



On Lie Groups and P- two Norm Algebras

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Abstract

The objective of this paper is studying Lie group $G(X)$ on algebra X and define two norms in this algebra and give some properties of two-norm algebras which are topological, and relate some concepts with the topological notions introduced by A. Alexiewicz.

Keywords: Lie group; Lie algebra; Normed algebra

1. Introduction:

A vector space or algebra over a field of real or complex numbers will be denote by X . For the sets A and B in X we write

$$\alpha A = \{\alpha a; a \in A\}, A \cdot B = \{ab; a \in A, b \in B\}, A^{-1} = \{a^{-1}; a \in A\}$$

A topological space T is called Hausdorff if for any two distinct points $p_1, p_2 \in M$ there exists open sets $U_1, U_2 \in T$ with

$$p_1 \in U_1; p_2 \in U_2; U_1 \cap U_2 = \emptyset.$$

A map between topological spaces is called continuous if the preimage of any open set is again open.

The important of research and its objectives:

The object of this paper is to generalize the notation of two-norm algebras introduced by A. Alexiewicz[1]. First of all we consider topologies in algebras which arise from two p -homogeneous norms.

Research methods and materials:

Let M be a manifold, An open chart on M is a pair $(U; \square)$, where U is an open subset of M and $\square$ is a homeomorphism of U onto an open subset of $\mathbb{R}^n$.

Definition 1:[2]. A smooth manifold M is an n -dimensional Hausdorff space where in the neighborhood of any point $p \in M$ there exists a chart of n -dimension.

Definition 2:[2] A Lie group is a group that is also a smooth manifold such that the multiplication map:

$$\begin{aligned} \mu: G \times G &\xrightarrow{(g,h) \rightarrow gh} G \\ \xi: G &\xrightarrow{g \rightarrow g^{-1}} G \end{aligned}$$

Are smooth. Space with a transitive G action for a Lie group G are called as is a homogeneous space. $S^2 = \{x \in \mathbb{R}^3; |x| = 1\}$ homogeneous space. Write a

We shall denote by e the unit element of the algebra if existing. $G(X)$ and $G_0(X)$ will denote the multiplicative Lie group of the algebra X and the set of quasi invertible

elements. $\gamma(\tau)$ will denote the neighborhood filter of zero for the topology τ .

Definition 3:[3]. The function $\|\cdot\|: X \rightarrow \mathbb{R}$ is said to be p -homogeneous norm or shortly p -norm where $0 < P \leq 1$ if $\|\cdot\|$ satisfy the following conditions:

- (1) $\|x\| = 0$ iff $x=0$
- (2) $\|x + y\| \leq \|x\| + \|y\|$
- (3) $\|\alpha x\| = |\alpha|^P \|x\|$

And $(X, \|\cdot\|)$ is said to be topological vector spaces or *t.v.s.* has a neighborhood basis of zero composed of bounded set.

Definition 4:[3]. The triplet $(X, \|\cdot\|, \|\cdot\|_0)$ is said to be p -two-norm space if X is a vector space and $\|\cdot\|, \|\cdot\|_0$ are two p -homogeneous norms, the first being finer than the second one. This is the case if and only if there exists a constant k such that $\|x\|_0 \leq k\|x\|$ for every $x \in X$.

Let τ_0 denote the topology generated in X by the metric $\rho(x, y) = \|x - y\|$, let $S_n = \{x \in X; \|x\| \leq n\}$. By τ_∞ we shall denote the finest vector topology on X which induced on S a topology coarser than τ_0 . such topology exist.

Definition 5. [3]: Let $(X, \|\cdot\|, \|\cdot\|_0)$ be p -two- norm space, suppose that in X defined multiplication of elements making X together with the vector operation an algebra. If multiplication is continuous for the topology τ_∞ then $(X, \|\cdot\|, \|\cdot\|_0)$ is called p -two norm algebra, the topology τ_∞ is then called multiplicative

Theorem 1. The topology τ_∞ is multiplicative if and only if

- (i) there exists a constant β such $\|xy\| \leq \beta\|x\|\|y\|$
- (ii) for every $U \in \gamma(\tau_0)$ there exists a $V \in \gamma(\tau_0)$ such that $S(V \cap S) \subset U$, $(V \cap S)S \subset U$

Proof: Necessity. The set S is τ_0 -bounded, for if $x_n \in S$, $\alpha_n \rightarrow 0$ then

$$\|\alpha_n x_n\| = |\alpha_n|^p \|x_n\| \rightarrow 0$$

The set SS is also τ_0 -bounded, Indeed let $\|x_n\| \leq 1$, $\|y_n\| \leq 1$, $\alpha_n \rightarrow 0$, then

$$\|\alpha_n x_n y_n\| = \|\sqrt{\alpha_n x_n} \sqrt{\alpha_n y_n}\| \rightarrow 0$$

Let us prove the necessity of the condition (i).

