varsigma_1,\varsigma_2,\dots,\varsigma_n) \leq B_n(\varsigma_1,\varsigma_1,\dots,\varsigma_1,\alpha)+B_n(\varsigma_2,\varsigma_2,\dots,\varsigma_2,\alpha)+\cdots+B_n(\varsigma_n,\varsigma_n,\dots,\varsigma_n,\alpha)$

Then we say that B_n is a $n-\mathit{dimensional}$ ball metric on Ω and (Ω, B_n) is a $n-\mathit{dimensional}$ ball metric space.

Clearly, this definition extends a $4-\mathit{dimensional}$ ball metric space to n dimensional ball metric space, for $n \geq 4$.

Various examples of $n-\mathit{dimensional}$ ball metric spaces can be found in [2]. The following are some more examples.

$\mathbf{2.10}$ **Example:** Suppose $\Omega = N \cup \{0\}$. Define $B_{\mathrm{n}}\colon \Omega^n \to \mathbb{R}^+ \cup \{0\}$ by

$$B_n(\varsigma_1,\varsigma_2,\dots,\varsigma_n)=\begin{cases}0, & \text{if } \varsigma_1=\varsigma_2=\cdots=\varsigma_n\\ \varsigma_1^2+\varsigma_2^2+\cdots+\varsigma_n^2, & \text{otherwise}\end{cases}$$

Then (Ω, B_{n}) is a $n-\mathit{dimensional}$ ball metric space.

$\mathbf{2.11}$ **Example:** Suppose $\Omega = N \cup \{0\}$. Define $B_{\mathrm{n}}\colon \Omega^n \to \mathbb{R}^+ \cup \{0\}$ by

$$B_n(\varsigma_1,\varsigma_2,\dots,\varsigma_n)=\begin{cases}0, & \text{if } \varsigma_1=\varsigma_2=\cdots=\varsigma_n\\ \varsigma_1+\varsigma_2+\cdots+\varsigma_n, & \text{otherwise}\end{cases}$$

Then (Ω, B_{n}) is a $n-dimensional$ ball metric space.

2.12 Example: Suppose $\Omega = N\cup\{0\}, \lambda > 0$. Define $B_{\mathrm{n}}\colon \Omega^n\to\mathbb{R}^+\cup\{0\}$ by

$$B_{\mathrm{n}}(\varsigma_1,\varsigma_2,\dots,\varsigma_n)=\begin{cases}0, & \text{if } \varsigma_1=\varsigma_2=\cdots=\varsigma_n\\ \lambda, & \text{otherwise}\end{cases}$$

Then (Ω, B_{n}) is a $n-dimensional$ ball metric space.

The definition of Convergence, Cauchy sequence, and completeness in a $n-dimensional$ ball metric space can be found in [2].

We now state a few lemmas, which we use in our further development.

2.13 Lemma: [2] *Let (Ω, B_n) be a $n-dimensional$ ball metric space.*

Then $B_n(\underbrace{\varsigma_1,\varsigma_1,\dots,\varsigma_1}_{(n-1)\ times},\varsigma_2) = B_n(\underbrace{\varsigma_2,\varsigma_2,\dots,\varsigma_2}_{(n-1)\ times},\varsigma_1)$, *for all* $\varsigma_1,\varsigma_2\in\Omega$.

2.14 Lemma: [2] $\varsigma_m\to\varsigma$ if and only if $B_n(\underbrace{\varsigma,\varsigma,\dots,\varsigma}_{(n-1)\text{ times}},\varsigma_m)\to 0$ as $m\to\infty$.

3. Main result for 4 - dimensional ball metric spaces

First, we prove the following Lemma and use it in our main result.

3.1 Lemma: *Let $\Omega\neq\emptyset$ and $B_4\colon\Omega^4\to\Re^+\cup\{0\}$ be a $4-dimensional\ ball\ metric\ space$ on Ω.*

Then $B_4(\varsigma,\beta,\beta,\beta)\le B_4(\varsigma,\gamma,\gamma,\gamma)+3B_4(\gamma,\beta,\beta,\beta), for\ all\ \varsigma,\beta,\gamma,\alpha\in\Omega$

Proof: Suppose (Ω, B_4) is a $4-dimensional$ ball metric space.

Replacing $y\varsigma_2,\varsigma_3,\varsigma_4$ by β and α by γ in definition 6.2.1 (ii), we get,

$$B_4(\varsigma,\beta,\beta,\beta)\le\begin{Bmatrix}B_4(\varsigma,\varsigma,\varsigma,\gamma)+B_4(\beta,\beta,\beta,\gamma)\\ +B_4(\beta,\beta,\beta,\gamma)+B_4(\beta,\beta,\beta,\gamma)\end{Bmatrix}$$

Therefore $B_4(\varsigma,\beta,\beta,\beta)\le B_4(\varsigma,\varsigma,\varsigma,\gamma)+3\,B_4(\beta,\beta,\beta,\gamma)$

Therefore $B_4(\varsigma,\beta,\beta,\beta)\le B_4(\varsigma,\gamma,\gamma,\gamma)+3\,B_4(\gamma,\beta,\beta,\beta)$ (by Lemma 6.2.5)

Now we state and prove our main theorem on 4-dimensional ball metric spaces.

