



## Positive Implicative and Commutative BS-Neutrosophic Ideals of BCK/BCI-Algebra

B. Satyanarayana<sup>1</sup>, Shake Baji<sup>2</sup>, D. Ramesh<sup>3\*</sup>, Said Broumi<sup>4</sup>

<sup>1</sup>Department of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar, Guntur-522 510, Andhra Pradesh, India

<sup>2</sup>Department of Mathematics, Sir C.R.Reddy College of Engineering, Eluru-534007, Andhra Pradesh, India

<sup>3</sup>Department Of Engineering Mathematics, College of Engineering, Koneru Lakshmaiah Educational Foundation, Vaddeswaram-522302, Guntur(DT), Andhra Pradesh, India

<sup>4</sup>Laboratory of Information Processing, Faculty of Science Ben M'Sik, University Hassan II, B.P 7955, Morocco

Emails: drbsn63@yahoo.co.in; shakebaji6@gmail.com; ram.fuzzy@gmail.com; broumisaid78@gmail.com;

### Abstract

The notion of Positive Implicative BS-Neutrosophic Ideal, Commutative BS-Neutrosophic Ideal and related properties are investigated. Characterizations of Positive Implicative BS-Neutrosophic Ideal, Commutative BS-Neutrosophic Ideal are discussed.

**Keywords:** BS-Neutrosophic Set (BS-NSS); BS-Neutrosophic Ideal (BS-NSI); Positive Implicative BS-Neutrosophic Ideal (PI BS-NSI); Commutative BS-Neutrosophic Ideal (C BS-NSI).

### 1 Introduction

L.A.Zadeh,<sup>1</sup> a professor of computer science at the University of California, introduced the concept of fuzzy set (FS) in 1965. Fuzzy sets analyzed the degree of membership of elements of set. In 1986 Atanassov<sup>2</sup> generalized a fuzzy set to an intuitionistic fuzzy set (IFS) by including another function called a non-membership function. The neutrosophic set concept was developed by Smarandache<sup>(3,4)</sup> and is a more general framework that extends the concepts of classical set, Fuzzy set, Intuitionistic Fuzzy set, Interval valued fuzzy (Intuitionistic) set. Neutrosophic algebraic structures in BCK/BCI-algebras is described in articles,<sup>5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15</sup> we know that Smarandache's neutrosophic set have many generalizations.

In<sup>16</sup> B.Satyanarayana et.al introduced the concept of BS-neutrosophic set, and applied it to BCK/BCI-algebras. The BS-neutrosophic set enhances the existing neutrosophic framework by including an interval-valued non-membership function, allowing for a dynamic representation of non-membership within a set. The purpose of this extension is to improve our ability to model and reason with uncertain or ambiguous information across different application domains. Also, the concept of BS-neutrosophic subalgebras, BS-NSI is introduced and the associated properties are investigated. The homomorphic inverse image of the BS-neutrosophic subalgebra and discussed the translation of the BS-neutrosophic subalgebra. In a BCI-algebra, they provided conditions for a BS-NSI to be a BS-neutrosophic subalgebra.

In this article, we introduced the concept of PI BS-NSI, C BS-NSI and investigated several properties. The introduction of PI BS-NSIs and C BS-NSIs represents significant development in the study of BCK algebras within the framework of the BS-neutrosophic set. These concepts open up new perspectives on how uncertainty and ambiguity can be effectively managed in the study of ideals, providing valuable insights for applications in logical systems, formal methods, and decision-making under uncertainty. Also we discussed relation between PI BS-NSI and BS-NSI, relation between C BS-NSI and BS-NSI. Characterization of Positive Implicative BS-NSI, C BS-NSI provided.

## 2 Preliminaries

**Definition 2.1.** (<sup>17,18,19</sup>) Let a non-empty set  $\mathcal{K}$  in which “\*” is a binary operation and “0” is a constant.  $\mathcal{K}$  is called a BCI-algebra if the following axioms satisfied by the  $\mathcal{K}$  for all  $\zeta_o, \eta_o, \theta_o \in \mathcal{K}$

$$((\zeta_o * \eta_o) * (\zeta_o * \theta_o)) * (\theta_o * \eta_o) = 0 \tag{1}$$

$$(\zeta_o * (\zeta_o * \eta_o)) * \eta_o = 0 \tag{2}$$

$$\zeta_o * \zeta_o = 0 \tag{3}$$

$$\zeta_o * \eta_o = 0, \eta_o * \zeta_o = 0 \Rightarrow \zeta_o = \eta_o \tag{4}$$

If the following axiom is satisfied by a BCI-algebra  $\mathcal{K}$

$$0 * \zeta_o = 0 \tag{5}$$

then it is called a BCK-algebra for all  $\zeta_o \in \mathcal{K}$

In any BCK/BCI-algebra the following properties are hold

$$\zeta_o * 0 = 0 \tag{6}$$

$$\zeta_o \leq \eta_o \Rightarrow \zeta_o * \theta_o \leq \eta_o * \theta_o, \theta_o * \eta_o \leq \theta_o * \zeta_o \tag{7}$$

$$(\zeta_o * \eta_o) * \theta_o = (\zeta_o * \theta_o) * \eta_o \tag{8}$$

$$(\zeta_o * \theta_o) * (\eta_o * \theta_o) \leq \zeta_o * \eta_o \tag{9}$$

for all  $\zeta_o, \eta_o, \theta_o \in \mathcal{K}$ . Where  $\zeta_o \leq \eta_o$  if and only if  $\zeta_o * \eta_o = 0$ . In any BCI-algebra  $\mathcal{K}$  the following conditions are hold<sup>17</sup>

$$\zeta_o * (\zeta_o * (\zeta_o * \eta_o)) = \zeta_o * \eta_o \tag{10}$$

$$0 * (\zeta_o * \eta_o) = (0 * \zeta_o) * (0 * \eta_o) \tag{11}$$

**Definition 2.2.** If the following condition is satisfied by a BCK-algebra  $\mathcal{K}$ ,  $\zeta_o * (\zeta_o * \eta_o) = \eta_o * (\eta_o * \zeta_o)$  for all  $\zeta_o, \eta_o \in \mathcal{K}$  then it is called a commutative BCK-algebra.

**Definition 2.3.** <sup>17</sup> If the following condition is holds in a BCI-algebra  $\mathcal{K}$ ,  $(\zeta_o * \eta_o) * \theta_o = (\zeta_o * \theta_o) * \eta_o$  for all  $\zeta_o, \eta_o, \theta_o \in \mathcal{K}$  then it is called associative.

**Definition 2.4.** Let  $\mathcal{K}$  be a BCK/BCI-algebra. A non-empty sub set  $\mathcal{H}$  of  $\mathcal{K}$  is said to be a sub algebra of  $\mathcal{K}$  if  $\zeta_o * \eta_o \in \mathcal{H}$  for all  $\zeta_o, \eta_o \in \mathcal{H}$ .