1- Suppose, if possible, that there exist x_n, y_n such that $\|x_n\| \leq 1$, $\|y_n\| \leq 1$, $\|x_n y_n\| > n\|x_n\|\|y_n\|$, then $\|x_n\| \neq 0 \neq \|y_n\|$ and

$$\left\| \frac{x_n}{n^{1/2p} \|x_n\|^{1/p}} \frac{y_n}{n^{1/2p} \|y_n\|^{1/p}} \right\| > 1$$

This is impossible, since $\|x_n/\|x_n\|^{1/p}\| = \|y_n/\|y_n\|^{1/p}\| = 1$

We now prove that the condition (ii) is necessary. Observe first that the set S is τ_0 -bounded. Indeed, let $x_n \in S$, $\alpha_n \rightarrow 0$ then $\|\alpha_n x_n\| = |\alpha_n|^p \|x_n\| \rightarrow 0$, therefore $\|\alpha_n x_n\|_0 \rightarrow 0$

Next observe that the topology τ_0 is coarser than τ_∞ . To see this choose a sequence (U_n) of τ_0 -neighborhoods of zero, such that $U_1 + U_1 \subset U$, $U_{n+1} + U_{n+1} \subset U_n$ for $n = 1, 2, 3, \dots$

Then $V := \sum_{i=1}^{\infty} U_n \cap S$ is in $\gamma(\tau_\infty)$ and $V \subset \sum_{i=1}^{\infty} U_n \subset U$.

Now let U be in $\gamma(\tau_0)$. Since the topology τ_∞ is multiplicative and $\tau_0 \leq \tau_\infty$, so there exists $W \in \gamma(\tau_\infty)$ such that $W \subset U$. Thus $WW = \sum V_n \cap S$ where $V_n \in \gamma(\tau_0)$ and therefore $(V_1 \cap S)W \subset U$. Since the set W is τ_0 -bounded, $\alpha W \subset V_1$ for some $\alpha > 0$ and it is enough to choose $V = \min(\alpha, \alpha^{-1})V_1$. The second part of (ii) is proved similarly.

2-The sufficiency of both conditions. Since $\|xy\| \leq \beta\|x\|\|y\|$ introducing a new p -norm $\|x\|' = \sqrt{\beta}\|x\|$ we obtain a sub- multiplicative p -norm equivalent to $\|x\|$ for which the unit ball equals $\beta^{-1/2p}$, and we can set $\sum U_n \cap \beta^{-1/2p}$ give also the neighborhood basis of zero for τ_∞ . So we can suppose freely that $\|xy\| \leq \|x\|\|y\|$ and therefore that $SS \subset S$

II- On continuity of the inverse

Let the algebra X has the unit element e and denote by $G(X)$ the multiplicative Lie group of X . We shall say that the inverse is ρ -continuous if:

- (a) $x_n \xrightarrow{\rho} e$ implies that almost all x_n are in $G(X)$
- (b) If $x_n \in G(X)$, $x_n \xrightarrow{\rho} e$, then $x_n^{-1} \xrightarrow{\rho} e$

From this condition it follows that if $x_n \xrightarrow{\rho} x_0 \in G(X)$, then almost all x_n are in $G(X)$ and $x_{n+k}^{-1} \xrightarrow{\rho} x_0^{-1}$ for some k in $\mathbb{N}$

The condition (b) is equivalent to the following one:

(b') Let $x_n \in G(X)$, $x_n \xrightarrow{\rho} e$, then $\sup_n \|x_n^{-1}\| < \infty$

The necessity of this condition being obvious, we only need to prove its sufficiency.

So let $x_n \in G(X)$, $x_n \xrightarrow{\rho} 1$. We distinguish two cases.