3.2 Theorem: *Suppose (Ω, B_4) is a complete 4-dimensional ball metric space and $\top\colon\Omega\to\Omega$ is a map. Suppose $0\le k<\frac{1}{3}$ is such that for $\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4\in\Omega$,*

$B_4(\top\varsigma_1,\top\varsigma_2,\top\varsigma_3,\top\varsigma_4)\le kB_4(\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4)$ (3.2.1)

Then $\top$ has only one fixed point.

Proof: For $\varsigma_1,\varsigma_2\in\Omega$, from (3.2.1), we have

$B_4(\top\varsigma_1,\top\varsigma_2,\top\varsigma_2,\top\varsigma_2)\le kB_4(\varsigma_1,\varsigma_2,\varsigma_2,\varsigma_2)$

Let $\varsigma_0 \in \Omega$. Define the sequence $\{\varsigma_n\}$ by $\varsigma_n = \top^n \varsigma_0$, for $n = 1,2,3, \ldots$

Then clearly $\varsigma_{n+1} = \top^n \varsigma_n$.

$B_4(\varsigma_n, \varsigma_n, \varsigma_n, \varsigma_{n+1}) = B_4(\top\varsigma_{n-1}, \top\varsigma_{n-1}, \top\varsigma_{n-1}, \top\varsigma_n) \leq kB_4(\varsigma_{n-1}, \varsigma_{n-1}, \varsigma_{n-1}, \varsigma_n)$

Write $s_n = B_4(\varsigma_n, \varsigma_n, \varsigma_n, \varsigma_{n+1})$

We have $s_n \leq ks_{n-1} \leq k^2 s_{n-2}$

Therefore $s_n \leq k^n s_0$for all $n \in N$.   (3.2.2)

Suppose $m > n$

This shows that $s_n \longrightarrow 0$ as $n \longrightarrow \infty$

By using of (ii) of Definition 2.1,

for $\varsigma_{n+1}, \varsigma_{n+2}, \ldots, \varsigma_{m-1}$, we have

$B_4(\varsigma_n, \varsigma_m, \varsigma_m, \varsigma_m) \leq \left\{\begin{matrix} B_4(\varsigma_n, \varsigma_n, \varsigma_n, \varsigma_{n+1}) + B_4(\varsigma_m, \varsigma_m, \varsigma_m, \varsigma_{n+1}) \\ +B_4(\varsigma_m, \varsigma_m, \varsigma_m, \varsigma_{n+1}) + B_4(\varsigma_m, \varsigma_m, \varsigma_m, \varsigma_{n+1}) \end{matrix}\right\}$

$= B_4(\varsigma_n, \varsigma_n, \varsigma_n, \varsigma_{n+1}) + 3B_4(\varsigma_m, \varsigma_m, \varsigma_m, \varsigma_{n+1})$

$= s_n + 3B_4(\varsigma_m, \varsigma_m, \varsigma_m, \varsigma_{n+1})$

$\leq s_n + 3s_{n+1} + 3^2 B_4(\varsigma_m, \varsigma_m, \varsigma_m, \varsigma_{n+2}) \leq s_n + 3s_{n+1} + 3^2 s_{n+2} + 3^3 B_4(\varsigma_m, \varsigma_m, \varsigma_m, \varsigma_{n+3})$

$\leq s_n + 3s_{n+1} + 3^2 s_{n+2} + 3^3 s_{n+3} + \cdots + 3^{m-n-1} s_{m-1}$ (b y lemma 3.1) (3.2.4)

From (3.2.3) and (3.2.4), we have

$B_4(\varsigma_n, \varsigma_m, \varsigma_m, \varsigma_m)$

$\leq s_n + 3ks_{n+1} + 3^2 k^2 s_{n+2} + 3^3 k^3 s_{n+3} + \cdots + 3^{m-n-1} k^{m-n-1} s_{m-1}$

$\leq s_n (1 + 3k + (3k)^2 + \cdots + (3k)^{m-n-1})$

$= s_n (1 + \lambda + \lambda^2 + \cdots + \lambda^{m-n-1})$, where $\lambda = 3k$

$\leq s_n \left(\frac{1}{1-\lambda}\right)$ (since $\lambda < 1$, by hypothesis)

$\leq k^{\mathrm{n}} s_0 \left(\frac{1}{1-\lambda}\right) \to 0$ as $n \to \infty$, since $k < \frac{1}{3}$

Hence $\{\varsigma_n\}$ is a Cauchy sequence.