**Definition 2.5.** Let  $\mathcal{K}$  be a BCK/BCI-algebra. A non-empty sub set  $\mathcal{H}$  of  $\mathcal{K}$  is said to be an ideal of  $\mathcal{K}$  if  $0 \in \mathcal{H}$ , and  $\eta_o, \zeta_o * \eta_o \in \mathcal{H} \Rightarrow \zeta_o \in \mathcal{H}$  for all  $\zeta_o, \eta_o \in \mathcal{K}$ .

**Definition 2.6.** Let  $\mathcal{K}$  be a BCK/BCI-algebra. A non-empty sub set  $\mathcal{H}$  of  $\mathcal{K}$  is said to be a Positive Implicative ideal of  $\mathcal{K}$  if  $0 \in \mathcal{H}$ , and  $(\zeta_o * \eta_o) * \theta_o, \eta_o * \theta_o \in \mathcal{H} \Rightarrow \zeta_o * \theta_o \in \mathcal{H}$  for all  $\zeta_o, \eta_o, \theta_o \in \mathcal{K}$ .

**Definition 2.7.** Let  $\mathcal{K}$  be a BCK/BCI-algebra. A non-empty sub set  $\mathcal{H}$  of  $\mathcal{K}$  is said to be a Commutative ideal of  $\mathcal{K}$  if  $0 \in \mathcal{H}$ , and  $(\zeta_o * \eta_o) * \theta_o, \theta_o \in \mathcal{H} \Rightarrow \zeta_o * (\eta_o * (\eta_o * \zeta_o)) \in \mathcal{H}$  for all  $\zeta_o, \eta_o, \theta_o \in \mathcal{K}$ .

**Definition 2.8.** <sup>1</sup> Let a non-empty set  $\mathcal{K}$ . In  $\mathcal{K}$  a fuzzy set is a mapping  $\mathcal{N}_t : \mathcal{K} \rightarrow [0, 1]$

**Definition 2.9.** A fuzzy set  $\mathcal{N}_t : \mathcal{K} \rightarrow [0, 1]$  is called fuzzy sub-algebra of  $\mathcal{K}$ , if  $\mathcal{N}_t(\zeta_o * \eta_o) \geq \min\{\mathcal{N}_t(\zeta_o), \mathcal{N}_t(\eta_o)\}$  for all  $\zeta_o, \eta_o \in \mathcal{K}$ .

An interval number is a closed subinterval  $\hat{h} = [h^-, h^+]$  of  $[I]$  where  $0 \leq h^- \leq h^+ \leq 1$ . The set of all interval numbers are Denote by  $[I]$ . We define refined minimum (rmin) and refined maximum (rmax) for any two elements of  $[I]$ . The symbols “ $\preceq$ ”, “ $\succ$ ”, “ $=$ ” are also defined for two elements of  $[I]$ . Suppose  $\hat{h}_1, \hat{h}_2 \in [I]$ . Then  $\hat{h}_1 = [h_1^-, h_1^+]$  and  $\hat{h}_2 = [h_2^-, h_2^+]$ .

$$rmin\{\hat{h}_1, \hat{h}_2\} = [\min\{h_1^-, h_2^-\}, \min\{h_1^+, h_2^+\}],$$

$$rmax \{ \widehat{h}_1, \widehat{h}_2 \} = [max \{ h_1^-, h_2^- \}, max \{ h_1^+, h_2^+ \}],$$

$\widehat{h}_1 \succ \widehat{h}_2 \Leftrightarrow h_1^- \geq h_2^-, h_1^+ \geq h_2^+$ , and similarly, we may have  $\widehat{h}_1 \preccurlyeq \widehat{h}_2$  and  $\widehat{h}_1 = \widehat{h}_2$ . To say  $\widehat{h}_1 \succ \widehat{h}_2$  (resp.  $\widehat{h}_1 \preccurlyeq \widehat{h}_2$ ) we mean  $\widehat{h}_1 \succ \widehat{h}_2$  and  $\widehat{h}_1 \neq \widehat{h}_2$  (resp.  $\widehat{h}_1 \preccurlyeq \widehat{h}_2$  and  $\widehat{h}_1 \neq \widehat{h}_2$ ). Let  $\widehat{h}_i \in [I]$  where  $i \in \Pi$ . We define

$$rinf_{i \in \Pi} \widehat{h}_i = [inf_{i \in \Pi} h_i^-, inf_{i \in \Pi} h_i^+] \text{ and } rsup_{i \in \Pi} \widehat{h}_i = [sup_{i \in \Pi} h_i^-, sup_{i \in \Pi} h_i^+].$$

**Definition 2.10.** <sup>20</sup> An interval-valued fuzzy set (IVFS) in a non-empty set  $\mathcal{K}$  is a function  $\widehat{\mathcal{N}} : \mathcal{K} \rightarrow [I]$  is called. The set of all IVFS in  $\mathcal{K}$  is denoted by  $[I]^\mathcal{K}$ . For every IVFS  $\widehat{\mathcal{N}} \in [I]^\mathcal{K}$  and  $\zeta_o \in \mathcal{K}, \widehat{\mathcal{N}}(\zeta_o) = [\mathcal{N}^-(\zeta_o), \mathcal{N}^+(\zeta_o)]$  is called the degree of belongingness of an element  $\zeta_o \in \widehat{\mathcal{N}}$ , where  $\mathcal{N}^- : \mathcal{K} \rightarrow [I]$  is called a lower fuzzy set and  $\mathcal{N}^+ : \mathcal{K} \rightarrow [I]$  is called an upper fuzzy set in  $\mathcal{K}$ . We denote IVFS by  $\widehat{\mathcal{N}} = [\mathcal{N}^-, \mathcal{N}^+]$ .

**Definition 2.11.** <sup>4</sup> Let  $\mathcal{K}$  be a non-empty set. A neutrosophic set (NSS) in  $\mathcal{K}$  is a structure of the form  $\mathcal{N} = \{ \langle \zeta_o; \mathcal{N}_T(\zeta_o), \mathcal{N}_I(\zeta_o), \mathcal{N}_F(\zeta_o) \rangle : \zeta_o \in \mathcal{K} \}$  where  $\mathcal{N}_T : \mathcal{K} \rightarrow [0, 1]$  is a degree of belongingness,  $\mathcal{N}_I : \mathcal{K} \rightarrow [0, 1]$  is a degree of indeterminacy of belongingness, and  $\mathcal{N}_F : \mathcal{K} \rightarrow [0, 1]$  degree of non-belongingness. For simplicity, we will use the symbol  $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$  for the neutrosophic set  $\mathcal{N} = \{ \langle \zeta_o; \mathcal{N}_T(\zeta_o), \mathcal{N}_I(\zeta_o), \mathcal{N}_F(\zeta_o) \rangle : \zeta_o \in \mathcal{K} \}$ .