- 1. There exists a $\delta > 0$ such that $\|x_n - 1\| \geq \delta$ for all n . Then

$$\|x_n^{-1} - 1\|_0 = \|x_n^{-1}(1 - x_n)\| = \|x_n^{-1}\| \|1 - x_n\| \left\| \frac{x_n^{-1}}{\|x_n^{-1}\|^{1/p}} \frac{1-x_n}{\|1-x_n\|^{1/p}} \right\|_0 \leq \sup_n \|x_n^{-1}\| \sup_n \|1 - x_n\| \varepsilon$$

Provided that $\|(1 - x_n)/\|1 - x_n\|^{1/p}\|_0 \leq \eta(\varepsilon)$ i.e when $\|1 - x_n\|_0 \leq \eta(\varepsilon)/\delta$

- 2. $\lim_{n \rightarrow \infty} \|x_n - 1\| = 0$, then $\vartheta > 0$ being small enough, $\|x_n - 1\| \leq \vartheta$ implies $\|x_n^{-1} - 1\| < \varepsilon$ When the inverse is ρ -continuous, there exist two functions $f, g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that:

- (c) $\|x\| \leq q$, $\|x\|_0 \leq f(q)$ implies $1 + x \in G(X)$
- (d) $\|x\| \leq q$, $\|x\|_0 \leq f(q)$ implies $\|(1 + x)^{-1}\| \leq g(q)$

Definition 6.[4] A topological algebra (X, τ) is called *inverse continuous* if the multiplicative lie group $G(X)$ is τ -open and the map $x \rightarrow x^{-1}$ is τ -continuous on $G(X)$

Theorem 2. A p-two norm algebra (with unite) equipped with the Wiweger topology is inverse continuous if and only if the inverse is ρ -continuous.

Proof: The necessity being obvious, we prove the sufficiency.

1- So let us suppose that the inverse is ρ -continuous. First we prove that $G(X)$ is τ_∞ - open. It is enough to show that $1 \in G(X)$. Let $\bar{w}_1 = f(1)$, $\sigma_1 = g(1)$, $\bar{w}_{n+1} = \rho(\frac{f(\sigma_n)}{\sigma_n})$, $\sigma_{n+1} = \sigma_n(1 + \sigma_{n+1})$

We shall prove that:

(*) if $\|x_i\| \leq 1$, $\|x_i\|_0 \leq \bar{w}_1$, then $1 + x_1 + \dots + x_n \in G(X)$ and $\|(1 + x_1 + \dots + x_n)^{-1}\| \leq \sigma_n$

For $n=1$, $1 + x_1 \in G(X)$ and $\|(1 + x_1)^{-1}\| \leq g(1) = \sigma_1$. Suppose now that (*) holds for

The element $x_1, \dots, x_n$ and let us suppose that $x_1, \dots, x_n, x_{n+1}$ fulfill the assumption (*).

Let $v = 1 + x_1 + \dots + x_n$, then $v \in G(X)$ and $\|(1 + v)^{-1}\| = \sigma_n$

Thus $w = 1 + x_1 + \dots + x_n = v(1 + v^{-1}x_{n+1})$ and $\|v^{-1}x_{n+1}\| \leq \|v^{-1}\|\|x_{n+1}\| \leq \|v^{-1}\| \leq \sigma_n$.

Moreover

$$\|v^{-1}x_{n+1}\|_0 = \left\| \|v^{-1}\|^{1/P} \frac{v^{-1}}{\|v^{-1}\|^{1/P}} x_{n+1} \right\|_0 \leq \|v^{-1}\| \left\| \frac{v^{-1}}{\|v^{-1}\|^{1/P}} x_{n+1} \right\|_0$$

And since $\|v^{-1}\|/\|v^{-1}\|^{1/P} \leq 1$, $\|x_{n+1}\| \leq 1$, $\|x_{n+1}\|_0 \leq w_{n+1} = \rho(f(\sigma_n)/\sigma_n)$

We obtain by (3) $\|v^{-1}x_{n+1}\|_0 \leq \|v^{-1}\| \frac{f(\sigma_n)}{\sigma_n} \leq f(\sigma_n)$, for $\|v^{-1}\| \leq \sigma_n$

Hence by (c) $1 + v^{-1}x_{n+1} \in G(X)$ and therefore $w \in G(X)$ and

$$\|w\| \leq \|v\|(\|1\| + \|v^{-1}x_{n+1}\|) \leq \sigma_n(1 + \sigma_n) = \sigma_{n+1}$$

From (*) it follows that $U_n = \{x \in X; \|x\|_0 \leq w\}$ then the set $U := 1 + \sum_{n=1}^{\infty} U_n \cap S$

Which is a τ -neighborhood of 1. is contained in $G(X)$.