Since Ω is complete, there exists $\varsigma^* \in \Omega$ such that $\varsigma_n \to \varsigma^*$ (3.2.5)

Now $B_4(\varsigma_{n+1}, \top\varsigma^*, \top\varsigma^*, \top\varsigma^*) = B_4(\top\varsigma_n, \top\varsigma^*, \top\varsigma^*, \top\varsigma^*)$

$\leq kB_4(\varsigma_{\mathrm{n}}, \varsigma^*, \varsigma^*, \varsigma^*)$

$\longrightarrow 0$ as $n \longrightarrow \infty$ (by 3.2.5)

Therefore $\varsigma_{n+1} \to \top\varsigma^*$

Therefore $\top\varsigma^* = \varsigma^*$ (by lemma 2.7)

Therefore ς^* is a fixed point of $\top$.

Suppose ς^{**} is a fixed point of $\top$.

Then $B_4(\top\varsigma^*, \top\varsigma^{**}, \top\varsigma^{**}, \top\varsigma^{**}) \leq kB_4(\varsigma^*, \varsigma^{**}, \varsigma^{**}, \varsigma^{**})$

Therefore $B_4(\varsigma^*, \varsigma^{**}, \varsigma^{**}, \varsigma^{**}) \leq kB_4(\varsigma^*, \varsigma^{**}, \varsigma^{**}, \varsigma^{**})$

Therefore $B_4(\varsigma^*, \varsigma^{**}, \varsigma^{**}, \varsigma^{**}) = 0$

Therefore $\varsigma^* = \varsigma^{**}$ (by (i) of definition 2.1)

Thus $\top$ *has only* one fixed point, namely ς^*.

4. Applications

In this section we obtain applications of the main theorem 3.2

4. 1 Theorem: *Suppose (Ω, B_4) is a complete 4-dimensional ball metric space and $\top: \Omega \to \Omega$ is a map. Suppose $0 \leq k < 0.1$ is such that for $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4 \in \Omega$.*

$$B_4(\top\varsigma_1, \top\varsigma_2, \top\varsigma_3, \top\varsigma_4) \leq k \begin{Bmatrix} B_4(\varsigma_1, \top\varsigma_1, \top\varsigma_1, \top\varsigma_1) + B_4(\varsigma_2, \top\varsigma_2, \top\varsigma_2, \top\varsigma_2) \\ +B_4(\varsigma_3, \top\varsigma_3, \top\varsigma_3, \top\varsigma_3) + B_4(\varsigma_4, \top\varsigma_4, \top\varsigma_4, \top\varsigma_4) \end{Bmatrix} \quad (4.1.1)$$

Then $\top$ has only one fixed point.

Proof: From (4.1.1), taking $\varsigma_2 = \varsigma_3 = \varsigma_4$, we have

$$B_4(\top\varsigma_1, \top\varsigma_2, \top\varsigma_2, \top\varsigma_2) \leq k\{B_4(\varsigma_1, \top\varsigma_1, \top\varsigma_1, \top\varsigma_1) + 3B_4(\varsigma_2, \top\varsigma_2, \top\varsigma_2, \top\varsigma_2)\}.$$

Let $\varsigma_0 \in \Omega$ and define the sequence $\{\varsigma_n\}$ by $\varsigma_{n+1} = \top^n\varsigma$ for $n = 0,1,2, \ldots$

We have $B_4(\varsigma_n, \varsigma_n, \varsigma_n, \varsigma_{n+1})$

$$\leq k \begin{pmatrix} B_4(\varsigma_{n-1}, \varsigma_{n-1}, \varsigma_{n-1}, \varsigma_n) + B_4(\varsigma_{n-1}, \varsigma_{n-1}, \varsigma_{n-1}, \varsigma_n) \\ +B_4(\varsigma_{n-1}, \varsigma_{n-1}, \varsigma_{n-1}, \varsigma_n) + B_4(\varsigma_n, \varsigma_n, \varsigma_n, \varsigma_{n+1}) \end{pmatrix}$$

so that, $B_4(\varsigma_n, \varsigma_n, \varsigma_n, \varsigma_{n+1}) \leq \frac{3k}{1-k} B_4(\varsigma_{n-1}, \varsigma_{n-1}, \varsigma_{n-1}, \varsigma_n)$

Since $\frac{3k}{1-k} < \frac{1}{3}$, by theorem 3.2, the result follows.