**Definition 2.12.** Let  $\mathcal{K}$  be a non-empty set. BS-NSS in  $\mathcal{K}$ , is a structure of the form  $\mathcal{N} = \{ \langle \zeta_o; \mathcal{N}_t(\zeta_o), \mathcal{N}_i(\zeta_o), \widehat{\mathcal{N}}_f(\zeta_o) \rangle : \zeta_o \in \mathcal{K} \}$  where  $\mathcal{N}_t, \mathcal{N}_i$ , are fuzzy sets in  $\mathcal{K}$ , which are called a degree of membership and degree of indeterminacy, respectively, and  $\widehat{\mathcal{N}}_f$  is an interval valued fuzzy set in  $\mathcal{K}$  which is called an interval valued degree of non-membership

**Definition 2.13.** Let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a BS-NSS in  $\mathcal{K}$ , we define the following sets  $\mathcal{U}(\mathcal{N}_t; l) = \{ \zeta_o \in \mathcal{K} : \mathcal{N}_t(\zeta_o) \geq l \}$ ,  $\mathcal{U}(\mathcal{N}_i; m) = \{ \zeta_o \in \mathcal{K} : \mathcal{N}_i(\zeta_o) \geq m \}$  and  $\mathcal{L}(\widehat{\mathcal{N}}_f; [n_1, n_2]) = \{ \zeta_o \in \mathcal{K} : \widehat{\mathcal{N}}_f(\zeta_o) \preccurlyeq [n_1, n_2] \}$  Where  $l, m \in [0, 1]$  and  $[n_1, n_2] \in [I]$

**Definition 2.14.** Let  $\mathcal{K}$  be a BCK/BCI algebra. A BS-NSS  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  in  $\mathcal{K}$  is called a BS - neutrosophic subalgebra of  $\mathcal{K}$  if it satisfies

- (BS-NSSA 1)  $\mathcal{N}_t(\zeta_o * \eta_o) \geq min\{\mathcal{N}_t(\zeta_o), \mathcal{N}_t(\eta_o)\}$
- (BS-NSSA 2)  $\mathcal{N}_i(\zeta_o * \eta_o) \geq min\{\mathcal{N}_i(\zeta_o), \mathcal{N}_i(\eta_o)\}$
- (BS-NSSA 3)  $\widehat{\mathcal{N}}_f(\zeta_o * \eta_o) \preccurlyeq rmax\{\widehat{\mathcal{N}}_f(\zeta_o), \widehat{\mathcal{N}}_f(\eta_o)\}$  for all  $\zeta_o, \eta_o \in \mathcal{K}$ .

**Definition 2.15.** Let  $\mathcal{K}$  be a BCK/BCI-algebra. A BS-NSS  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  in  $\mathcal{K}$  is called a BS-NSI of  $\mathcal{K}$  if it satisfies

- (BS-NSI 1)  $\mathcal{N}_t(0) \geq \mathcal{N}_t(\zeta_o), \mathcal{N}_i(0) \geq \mathcal{N}_i(\zeta_o)$  and  $\widehat{\mathcal{N}}_f(0) \preccurlyeq \widehat{\mathcal{N}}_f(x)$  for all  $\zeta_o \in \mathcal{K}$
- (BS-NSI 2)  $\mathcal{N}_t(\zeta_o) \geq min\{\mathcal{N}_t(\zeta_o * \eta_o), \mathcal{N}_t(\eta_o)\}$
- (BS-NSI 3)  $\mathcal{N}_i(\zeta_o) \geq min\{\mathcal{N}_i(\zeta_o * \eta_o), \mathcal{N}_i(\eta_o)\}$
- (BS-NSI 4)  $\widehat{\mathcal{N}}_f(\zeta_o) \preccurlyeq rmax\{\widehat{\mathcal{N}}_f(\zeta_o * \eta_o), \widehat{\mathcal{N}}_f(\eta_o)\}$  for all  $\zeta_o, \eta_o \in \mathcal{K}$ .

### 3 Positive implicative BS-Neutrosophic ideal

**Definition 3.1.** Let  $\mathcal{K}$  be a BCK-algebra. A BS-NSS  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  in  $\mathcal{K}$  is called a Positive implicative BS-Neutrosophic ideal (PI BS-NSI) of  $\mathcal{K}$  if it satisfies (BS-NSI 1) and

- (PI BS-NSI 1)  $\mathcal{N}_t(\zeta_o * \theta_o) \geq min\{\mathcal{N}_t((\zeta_o * \eta_o) * \theta_o), \mathcal{N}_t(\eta_o * \theta_o)\}$
- (PI BS-NSI 2)  $\mathcal{N}_i(\zeta_o * \theta_o) \geq min\{\mathcal{N}_i((\zeta_o * \eta_o) * \theta_o), \mathcal{N}_i(\eta_o * \theta_o)\}$
- (PI BS-NSI 3)  $\widehat{\mathcal{N}}_f(\zeta_o * \theta_o) \preccurlyeq rmax\{\widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \theta_o), \widehat{\mathcal{N}}_f(\eta_o * \theta_o)\}$  for all  $\zeta_o, \eta_o, \theta_o \in \mathcal{K}$ .

Table 1: BCK-algebra

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| * | 0 | a | b | c | d |
| 0 | 0 | 0 | 0 | 0 | 0 |
| a | a | 0 | 0 | 0 | 0 |
| b | b | b | 0 | 0 | b |
| c | c | c | c | 0 | c |
| d | d | d | d | d | 0 |

**Example 3.2.** Consider a set  $\mathcal{K} = \{0, a, b, c, d\}$  with the binary operation ‘\*’ which is given in the Table.1 Then  $(\mathcal{K}; *, 0)$  is a BCK-algebra.

Let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a BS-NSS in  $\mathcal{K}$  defined in Table.2

Table 2: BS-Neutrosophic Set

| $\mathcal{K}$ | $\mathcal{N}_t(\zeta_o)$ | $\mathcal{N}_i(\zeta_o)$ | $\widehat{\mathcal{N}}_f(\zeta_o)$ |
|---------------|--------------------------|--------------------------|------------------------------------|
| 0             | 1                        | 0.9                      | [0.2, 0.5]                         |
| a             | 0.8                      | 0.6                      | [0.4, 0.6]                         |
| b             | 0.4                      | 0.3                      | [0.5, 0.7]                         |
| c             | 0.4                      | 0.3                      | [0.5, 0.7]                         |
| d             | 0.2                      | 0.1                      | [0.7, 1]                           |

It is routine to verify that  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a PI BS-NSI of  $\mathcal{K}$ .