2-We now prove that the inverse is τ_∞ - continuous at 1. we shall consider two neighborhood bases for τ_∞ the first B_1 composed of sets of form $\sum_{n=1}^{\infty} P_n \cap S$

Where $P_n \in \gamma(\tau_0)$, the second B_2 composed of sets of form $\sum_{n=1}^{\infty} Q_n \cap \sigma_n(1 + \sigma_n)S$.

$Q_n \in \gamma(\tau_0)$ It is enough to show that for every U belongs to B_2 there exists V belongs to B_1 such that $1 + V \in G(X)$, and $(1 + V)^{-1} \in 1 + U$.

We can choose $Q_n = \{x \in X; \|x\|_0 \leq \varepsilon_n\}$, $\varepsilon_n > 0$.

Let $\eta_1 = \min(w_1, \rho(\varepsilon_1/\sigma_1))$, $\eta_n = \min(1, w_n, \rho(\varepsilon_n/\sigma_n\sigma_{n+1}))$

We shall show that for $\|x_i\| \leq 1$, $\|x_i\|_0 \leq \eta_i$, $i=1,2,\dots$, we have

(I) $1 + x_1 + \dots + x_n \in G(X)$

(II) $(1 + x_1 + \dots + x_n)^{-1} = 1 + u_1 + \dots + u_n$; $\|u_i\| \leq \sigma_i(1 + \sigma_i)$, $\|u_i\|_0 \leq \eta_i$.

From $\eta_i \leq w_i$, (I) follows immediately

For $n = 1$, $(1 + x_1)^{-1} = 1 + u_1$, where $u_1 = (1 + x_1)^{-1} - 1 = -(1 + x_1)^{-1}x_1$, whence

$\|u_1\| \leq \|(1 + x_1)^{-1}\|\|x_1\| \leq \sigma_1 \leq \sigma_1(1 + \sigma_1)$ and

$$\|u_1\|_0 = \|(1 + x_1)^{-1}x_1\|_0 = \left\| \|(1 + x_1)^{-1}\|^{1/P} \frac{(1 + x_1)^{-1}}{\|(1 + x_1)^{-1}\|^{1/P}} x_1 \right\|_0 \leq$$

$$\|(1 + x_1)^{-1}\| \left\| \frac{(1 + x_1)^{-1}}{\|(1 + x_1)^{-1}\|^{1/P}} x_1 \right\|_0$$

And since $\left\| \frac{(1 + x_1)^{-1}}{\|(1 + x_1)^{-1}\|^{1/P}} \right\| \leq 1$, $\|x_1\| \leq 1$, $\|x_1\|_0 \leq \rho(\varepsilon_1/\sigma_1)$ we obtain by (3)

$$\|(1 + x_1)^{-1}\| \left\| \frac{(1 + x_1)^{-1}}{\|(1 + x_1)^{-1}\|^{1/P}} \right\|_0 \leq \|(1 + x_1)^{-1}\| \frac{\varepsilon_1}{\sigma_1} \leq \sigma_1 \frac{\varepsilon_1}{\sigma_1} = \varepsilon_1$$

So (II) is true for $n = 1$. Suppose now it is valid for any set of n elements, and let

$\|x_i\| \leq 1$, $\|x_i\|_0 \leq \eta_i$; $i = 1, 2, \dots, n + 1$, then $x := 1 + x_1 + \dots + x_n \in G(X)$, $x^{-1} = 1 + u_1 + \dots + u_n$; $\|u_i\| \leq$

$\sigma_i(1 + \sigma_i)$, $\|u_i\|_0 \leq \varepsilon_i$ and

$(1 + x_1 + \dots + x_n)^{-1} = 1 + u_1 + \dots + u_n + u_{n+1}$ where

$u_{n+1} = (1 + x_1 + \dots + x_n + x_{n+1})^{-1} - (1 + x_1 + \dots + x_n)^{-1} =$

$= -(1 + x_1 + \dots + x_{n+1})^{-1}x_{n+1}(1 + x_1 + \dots + x_n)^{-1}$.