4. 2 Theorem: *Suppose (Ω, B_4) is a complete $4 - dimensional\ ball\ metric\ space$ and $\top: \Omega \to \Omega$ is a map. Suppose the real numbers a, b, c, d are such that $0 \leq a < \frac{1}{3}$, $0 \leq b < \frac{1}{4}$, $0 \leq c < \frac{1}{4}$, $0 \leq d < \frac{1}{4}$. Write $\delta = max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}, \frac{d}{1-d}\right\}$.*

Assume that for $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4 \in \Omega$,

$$B_4(\top\varsigma_1, \top\varsigma_2, \top\varsigma_3, \top\varsigma_4) \leq \left(\frac{\delta}{4} B_4(\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4) + 3\frac{\delta}{4} B_4(\varsigma_1, \varsigma_1, \varsigma_1, \top\varsigma_1)\right) \quad (4.2.1)$$

Then $\top$ has only one fixed point.

Proof: From (4.2.1).

$B_4(T\varsigma_1, T\varsigma_2, T\varsigma_2, T\varsigma_2) \leq \left(\frac{\delta}{4} B_4(\varsigma, \varsigma_2, \varsigma_2, \varsigma_2) + \frac{3\delta}{4} B_4(\varsigma_1, \varsigma_1, \varsigma_1, T\varsigma_1)\right)$
Let $\varsigma_0 \in \Omega$ be and define the sequence $\{\varsigma_n\}$ by $\varsigma_{n+1} = \top^n \varsigma_0$.
Then $B_4(\varsigma_{n+1}, \varsigma_n, \varsigma_n, \varsigma_n) = B_4(T\varsigma_n, \top\varsigma_{n-1}, \top\varsigma_{n-1}, \top\varsigma_{n-1})$

$$\leq \left(\frac{\delta}{4} B_4(\varsigma_n, \varsigma_{n-1}, \varsigma_{n-1}, \varsigma_{n-1}) + \frac{3\delta}{4} B_4(\varsigma_n, \varsigma_{n-1}, \varsigma_{n-1}, \varsigma_{n-1})\right)$$

$$= \delta B_4(\varsigma_n, \varsigma_{n-1}, \varsigma_{n-1}, \varsigma_{n-1})$$

Since $\delta < \frac{1}{3}$, by Theorem 3.2, the result follows.

4. 3 Theorem: *Suppose* (Ω, B_4) *is a complete* $4 - dimensional$ *ball metric space and* $\top\colon \Omega \to \Omega$ *is a map. Suppose* a, b, c, d *are such that* $0 \leq a < \frac{1}{4}, 0 \leq b < \frac{1}{5}, 0 \leq c < \frac{1}{5}, 0 \leq d < \frac{1}{5}$. *Write* $\delta = max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}, \frac{d}{1-d}\right\}$.

Suppose, for $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4 \in \Omega$

$B_4(\top\varsigma_1, \top\varsigma_2, \top\varsigma_3, \top\varsigma_4) \leq \left(\frac{\delta}{4} B_4(\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4) + 3\frac{\delta}{3} B_4(\varsigma_1, \varsigma_1, \varsigma_1, T\varsigma_1)\right)$ (4.3.1)

Then $\top$ *has only one fixed point.*

Proof: From (4.3.1).

$$B_4(\top\varsigma_1, \top\varsigma_2, \top\varsigma_2, \top\varsigma_2) \leq \left(\frac{\delta}{4} B_4(\varsigma_1, \varsigma_2, \varsigma_2, \varsigma_2) + \frac{3\delta}{3} B_4(\varsigma_1, \varsigma_1, \varsigma_1, \top\varsigma_1)\right)$$

Let $\varsigma_0 \in \Omega$ and define the sequence $\{\varsigma_n\}$ by $\varsigma_{n+1} = \top\varsigma_n$.

Then $B_4(\varsigma_{n+1}, \varsigma_n, \varsigma_n, \varsigma_n) = B_4(T\varsigma_n, \top\varsigma_{n-1}, \top\varsigma_{n-1}, \top\varsigma_{n-1})$

$$\leq \left(\frac{\delta}{4} B_4(\varsigma_n, \varsigma_{n-1}, \varsigma_{n-1}, \varsigma_{n-1}) + \frac{3\delta}{3} B_4(\varsigma_n, \varsigma_{n-1}, \varsigma_{n-1}, \varsigma_{n-1})\right)$$
$$= \frac{5}{4} \delta B_4(\varsigma_n, \varsigma_{n-1}, \varsigma_{n-1}, \varsigma_{n-1})$$

Since $\frac{5}{4}\delta < \frac{1}{3}$, by Theorem 3.2, the result follows.

5. Main result for n - dimensional ($n \geq 4$) ball metric spaces

Some fixed point theorems in $n - dimensional$ $(n \geq 4)$ ball metric spaces are proved in [4].