**Theorem 3.3.** Every PI BS-NSI is a BS-NSI.

*Proof.* Let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a PI BS-NSI of  $\mathcal{K}$ . If we take  $\theta_o = 0$  in (PI BS-NSI 1), (PI BS-NSI 2), (PI BS-NSI 3) and use (2.6) then we have

$$\begin{aligned} \mathcal{N}_t(\zeta_o * 0) &\geq \min\{\mathcal{N}_t((\zeta_o * \eta_o) * 0), \mathcal{N}_t(\eta_o * 0)\} \\ &\Rightarrow \mathcal{N}_t(\zeta_o) \geq \min\{\mathcal{N}_t(\zeta_o * \eta_o), \mathcal{N}_t(\eta_o)\}, \\ \mathcal{N}_i(\zeta_o * 0) &\geq \min\{\mathcal{N}_i((\zeta_o * \eta_o) * 0), \mathcal{N}_i(\eta_o * 0)\} \\ &\Rightarrow \mathcal{N}_i(\zeta_o) \geq \min\{\mathcal{N}_i(\zeta_o * \eta_o), \mathcal{N}_i(\eta_o)\}, \\ \widehat{\mathcal{N}}_f(\zeta_o * 0) &\leq rmax\{\widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * 0), \widehat{\mathcal{N}}_f(\eta_o * 0)\} \\ &\Rightarrow \widehat{\mathcal{N}}_f(\zeta_o) \leq rmax\{\widehat{\mathcal{N}}_f(\zeta_o * \eta_o), \widehat{\mathcal{N}}_f(\eta_o)\} \end{aligned}$$

for all  $\zeta_o, \eta_o \in \mathcal{K}$ . Thus, every PI BS-NSI is a BS-NSI of  $\mathcal{K}$ . □

The converse of Theorem.3.3 is not true as seen in the following example.

**Example 3.4.** Consider a BCK-algebra  $\mathcal{K} = \{0, 1, 2, 3\}$  with the binary operation ‘\*’ which is given in Table.3

Table 3: BCK-Algebra

|   |   |   |   |   |
|---|---|---|---|---|
| * | 0 | 1 | 2 | 3 |
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 |
| 2 | 2 | 1 | 0 | 2 |
| 3 | 3 | 3 | 3 | 0 |

Let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a BS-NSS in  $\mathcal{K}$  defined by Table.4

It is routine to verify that  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a BS-NSI of  $\mathcal{K}$ .

Since  $\mathcal{N}_t(2 * 1) < \min\{\mathcal{N}_t((2 * 1) * 1), \mathcal{N}_t(1 * 1)\}$ ,  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is not a PI BS-NSI of  $\mathcal{K}$ .

Table 4: BS-Neutrosophic Set

| $\mathcal{K}$ | $\mathcal{N}_t(\zeta_o)$ | $\mathcal{N}_i(\zeta_o)$ | $\widehat{\mathcal{N}}_f(\zeta_o)$ |
|---------------|--------------------------|--------------------------|------------------------------------|
| 0             | 0.91                     | 1                        | [0.21, 0.63]                       |
| 1             | 0.54                     | 0.35                     | [0.53, 0.75]                       |
| 2             | 0.54                     | 0.35                     | [0.53, 0.75]                       |
| 3             | 0.25                     | 0.17                     | [0.78, 1]                          |

**Lemma 3.5.** Let  $\mathcal{K}$  be a BCK/BCI-algebra. Then every BS-NSI  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  of  $\mathcal{K}$  satisfies the following assertion  $\zeta_o * \eta_o \preceq \theta_o \Rightarrow \mathcal{N}_t(\zeta_o) \geq \min\{\mathcal{N}_t(\eta_o), \mathcal{N}_t(\theta_o)\}$ ,  $\mathcal{N}_i(\zeta_o) \geq \min\{\mathcal{N}_i(\eta_o), \mathcal{N}_i(\theta_o)\}$ ,  $\widehat{\mathcal{N}}_f(\zeta_o) \preceq rmax\{\widehat{\mathcal{N}}_f(\eta_o), \widehat{\mathcal{N}}_f(\theta_o)\}$  for all  $\zeta_o, \eta_o, \theta_o \in \mathcal{K}$ .

**Theorem 3.6.** A BS-NSS  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  in a BCK-algebra  $\mathcal{K}$  is a PI BS-NSI of  $\mathcal{K}$  if and only if it is a BS-NSI of  $\mathcal{K}$  satisfying the following conditions

$$\left( \begin{array}{l} \mathcal{N}_t(\zeta_o * \eta_o) \geq \mathcal{N}_t((\zeta_o * \eta_o) * \eta_o) \\ \mathcal{N}_i(\zeta_o * \eta_o) \geq \mathcal{N}_i((\zeta_o * \eta_o) * \eta_o) \\ \widehat{\mathcal{N}}_f(\zeta_o * \eta_o) \preceq \widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \eta_o) \end{array} \right) \text{ (for all } \zeta_o, \eta_o \in \mathcal{K} \text{) (A)}$$

*Proof.* Assume that  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is PI BS-NSI of  $\mathcal{K}$ . If  $\theta_o$  is replaced by  $\eta_o$  in (PI BS-NSI 1), (PI BS-NSI 2) and (PI BS-NSI 3) we get

$$\begin{aligned} \mathcal{N}_t(\zeta_o * \eta_o) &\geq \min\{\mathcal{N}_t((\zeta_o * \eta_o) * \eta_o), \mathcal{N}_t(\eta_o * \eta_o)\} \\ &= \min\{\mathcal{N}_t((\zeta_o * \eta_o) * \eta_o), \mathcal{N}_t(0)\} = \mathcal{N}_t((\zeta_o * \eta_o) * \eta_o), \\ \mathcal{N}_i(\zeta_o * \eta_o) &\geq \min\{\mathcal{N}_i((\zeta_o * \eta_o) * \eta_o), \mathcal{N}_i(\eta_o * \eta_o)\} \\ &= \min\{\mathcal{N}_i((\zeta_o * \eta_o) * \eta_o), \mathcal{N}_i(0)\} = \mathcal{N}_i((\zeta_o * \eta_o) * \eta_o), \\ \widehat{\mathcal{N}}_f(\zeta_o * \eta_o) &\preceq rmax\{\widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \eta_o), \widehat{\mathcal{N}}_f(\eta_o * \eta_o)\} \\ &= rmax\{\widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \eta_o), \widehat{\mathcal{N}}_f(0)\} = \widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \eta_o) \text{ for all } \zeta_o, \eta_o \in \mathcal{K}. \end{aligned}$$

Conversely, let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a BS-NSI of  $\mathcal{K}$  satisfying the condition (A) since