Now

$\|u_{n+1}\| \leq \|(1 + x_1 + \dots + x_{n+1})^{-1}\|\|x_{n+1}\|\|(1 + x_1 + \dots + x_n)^{-1}\| \leq \sigma_n\sigma_{n+1}$,

$u_{n+1} = v_1x_{n+1} + v_2$; $v_1 = (1 + x_1 + \dots + x_{n+1})^{-1}$, $v_2 = (1 + x_1 + \dots + x_n)^{-1}$

Whence

$$\begin{aligned} \|u_{n+1}\|_0 &\leq \|v_1x_{n+1} + v_2\|_0 = \left\| \|v_1\|^{1/P} \frac{v_1}{\|v_1\|^{1/P}} x_{n+1} + \|v_2\|^{1/P} \frac{v_2}{\|v_2\|^{1/P}} \right\|_0 \leq \\ &\leq \|v_1\| \|v_2\| \left\| \frac{v_1}{\|v_1\|^{1/P}} x_{n+1} + \frac{v_2}{\|v_2\|^{1/P}} \right\|_0 \end{aligned}$$

Since $\|v_1\|/\|v_1\|^{1/P} \leq 1$, $\|v_1\|/\|v_1\|^{1/P} \leq 1$, we get

$$\|u_{n+1}\|_0 \leq \|v_1\| \|v_2\| \frac{\varepsilon_n}{\sigma_n \sigma_{n+1}} \leq \sigma_n \sigma_{n+1} \frac{\varepsilon_n}{\sigma_n \sigma_{n+1}} = \varepsilon_n$$

$P_n = \{x \in X; \|x_n\| \leq \eta_n\}$ To obtain the desired result it is enough to choose ε_n . This concludes the proof

Example: An example a p -two norm algebra may serve the algebra V^p for $0 < p < 1$. The element of V^p are function x from $[0,1]$ to $\mathbb{R}$ whose p -variation is finite.

The p -variation, $\text{var}_p x$, of the function x is defined as the supremum of all the sum

$$\sum_{i=1}^n |x(t_i) - x(t_{i-1})|^p; 0 = t_0 \leq t_1 \leq \dots \leq t_n = 1$$

All three algebra operation are defined point wise. Let us introduce the p -norms

$$\begin{aligned} \|x\| &= \|x(0)\|^p + \text{var}_p(x) \\ \|x\|_0 &= \sup\{|x(t)|^p; 0 \leq t \leq 1\} \end{aligned}$$

We obtain p -two norm spaces. From the obvious inequality

$$\begin{aligned} \|xy\| &\leq \|x\|_0 \|y\| + \|x\|_0 \|y\| \leq 2\|x\| \|y\| \\ \|xy\|_0 &\leq \|x\|_0 \|y\|_0 \end{aligned}$$

It follows that $(V^p, \|\cdot\|, \|\cdot\|_0)$ is a p -two-norm algebra.

The function equals to 1 its unit element. The sequence (x_n) of element V^p is $\text{var}_p(x_n) < \infty$ and $x_n(t)$ tend to $x_0(t)$ uniformly γ -convergent to x_0 if and only if $x_0 \in V^p$.

A simple calculation show that if $\inf\{|x(t)|; 0 \leq t \leq 1\} = \varepsilon > 0$, then $\text{var}_p \frac{1}{x} \leq \frac{\text{var}_p x}{\varepsilon^{2p}}$,

It follows that the inverse is γ -continuous, and $G(V^p)$ consist of functions different from zero.

The neighborhood basis of zero for the topology τ_∞ consists of sets of form $\sum_{i=1}^n W_n$;

$W_n = \{x \in V^p; \text{var}_p x \leq 1, \|x\|_0 \leq \varepsilon_n\}$ with some $\varepsilon_n > 0$.

Conclusion:

The problem arises as to the condition $\|\alpha x\| = |\alpha|^p \|x\|$ Characterizing p -normed spaces could be replaced by $\|\alpha x\| \leq f(\alpha) \|x\|$ with some real function f , $\|\cdot\|$ being a Fréchet norm?

In this case we must have $f(\alpha\beta) \|x\| = \|\alpha\beta x\| = f(\alpha) \|\beta x\| = f(\alpha) f(\beta) \|x\|$, whence $f(\alpha\beta) = f(\alpha) f(\beta)$ for arbitrary α, β and $f(\alpha) > 0$ for $\alpha \neq 0$. since the function $f \rightarrow \|f(x)\|$ is continuous, the function f also must be so.

it is well known that this is the case if and only if $f(x) = |x|^p$

Also we can ask if the p -homogeneity of the norm might be replace $\|\alpha x\| \leq f(\alpha) \|x\|$ together with $\lim_{\alpha \rightarrow 0} f(\alpha) = 0$ This also gives nothing new, since in this case the ball $\{x \in X; \|x\| \leq 1\}$ must be bounded and the theorem of Aoki-Rolewicz the Fréchet norm $\|\cdot\|$ must be equivalent to p -norm.

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