Now we establish one more fixed point theorem. For this, we first prove the following Lemma.

5. 1 Lemma: *Let* (Ω, B_n) *be a* $n - dimensional$ *ball metric spaces. Then, for* $\varsigma_1, \varsigma_2, \varsigma_3 \in \Omega$, *we have*

$$B_n(\varsigma_1, \underbrace{\varsigma_2, \varsigma_2, \dots, \varsigma_2}_{(n-1)\ times}) \leq B_n(\varsigma_1, \underbrace{\varsigma_3, \varsigma_3, \dots, \varsigma_3}_{(n-1)\ times}) + (n-1) B_n(\varsigma_3, \underbrace{\varsigma_2, \varsigma_2, \dots, \varsigma_2}_{(n-1)\ times})$$

Proof: Suppose (Ω, B_n) is a B_n-metric space.

By definition 2.9 (ii),

$$B_n\left(\varsigma_1,\underbrace{\varsigma_2,\varsigma_2,\dots,\varsigma_2}_{(n-1)\text{ times}}\right) \leq \left\{\begin{array}{l} B_n\left(\underbrace{\varsigma_1,\varsigma_1,\dots,\varsigma_1}_{(n-1)\, times},\varsigma_3\right)+\ B_n\left(\underbrace{\varsigma_2,\varsigma_2,\dots,\varsigma_2}_{(n-1)\text{ times}},\varsigma_3\right) \\ +B_n\left(\underbrace{\varsigma_2,\varsigma_2,\dots,\varsigma_2}_{(n-1)\text{ times}},\varsigma_3\right)+B_n\left(\underbrace{\varsigma_2,\varsigma_2,\dots,\varsigma_2}_{(n-1)\text{ times}},\varsigma_3\right)\end{array}\right\}$$

$$B_n\left(\varsigma_1,\underbrace{\varsigma_2,\varsigma_2,\dots,\varsigma_2}_{(n-1)\text{ times}}\right) \leq B_n\left(\underbrace{\varsigma_1,\varsigma_1,\dots,\varsigma_1}_{(n-1)\, times},\varsigma_3\right)+(n-1)\,B_n\left(\underbrace{\varsigma_2,\varsigma_2,\dots,\varsigma_2}_{(n-1)\text{ times}},\varsigma_3\right)$$

$$=B_n\left(\varsigma_1,\underbrace{\varsigma_3,\varsigma_3,\dots,\varsigma_3}_{(n-1)\, times}\right)+(n-1)B_n\left(\varsigma_3,\underbrace{\varsigma_2,\varsigma_2,\dots,\varsigma_2}_{(n-1)\text{ times}}\right), \text{(by Lemma 2.13)}$$

Now we state and prove our primary result in $n-dimensional$ $(n \geq 4)$ ball metric spaces.

5.2 Theorem: *Suppose* (Ω, B_n) *is a complete* n*-dimensional ball metric space and* $T\colon \Omega \to \Omega$ *is a map. Suppose* $0 \leq k < \frac{1}{(n-1)}$. *Assume that for* $\varsigma_1, \varsigma_2, \dots, \varsigma_n \in \Omega$.

$$B_n(\mathrm{T}\varsigma_1, \mathrm{T}\varsigma_2, \dots, \mathrm{T}\varsigma_n) \leq kB_n(\varsigma_1, \varsigma_2, \dots, \varsigma_n) \qquad (5.2.1)$$

Then T *has only one fixed point.*

Proof: For $\varsigma_1, \varsigma_2 \in \Omega$, from (5.2.1) we have

$$B_{\mathrm{n}}(\mathrm{T}\varsigma_1,\underbrace{\mathrm{T}\varsigma_2,\mathrm{T}\varsigma_2,\dots,\mathrm{T}\varsigma_2}_{(n-1)\text{ times}}) \leq kB_{\mathrm{n}}(\varsigma_1,\underbrace{\varsigma_2,\varsigma_2,\dots,\varsigma_2}_{(n-1)\text{ times}})$$

Let $\varsigma^0 \in \Omega$. Define the sequence $\{\varsigma^m\}$ by $\varsigma^m = \mathrm{T}^m\varsigma^0$, for $m = 1,2,3,\dots$

Then clearly $\varsigma^{m+1} = \mathrm{T}\varsigma^m$.

$$B_{\mathrm{n}}(\varsigma^{m+1},\underbrace{\varsigma^m,\varsigma^m,\dots,\varsigma^m}_{(n-1)\text{ times}}) = B_{\mathrm{n}}(\mathrm{T}\varsigma^m,\underbrace{\mathrm{T}\varsigma^{m-1},\mathrm{T}\varsigma^{m-1},\dots,\mathrm{T}\varsigma^{m-1}}_{(n-1)\text{ times}})$$

$$\leq kB_{\mathrm{n}}(\varsigma^m,\underbrace{\varsigma^{m-1},\varsigma^{m-1},\dots,\varsigma^{m-1}}_{(n-1)\text{ times}}) \qquad (5.2.2)$$