$((\zeta_o * \theta_o) * \theta_o) * (\eta_o * \theta_o) \preceq ((\zeta_o * \theta_o) * \eta_o) = ((\zeta_o * \eta_o) * \theta_o)$  for all  $\zeta_o, \eta_o, \theta_o \in \mathcal{K}$ . It follows from Lemma 3.5 that

$$\left( \begin{array}{l} \mathcal{N}_t(((\zeta_o * \theta_o) * \theta_o) * (\eta_o * \theta_o)) \geq \mathcal{N}_t((\zeta_o * \eta_o) * \theta_o) \\ \mathcal{N}_i(((\zeta_o * \theta_o) * \theta_o) * (\eta_o * \theta_o)) \geq \mathcal{N}_i((\zeta_o * \eta_o) * \theta_o) \\ \widehat{\mathcal{N}}_f(((\zeta_o * \theta_o) * \theta_o) * (\eta_o * \theta_o)) \preceq \widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \theta_o) \end{array} \right) \text{ (B) for all } \zeta_o, \eta_o, \theta_o \in \mathcal{K}. \text{ Now using (A), (BS-NSI 2), (BS-NSI 3), (BS-NSI 4) and (B), we have}$$

$$\begin{aligned} \mathcal{N}_t(\zeta_o * \theta_o) &\geq \mathcal{N}_t((\zeta_o * \theta_o) * \theta_o) \\ &\geq \min\{\mathcal{N}_t(((\zeta_o * \theta_o) * \theta_o) * (\eta_o * \theta_o)), \mathcal{N}_t(\eta_o * \theta_o)\} \\ &\geq \min\{\mathcal{N}_t((\zeta_o * \eta_o) * \theta_o), \mathcal{N}_t(\eta_o * \theta_o)\}, \\ \mathcal{N}_i(\zeta_o * \theta_o) &\geq \mathcal{N}_i((\zeta_o * \theta_o) * \theta_o) \\ &\geq \min\{\mathcal{N}_i(((\zeta_o * \theta_o) * \theta_o) * (\eta_o * \theta_o)), \mathcal{N}_i(\eta_o * \theta_o)\} \\ &\geq \min\{\mathcal{N}_i((\zeta_o * \eta_o) * \theta_o), \mathcal{N}_i(\eta_o * \theta_o)\}, \\ \widehat{\mathcal{N}}_f(\zeta_o * \theta_o) &\preceq \widehat{\mathcal{N}}_f((\zeta_o * \theta_o) * \theta_o) \\ &\preceq rmax\{\widehat{\mathcal{N}}_f(((\zeta_o * \theta_o) * \theta_o) * (\eta_o * \theta_o)), \widehat{\mathcal{N}}_f(\eta_o * \theta_o)\} \\ &\preceq rmax\{\widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \theta_o), \widehat{\mathcal{N}}_f(\eta_o * \theta_o)\} \text{ for all } \zeta_o, \eta_o, \theta_o \in \mathcal{K}. \end{aligned}$$

Thus, Positive Implicative  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a BS-NSI of  $\mathcal{K}$ . □

**Theorem 3.7.** Let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a BS-NSI of  $\mathcal{K}$ . Then  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is PI BS-NSI of  $\mathcal{K}$  if and only if it satisfies the following conditions

$$\left( \begin{array}{l} \mathcal{N}_t((\zeta_o * \theta_o) * (\eta_o * \theta_o)) \geq \mathcal{N}_t((\zeta_o * \eta_o) * \theta_o) \\ \mathcal{N}_i((\zeta_o * \theta_o) * (\eta_o * \theta_o)) \geq \mathcal{N}_i((\zeta_o * \eta_o) * \theta_o) \\ \widehat{\mathcal{N}}_f((\zeta_o * \theta_o) * (\eta_o * \theta_o)) \preceq \widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \theta_o) \end{array} \right) \text{ (C) for all } \zeta_o, \eta_o, \theta_o \in \mathcal{K}.$$

*Proof.* Suppose that  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is PI BS-NSI of  $\mathcal{K}$ . Then  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a BS-NSI of  $\mathcal{K}$  by Theorem 3.3 and satisfies the condition (A) by Theorem 3.6. Since  $((\zeta_o * (\eta_o * \theta_o)) * \theta_o) * \theta_o =$

$((\zeta_o * \theta_o) * (\eta_o * \theta_o)) * \theta_o \preceq ((\zeta_o * \eta_o) * \theta_o)$  for all  $\zeta_o, \eta_o, \theta_o \in \mathcal{K}$ . It follows from Lemma 3.5 that

$$\left( \begin{array}{l} \mathcal{N}_t(((\zeta_o * \theta_o) * (\eta_o * \theta_o)) * \theta_o) \geq \mathcal{N}_t((\zeta_o * \eta_o) * \theta_o) \\ \mathcal{N}_i(((\zeta_o * \theta_o) * (\eta_o * \theta_o)) * \theta_o) \geq \mathcal{N}_i((\zeta_o * \eta_o) * \theta_o) \\ \widehat{\mathcal{N}}_f(((\zeta_o * \theta_o) * (\eta_o * \theta_o)) * \theta_o) \preceq \widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \theta_o) \end{array} \right) \text{ (D) Now using (2.8), (A) and (D), we have}$$

$$\begin{aligned} \mathcal{N}_t((\zeta_o * \theta_o) * (\eta_o * \theta_o)) &= \mathcal{N}_t((\zeta_o * (\eta_o * \theta_o)) * \theta_o) \\ &\geq \mathcal{N}_t(((\zeta_o * (\eta_o * \theta_o)) * \theta_o) * \theta_o) \geq \mathcal{N}_t((\zeta_o * \eta_o) * \theta_o), \\ \mathcal{N}_i((\zeta_o * \theta_o) * (\eta_o * \theta_o)) &= \mathcal{N}_i((\zeta_o * (\eta_o * \theta_o)) * \theta_o) \\ &\geq \mathcal{N}_i(((\zeta_o * (\eta_o * \theta_o)) * \theta_o) * \theta_o) \geq \mathcal{N}_i((\zeta_o * \eta_o) * \theta_o), \\ \widehat{\mathcal{N}}_f((\zeta_o * \theta_o) * (\eta_o * \theta_o)) &= \widehat{\mathcal{N}}_f((\zeta_o * (\eta_o * \theta_o)) * \theta_o) \\ &\preceq \widehat{\mathcal{N}}_f(((\zeta_o * (\eta_o * \theta_o)) * \theta_o) * \theta_o) \preceq \widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \theta_o). \end{aligned}$$

Hence condition (C) is valid. Conversely, let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a BS-NSI of  $\mathcal{K}$  which satisfies the condition (C) If we put  $\theta_o = \eta_o$  in (C) and use (2.3), (2.6) then we have

$$\begin{aligned} \mathcal{N}_t((\zeta_o * \eta_o) * (\eta_o * \eta_o)) &\geq \mathcal{N}_t((\zeta_o * \eta_o) * \eta_o) \\ \Rightarrow \mathcal{N}_t((\zeta_o * \eta_o)) &\geq \mathcal{N}_t((\zeta_o * \eta_o) * \eta_o), \\ \mathcal{N}_i((\zeta_o * \eta_o) * (\eta_o * \eta_o)) &\geq \mathcal{N}_i((\zeta_o * \eta_o) * \eta_o) \\ \Rightarrow \mathcal{N}_i((\zeta_o * \eta_o)) &\geq \mathcal{N}_i((\zeta_o * \eta_o) * \eta_o), \\ \widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * (\eta_o * \eta_o)) &\preceq \widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \eta_o) \\ \Rightarrow \widehat{\mathcal{N}}_f((\zeta_o * \eta_o)) &\preceq \widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \eta_o). \end{aligned}$$

for all  $\zeta_o, \eta_o, \theta_o \in \mathcal{K}$ .

Therefore, by Theorem 3.6  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is PI BS-NSI of  $\mathcal{K}$ . □

**Theorem 3.8.** Let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a BS-NSS of  $\mathcal{K}$ . Then  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is PI BS-NSI of  $\mathcal{K}$  if and only if it satisfies the conditions (BS-NSI 1) and

$$\left( \begin{array}{l} \mathcal{N}_t(\zeta_o * \eta_o) \geq \min \{ \mathcal{N}_t(((\zeta_o * \eta_o) * \eta_o) * \theta_o), \mathcal{N}_t(\theta_o) \} \\ \mathcal{N}_i(\zeta_o * \eta_o) \geq \min \{ \mathcal{N}_i(((\zeta_o * \eta_o) * \eta_o) * \theta_o), \mathcal{N}_i(\theta_o) \} \\ \widehat{\mathcal{N}}_f(\zeta_o * \eta_o) \preceq \text{rmax} \{ \widehat{\mathcal{N}}_f(((\zeta_o * \eta_o) * \eta_o) * \theta_o), \widehat{\mathcal{N}}_f(\theta_o) \} \end{array} \right) \text{ (E) (for all } \zeta_o, \eta_o, \theta_o \in \mathcal{K}.)$$

*Proof.* Suppose that  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a PI BS-NSI of  $\mathcal{K}$ . Then  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a BS-NSI of  $\mathcal{K}$  by Theorem 3.3. and so, condition (BS-NSI 1) is valid. Using (BS-NSI 2), (BS-NSI 3), (BS-NSI 4), (2.3), (2.6), (2.8), and (C), we have

$$\begin{aligned} \mathcal{N}_t(\zeta_o * \eta_o) &\geq \min \{ \mathcal{N}_t((\zeta_o * \eta_o) * \theta_o), \mathcal{N}_t(\theta_o) \} \\ &= \min \{ \mathcal{N}_t(((\zeta_o * \theta_o) * \eta_o) * (\eta_o * \eta_o)), \mathcal{N}_t(\theta_o) \} \\ &\geq \min \{ \mathcal{N}_t(((\zeta_o * \theta_o) * \eta_o) * \eta_o), \mathcal{N}_t(\theta_o) \} \\ &= \min \{ \mathcal{N}_t(((\zeta_o * \eta_o) * \eta_o) * \theta_o), \mathcal{N}_t(\theta_o) \}, \\ \mathcal{N}_i(\zeta_o * \eta_o) &\geq \min \{ \mathcal{N}_i((\zeta_o * \eta_o) * \theta_o), \mathcal{N}_i(\theta_o) \} \\ &= \min \{ \mathcal{N}_i(((\zeta_o * \theta_o) * \eta_o) * (\eta_o * \eta_o)), \mathcal{N}_i(\theta_o) \} \\ &\geq \min \{ \mathcal{N}_i(((\zeta_o * \theta_o) * \eta_o) * \eta_o), \mathcal{N}_i(\theta_o) \} \\ &= \min \{ \mathcal{N}_i(((\zeta_o * \eta_o) * \eta_o) * \theta_o), \mathcal{N}_i(\theta_o) \}, \\ \widehat{\mathcal{N}}_f(\zeta_o * \eta_o) &\preceq \text{rmax} \{ \widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \theta_o), \widehat{\mathcal{N}}_f(\theta_o) \} \\ &= \text{rmax} \{ \widehat{\mathcal{N}}_f(((\zeta_o * \theta_o) * \eta_o) * (\eta_o * \eta_o)), \widehat{\mathcal{N}}_f(\theta_o) \} \\ &\preceq \text{rmax} \{ \widehat{\mathcal{N}}_f(((\zeta_o * \theta_o) * \eta_o) * \eta_o), \widehat{\mathcal{N}}_f(\theta_o) \} \\ &= \text{rmax} \{ \widehat{\mathcal{N}}_f(((\zeta_o * \eta_o) * \eta_o) * \theta_o), \widehat{\mathcal{N}}_f(\theta_o) \} \text{ for all } \zeta_o, \eta_o, \theta_o \in \mathcal{K}. \end{aligned}$$

Conversely, let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a BS-NSS in  $\mathcal{K}$  which satisfies condition (BS-NSI 1) and condition (E) then

$$\begin{aligned} \mathcal{N}_t(\zeta_o) &= \mathcal{N}_t(\zeta_o * 0) \geq \min \{ \mathcal{N}_t(((\zeta_o * 0) * 0) * \theta_o), \mathcal{N}_t(\theta_o) \} \\ &= \min \{ \mathcal{N}_t(\zeta_o * \theta_o), \mathcal{N}_t(\theta_o) \}, \\ \mathcal{N}_i(\zeta_o) &= \mathcal{N}_i(\zeta_o * 0) \geq \min \{ \mathcal{N}_i(((\zeta_o * 0) * 0) * \theta_o), \mathcal{N}_i(\theta_o) \} \\ &= \min \{ \mathcal{N}_i(\zeta_o * \theta_o), \mathcal{N}_i(\theta_o) \}, \text{ and} \\ \widehat{\mathcal{N}}_f(\zeta_o) &= \widehat{\mathcal{N}}_f(\zeta_o * 0) \preceq \text{rmax} \{ \widehat{\mathcal{N}}_f(((\zeta_o * 0) * 0) * \theta_o), \widehat{\mathcal{N}}_f(\theta_o) \} \\ &\preceq \text{rmax} \{ \widehat{\mathcal{N}}_f(\zeta_o * \theta_o), \widehat{\mathcal{N}}_f(\theta_o) \} \text{ for all } \zeta_o, \theta_o \in \mathcal{K}. \end{aligned}$$