Write $\propto_m = B_{\mathrm{n}}(\varsigma^{m+1},\underbrace{\varsigma^m,\varsigma^m,\dots,\varsigma^m}_{(n-1)\text{ times}})$

Then $\propto_m = B_{\mathrm{n}}\left(\underbrace{\varsigma^{m+1},\varsigma^{m+1},\dots,\varsigma^{m+1}}_{(n-1)\text{ times}},\varsigma^m\right)$ (by Lemma 2.13)

We have $\propto_m \leq k \propto_{m-1}$ (by (5.2.2))

Now by induction follows that $\propto_m \leq k^m \propto_0$ for all $m \in N$,

This shows that $\propto_m \to 0$ as $m \to \infty$

Suppose p is a positive integer.

Then using (ii) of Definition 2.10 and Lemma (2.15), we get,

$$B_{\mathrm{n}}(\varsigma^{m+p},\underbrace{\varsigma^{m},\varsigma^{m},\dots,\varsigma^{m}}_{(n-1)\text{ times}}) = B_{\mathrm{n}}(\varsigma^{m},\underbrace{\varsigma^{m+p},\varsigma^{m+p},\dots,\varsigma^{m+p}}_{(n-1)\text{ times}})$$

$$\leq B_{\mathrm{n}}\left(\underbrace{\varsigma^{m},\varsigma^{m},\dots,\varsigma^{m}}_{(n-1)\ times},\varsigma^{m+1}\right) + \underbrace{B_{\mathrm{n}}\left(\underbrace{\varsigma^{m+p},\varsigma^{m+p},\dots,\varsigma^{m+p}}_{(n-1)\ times},\varsigma^{m+1}\right) + \cdots + B_{\mathrm{n}}\left(\underbrace{\varsigma^{m+p},\varsigma^{m+p},\dots,\varsigma^{m+p}}_{(n-1)\ times},\varsigma^{m+1}\right)}_{(n-1)terms}$$

$$=\propto_{m} + (n-1)B_{\mathrm{n}}(\underbrace{\varsigma^{m+p},\varsigma^{m+p},\dots,\varsigma^{m+p}}_{(n-1)\text{ times}},\varsigma^{m+1})$$

$$=\propto_{m} + (n-1)B_{\mathrm{n}}(\underbrace{\varsigma^{m+1},\varsigma^{m+1},\dots,\varsigma^{m+1}}_{(n-1)\text{ times}},\varsigma^{m+p})$$

$\leq \propto_{m} + (n-1)(\propto_{m+1} + (n-1)B_{\mathrm{n}}(\varsigma^{m+2},\varsigma^{m+2},\dots,\varsigma^{m+p})$ (by Lemma 5.1)

$\leq \propto_{m} + (n-1)\propto_{m+1} + (n-1)^2 B_{\mathrm{n}}(\varsigma^{m+2},\varsigma^{m+2},\dots,\varsigma^{m+p})$

$\leq \propto_{m} + (n-1)\propto_{m+1} + (n-1)^2\propto_{m+2} + (n-1)^3\propto_{m+3} + \cdots + (n-1)\propto_{m+p-1}$

$\leq \propto_{m} + k(n-1)\propto_{m} + k^2(n-1)^2\propto_{m} + \cdots + k^{m+p-1}(n-1)^{m+p-1}\propto_{m}$

$= \propto_{m}(1 + k(n-1) + (k(n-1))^2 + \cdots + (k(n-1))^{p-1})$

$\leq k^m \propto_0 (1 + k(n-1) + (k(n-1))^2 + \cdots + (k(n-1))^{p-1})$

$\leq k^m \propto_0 \left(\frac{1}{1-k(n-1)}\right)$, since $k(n-1) < 1$

$\longrightarrow 0$ as $m \to \infty$

Hence $\{\varsigma^m\}$ is a Cauchy sequence.

Since Ω is complete, there exists $\varsigma^* \in \Omega$ such that $\varsigma^m \to \varsigma^*$

Now $B_{\mathrm{n}}(\varsigma^{m+1},\top\varsigma^*,\dots,\top\varsigma^*) = B_{\mathrm{n}}(\top\varsigma^m,\top\varsigma^*,\dots,\top\varsigma^*)$

$$\leq k\, B_{\mathrm{n}}(\varsigma^m,\varsigma^*,\dots,\varsigma^*) \to 0 \text{ as } m \to \infty$$

Therefore $\varsigma^{m+1} \to \top\varsigma^*$

Therefore $\top\varsigma^* = \varsigma^*$ (by Lemma 2.7)

Therefore ς^* is a fixed point of $\top$.