Thus  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a BS-NSI of  $\mathcal{K}$ . Taking  $\theta_o = 0$  in conditions (E) and using (2.6) and (BS-NSI 1)

$$\begin{aligned} \mathcal{N}_t(\zeta_o * \eta_o) &\geq \min \{ \mathcal{N}_t( ((\zeta_o * \eta_o) * \eta_o) * 0 ), \mathcal{N}_t(0) \} \\ &= \min \{ \mathcal{N}_t( (\zeta_o * \eta_o) * \eta_o ), \mathcal{N}_t(0) \} = \mathcal{N}_t( (\zeta_o * \eta_o) * \eta_o ), \\ \mathcal{N}_i(\zeta_o * \eta_o) &\geq \min \{ \mathcal{N}_i( ((\zeta_o * \eta_o) * \eta_o) * 0 ), \mathcal{N}_i(0) \} \\ &= \min \{ \mathcal{N}_i( (\zeta_o * \eta_o) * \eta_o ), \mathcal{N}_i(0) \} = \mathcal{N}_i( (\zeta_o * \eta_o) * \eta_o ), \\ \widehat{\mathcal{N}}_f(\zeta_o * \eta_o) &\leq rmax \{ \widehat{\mathcal{N}}_f( ((\zeta_o * \eta_o) * \eta_o) * 0 ), \widehat{\mathcal{N}}_f(0) \} \\ &= rmax \{ \widehat{\mathcal{N}}_f( (\zeta_o * \eta_o) * \eta_o ), \widehat{\mathcal{N}}_f(0) \} = \widehat{\mathcal{N}}_f( (\zeta_o * \eta_o) * \eta_o ), \end{aligned}$$

for all  $\zeta_o, \eta_o \in \mathcal{K}$ . It follows from Theorem 3.6 that  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is PI BS-NSI of  $\mathcal{K}$ . □

**Lemma 3.9.** Let  $\mathcal{K}$  be a BCK\BCI-algebra. Then every BS-NSI  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  of  $\mathcal{K}$  satisfies the assertion

$$\zeta_o * \eta_o \leq \theta_o \Rightarrow \left( \begin{array}{l} \mathcal{N}_t(\zeta_o) \geq \min \{ \mathcal{N}_t(\eta_o), \mathcal{N}_t(\theta_o) \} \\ \mathcal{N}_i(\zeta_o) \geq \min \{ \mathcal{N}_i(\eta_o), \mathcal{N}_i(\theta_o) \} \\ \widehat{\mathcal{N}}_f(\zeta_o) \leq rmax \{ \widehat{\mathcal{N}}_f(\eta_o), \widehat{\mathcal{N}}_f(\theta_o) \} \end{array} \right) \text{ (F) for all } \zeta_o, \eta_o, \theta_o \in \mathcal{K}.$$

**Lemma 3.10.** If a BS-NSS  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  in  $\mathcal{K}$  satisfies the condition (F) then  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a BS-NSI of  $\mathcal{K}$ .

*Proof.* Since  $0 * \zeta_o \leq \zeta_o$  and  $\zeta_o * (\zeta_o * \eta_o) \leq \eta_o$  for all  $\zeta_o, \eta_o \in \mathcal{K}$ , it follows from (F) that  $\mathcal{N}_t(0) \geq \mathcal{N}_t(\zeta_o)$ ,  $\mathcal{N}_i(0) \geq \mathcal{N}_i(\zeta_o)$  and  $\widehat{\mathcal{N}}_f(0) \leq \widehat{\mathcal{N}}_f(x)$  for all  $\zeta_o \in \mathcal{K}$

$$\begin{aligned} \mathcal{N}_t(\zeta_o) &\geq \min \{ \mathcal{N}_t(\zeta_o * \eta_o), \mathcal{N}_t(\eta_o) \}, \mathcal{N}_i(\zeta_o) \geq \min \{ \mathcal{N}_i(\zeta_o * \eta_o), \mathcal{N}_i(\eta_o) \} \\ \widehat{\mathcal{N}}_f(\zeta_o) &\leq rmax \{ \widehat{\mathcal{N}}_f(\zeta_o * \eta_o), \widehat{\mathcal{N}}_f(\eta_o) \} \text{ for all } \zeta_o, \eta_o \in \mathcal{K}. \end{aligned}$$

Thus  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a BS-NSI of  $\mathcal{K}$ . □

**Theorem 3.11.** Let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a BS-NSS of  $\mathcal{K}$ . Then  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is PI BS-NSI of  $\mathcal{K}$  if and only if it satisfies the following conditions

$$(((\zeta_o * \eta_o) * \eta_o) * a) * b = 0 \Rightarrow \left( \begin{array}{l} \mathcal{N}_t(\zeta_o * \eta_o) \geq \min \{ \mathcal{N}_t(a), \mathcal{N}_t(b) \} \\ \mathcal{N}_i(\zeta_o * \eta_o) \geq \min \{ \mathcal{N}_i(a), \mathcal{N}_i(b) \} \\ \widehat{\mathcal{N}}_f(\zeta_o * \eta_o) \leq rmax \{ \widehat{\mathcal{N}}_f(a), \widehat{\mathcal{N}}_f(b) \} \end{array} \right) \text{ (G) for all } \zeta_o, \eta_o, a, b \in \mathcal{K}.$$

*Proof.* Suppose that  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is PI BS-NSI of  $\mathcal{K}$ . Then  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a BS-NSI of  $\mathcal{K}$  by Theorem 3.3. Let  $\zeta_o, \eta_o, a, b \in \mathcal{K}$  be such that  $(((\zeta_o * \eta_o) * \eta_o) * a) * b = 0$  then using Theorem 3.6 and Lemma 3.9. we have

$$\begin{aligned} \mathcal{N}_t(\zeta_o * \eta_o) &\geq \mathcal{N}_t( ((\zeta_o * \eta_o) * \eta_o) * a ) \geq \min \{ \mathcal{N}_t(a), \mathcal{N}_t(b) \}, \\ \mathcal{N}_i(\zeta_o * \eta_o) &\geq \mathcal{N}_i( ((\zeta_o * \eta_o) * \eta_o) * a ) \geq \min \{ \mathcal{N}_i(a), \mathcal{N}_i(b) \}, \\ \widehat{\mathcal{N}}_f(\zeta_o * \eta_o) &\leq \widehat{\mathcal{N}}_f( ((\zeta_o * \eta_o) * \eta_o) * a ) \leq rmax \{ \widehat{\mathcal{N}}_f(a), \widehat{\mathcal{N}}_f(b) \} \end{aligned}$$

for all  $\zeta_o, \eta_o \in \mathcal{K}$ . Thus, condition (G) is valid.