Suppose ς^{**} is a fixed point of $\top$.

Then $B_{\mathrm{n}}(\top\varsigma^*,\top\varsigma^{**},\dots,\top\varsigma^{**}) \leq kB_{\mathrm{n}}(\varsigma^*,\varsigma^{**},\dots,\varsigma^{**})$

Therefore $B_{\mathrm{n}}(\varsigma^*,\varsigma^{**},\dots,\varsigma^{**}) \leq kB_{\mathrm{n}}(\varsigma^*,\varsigma^{**},\dots,\varsigma^{**})$

Therefore $B_{\mathrm{n}}(\varsigma^*,\varsigma^{**},\dots,\varsigma^{**}) = 0$

Therefore $\varsigma^* = \varsigma^{**}$

Thus $\top$ has only one fixed point, namely ς^*.

6. Applications

In this section, we give three applications of Theorem 5.2

6. 1 Theorem: *Suppose* (Ω, B_n) *is a complete* n*-dimensional ball metric space and* $T: \Omega \to \Omega$ *is a map. Suppose* $0 \le k < \frac{1}{(n-1)^2+1}$.

Assume that for $\varsigma^1, \varsigma^2, \dots, \varsigma^m \in \Omega$.

$$B_n(\top\varsigma^1, \top\varsigma^2, \dots, \top\varsigma^n) \le k \left\{ \begin{matrix} B_n(\varsigma^1, \top\varsigma^1, \dots, \top\varsigma^1) + \\ B_n(\varsigma^2, \top\varsigma^2, \dots, \top\varsigma^2) \\ + \cdots + \\ B_n(\varsigma^n, \top\varsigma^n, \dots, \top\varsigma^n) \end{matrix} \right\} \quad (6.1.1)$$

Then $\top$ *has only one fixed point.*

Proof: From (6.1.1),

$$B_n(\top\varsigma^1, \top\varsigma^2, \dots, \top\varsigma^2) \le k\{B_\mathrm{n}(\varsigma^1, \top\varsigma^1, \dots, \top\varsigma^1) + (n-1)B_\mathrm{n}(\varsigma^2, \top\varsigma^2, \dots, \top\varsigma^2)\}$$

Let $\varsigma^0 \in \Omega$ and define the sequence $\{\varsigma^m\}$ by $\varsigma^{m+1} = \top\varsigma^m$ for $m = 0,1,2, \dots$

We have $B_n(\varsigma^m, \varsigma^m, \dots, \varsigma^m, \varsigma^{m+1})$

$$\le k \begin{pmatrix} B_n(\varsigma^{m-1}, \varsigma^{m-1}, \dots, \varsigma^{m-1}, \varsigma^m) + B_n(\varsigma^{m-1}, \varsigma^{m-1}, \dots, \varsigma^{m-1}, \varsigma^m) \\ + \cdots + B_n(\varsigma^{m-1}, \varsigma^{m-1}, \dots, \varsigma^{m-1}, \varsigma^m) + B_n(\varsigma^m, \varsigma^m, \dots, \varsigma^m, \varsigma^{m+1}) \end{pmatrix}$$

so that $B_\mathrm{n}\left(\underbrace{\varsigma^m, \varsigma^m, \dots, \varsigma^m}_{(n-1)\text{ times}}, \varsigma^{m+1}\right) \le \frac{(n-1)k}{1-\mathrm{k}} B_\mathrm{n}\left(\underbrace{\varsigma^{m-1}, \varsigma^{m-1}, \dots, \varsigma^{m-1}}_{(n-1)\text{ times}}, \varsigma^m\right)$

Since $\frac{(n-1)k}{1-k} < \frac{1}{(n-1)}$, by Theorem 5.2, the result follows.

6. 2 Theorem: *Suppose* (Ω, B_n) *is a complete* n*-dimensional ball metric space and* $\top: \Omega \to \Omega$ *is a map. Suppose* $a_1, a_2, \dots, a_n$ *are such that* $0 \le a_1 < \frac{1}{(n-1)}, 0 \le a_2 < \frac{1}{n}, \dots, 0 \le a_n < \frac{1}{n}$ *.With* $\delta = max\left\{a_1, \frac{a_2}{1-a_2}, \frac{a_3}{1-a_3}, \dots, \frac{a_n}{1-a_n}\right\}$.

Assume that for $\varsigma^1, \varsigma^2, \dots, \varsigma^n \in \Omega$,

$$B_n(\top\varsigma^1, \top\varsigma^2, \dots, \top\varsigma^m) \le \left(\frac{\delta}{n} B_n(\varsigma^1, \varsigma^2, \dots, \varsigma^n) + (n-1)\frac{\delta}{n} B_n(\varsigma^1, \varsigma^1, \dots, \varsigma^1, \top\varsigma^1)\right) \qquad (6.2.1)$$

Then $\top$ *has only one fixed point.*

Proof: From (6.2.1).

$$B_n(\top\varsigma^1, \top\varsigma^2, \dots, \top\varsigma^2) \le \left(\frac{\delta}{n} B_n(\varsigma^1, \varsigma^2, \dots, \varsigma^2) + \frac{(n-1)\delta}{n} B_n(\varsigma^1, \varsigma^1, \dots, \varsigma^1, \top\varsigma^1)\right)$$

Let $\varsigma^0 \in \Omega$ and define the sequence $\{\varsigma^m\}$ by $\varsigma^{m+1} = \top\varsigma^m$.

Then $B_n(\varsigma^{m+1}, \varsigma^m, \dots, \varsigma^m, \varsigma^m) = B_n(\top\varsigma^m, \top\varsigma^{m-1}, \dots, \top\varsigma^{m-1}, \top\varsigma^{m-1})$

$$\begin{aligned} &\le \left(\frac{\delta}{n} B_n(\varsigma^m, \varsigma^{m-1}, \dots, \varsigma^{m-1}) + \frac{(n-1)\delta}{n} B_n(\varsigma^m, \varsigma^{m-1}, \dots, \varsigma^{m-1})\right) \\ &= \delta B_n(\varsigma^m, \varsigma^{m-1}, \dots, \varsigma^{m-1}) \end{aligned}$$

Since $\delta < \frac{1}{(n-1)}$, by Theorem 5.2, the result follows.

6.3 Theorem: *Suppose* (Ω, B_n) *is a complete* $n-dimensional$ *ball metric space and* $\top: \Omega \rightarrow \Omega$ *is a map. Suppose* $a_1, a_2, \dots, a_n$ *are such that* $0 \le a_1 < \frac{1}{n}, 0 \le a_2 < \frac{1}{(n+1)}, \dots, 0 \le a_n < \frac{1}{(n+1)}$.

Write $\delta = max\left\{a_1, \frac{a_2}{1-a_2}, \frac{a_3}{1-a_3}, \dots, \frac{a_n}{1-a_n}\right\}$.

Assume that for $\varsigma^1, \varsigma^2, \dots, \varsigma^n \in \Omega$,

$$B_n(\top\varsigma^1, \top\varsigma^2, \dots, \top\varsigma^n) \le \left(\frac{\delta}{n} B_n(\varsigma^1, \varsigma^2, \dots, \varsigma^n) + \frac{(n-1)\delta}{(n-1)} B_n(\varsigma^1, \varsigma^1, \dots, \varsigma^1, \top\varsigma^1)\right) \qquad (6.3.1)$$

Then $\top$ *has only one fixed point.*

Proof: From (6.3.1),

$$B_n(\top\varsigma^1, \top\varsigma^2, \dots, \top\varsigma^2) \le \left(\frac{\delta}{n} B_n(\varsigma^1, \varsigma^2, \dots, \varsigma^2, \varsigma^2) + \frac{(n-1)\delta}{n} B_n(\varsigma^1, \varsigma^1, \dots, \varsigma^1, \top\varsigma^1)\right)$$

Let $\varsigma^0 \in \Omega$ and define the sequence $\{\varsigma^m\}$ by $\varsigma^{m+1} = \top\varsigma^m$.

Then $B_n\left(\varsigma^{m+1}, \underbrace{\varsigma^m, \varsigma^m, \dots, \varsigma^m}_{(n-1)\text{ times}}\right) = B_n\left(\top\varsigma^m, \underbrace{\top\varsigma^{m-1}, \top\varsigma^{m-1}, \dots, \top\varsigma^{m-1}}_{(n-1)\text{ times}}\right)$

$$\le \left(\frac{\delta}{n} B_n(\varsigma^m, \varsigma^{m-1}, \dots, \varsigma^{m-1}, \varsigma^{m-1}) + \frac{(n-1)\delta}{(n-1)} B_n(\varsigma^m, \varsigma^{m-1}, \dots, \varsigma^{m-1}, \varsigma^{m-1})\right)$$

$$= \frac{(n+1)\delta}{n} B_n(\varsigma^m, \varsigma^{m-1}, \dots, \varsigma^{m-1}, \varsigma^{m-1})$$

Since $\frac{(n+1)\delta}{n} < \frac{1}{(n-1)}$, by Theorem 5.2, the result follows.

7. Conclusion

Some fixed point results in 4-dimensional ball metric spaces are established and applications are obtained. We also obtain fixed point results in $n - dimensional$ $(n \ge 4)$ ball metric spaces. Applications to fixed point results are shown.

8. References

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