Conversely, let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a BS-NSS in  $\mathcal{K}$  which satisfies the conditions (G). Let  $\zeta_o, a, b \in \mathcal{K}$  be such that  $(((\zeta_o * 0) * 0) * a) * b = 0$  and so

$$\begin{aligned} \mathcal{N}_t(\zeta_o) &= \mathcal{N}_t(\zeta_o * 0) \geq \min \{ \mathcal{N}_t(a), \mathcal{N}_t(b) \}, \\ \mathcal{N}_i(\zeta_o) &\geq \mathcal{N}_i(\zeta_o * 0) \geq \min \{ \mathcal{N}_i(a), \mathcal{N}_i(b) \}, \\ \widehat{\mathcal{N}}_f(\zeta_o) &\leq \widehat{\mathcal{N}}_f(\zeta_o * 0) \leq rmax \{ \widehat{\mathcal{N}}_f(a), \widehat{\mathcal{N}}_f(b) \}. \end{aligned}$$

Thus  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a BS-NSI of  $\mathcal{K}$  by Lemma 3.10.

Since  $(((\zeta_o * \eta_o) * \eta_o) * ((\zeta_o * \eta_o) * \eta_o)) * 0 = 0$  for all  $\zeta_o, \eta_o \in \mathcal{K}$  we have

$$\begin{aligned} \mathcal{N}_t(\zeta_o * \eta_o) &\geq \min \{ \mathcal{N}_t( ((\zeta_o * \eta_o) * \eta_o) * 0 ), \mathcal{N}_t(0) \} = \mathcal{N}_t( ((\zeta_o * \eta_o) * \eta_o) * 0 ), \\ \mathcal{N}_i(\zeta_o * \eta_o) &\geq \min \{ \mathcal{N}_i( ((\zeta_o * \eta_o) * \eta_o) * 0 ), \mathcal{N}_i(0) \} = \mathcal{N}_i( ((\zeta_o * \eta_o) * \eta_o) * 0 ), \\ \widehat{\mathcal{N}}_f(\zeta_o * \eta_o) &\leq rmax \{ \widehat{\mathcal{N}}_f( ((\zeta_o * \eta_o) * \eta_o) * 0 ), \widehat{\mathcal{N}}_f(0) \} = \widehat{\mathcal{N}}_f( ((\zeta_o * \eta_o) * \eta_o) * 0 ). \end{aligned}$$

It follows from Theorem 3.6 that  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is PI BS-NSI of  $\mathcal{K}$ . □

**Theorem 3.12.** Let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a BS-NSS of  $\mathcal{K}$ . Then  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is PI BS-NSI of  $\mathcal{K}$  if and only if it satisfies the following conditions

$$\left( \begin{array}{l} \mathcal{N}_t((\zeta_o * \theta_o) * (\eta_o * \theta_o)) \geq \min\{\mathcal{N}_t(a), \mathcal{N}_t(b)\} \\ \mathcal{N}_i((\zeta_o * \theta_o) * (\eta_o * \theta_o)) \geq \min\{\mathcal{N}_i(a), \mathcal{N}_i(b)\} \\ \widehat{\mathcal{N}}_f((\zeta_o * \theta_o) * (\eta_o * \theta_o)) \preceq rmax\{\widehat{\mathcal{N}}_f(a), \widehat{\mathcal{N}}_f(b)\} \end{array} \right) (H)$$

for all  $\zeta_o, \eta_o, \theta_o, a, b \in \mathcal{K}$  be such that  $((\zeta_o * \eta_o) * \theta_o) * a * b = 0$ .

*Proof.* Suppose that  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a PI BS-NSI of  $\mathcal{K}$ . Then  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a BS-NSI of  $\mathcal{K}$  by Theorem 3.3. let  $\zeta_o, \eta_o, \theta_o, a, b \in \mathcal{K}$  be such that  $((\zeta_o * \eta_o) * \theta_o) * a * b = 0$ . Using Theorem 3.7 and Lemma 3.9, we have

$$\begin{aligned} \mathcal{N}_t((\zeta_o * \theta_o) * (\eta_o * \theta_o)) &\geq \mathcal{N}_t((\zeta_o * \eta_o) * \theta_o) \geq \min\{\mathcal{N}_t(a), \mathcal{N}_t(b)\}, \\ \mathcal{N}_i((\zeta_o * \theta_o) * (\eta_o * \theta_o)) &\geq \mathcal{N}_i((\zeta_o * \eta_o) * \theta_o) \geq \min\{\mathcal{N}_i(a), \mathcal{N}_i(b)\}, \\ \widehat{\mathcal{N}}_f((\zeta_o * \theta_o) * (\eta_o * \theta_o)) &\preceq \widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \theta_o) \preceq rmax\{\widehat{\mathcal{N}}_f(a), \widehat{\mathcal{N}}_f(b)\}. \end{aligned}$$

Conversely, let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a BS-NSS in  $\mathcal{K}$  which satisfies the conditions (H) Let  $\zeta_o, \eta_o, a, b \in \mathcal{K}$  be such that  $((\zeta_o * \eta_o) * \theta_o) * a * b = 0$ . Then

$$\begin{aligned} \mathcal{N}_t(\zeta_o * \eta_o) &= \mathcal{N}_t((\zeta_o * \eta_o) * (\eta_o * \eta_o)) \geq \min\{\mathcal{N}_t(a), \mathcal{N}_t(b)\}, \\ \mathcal{N}_i(\zeta_o * \eta_o) &\geq \mathcal{N}_i((\zeta_o * \eta_o) * (\eta_o * \eta_o)) \geq \min\{\mathcal{N}_i(a), \mathcal{N}_i(b)\}, \\ \widehat{\mathcal{N}}_f(\zeta_o * \eta_o) &\preceq \widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * (\eta_o * \eta_o)) \preceq rmax\{\widehat{\mathcal{N}}_f(a), \widehat{\mathcal{N}}_f(b)\}. \end{aligned}$$

It follows from Theorem 3.11 that  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is PI BS-NSI of  $\mathcal{K}$ . □

**Theorem 3.13.** Let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a BS-NSS of  $\mathcal{K}$ . Then  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is PI BS-NSI of  $\mathcal{K}$  if and only if it satisfies the following conditions

$$\begin{aligned} \mathcal{N}_t(\zeta_o * \eta_o) &\geq \min\{\mathcal{N}_t(a_1), \mathcal{N}_t(a_2), \dots, \mathcal{N}_t(a_n)\}, \\ \mathcal{N}_i(\zeta_o * \eta_o) &\geq \min\{\mathcal{N}_i(a_1), \mathcal{N}_i(a_2), \dots, \mathcal{N}_i(a_n)\}, \\ \widehat{\mathcal{N}}_f(\zeta_o * \eta_o) &\preceq rmax\{\widehat{\mathcal{N}}_f(a_1), \widehat{\mathcal{N}}_f(a_2), \dots, \widehat{\mathcal{N}}_f(a_n)\} \end{aligned}$$

for all  $\zeta_o, \eta_o, a_1, a_2, \dots, a_n \in \mathcal{K}$ .with

$$(\dots\dots\dots((\zeta_o * \eta_o) * \eta_o) * a_1 * a_2 \dots\dots\dots) * a_n = 0$$

**Theorem 3.14.** Let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a BS-NSS of  $\mathcal{K}$ . Then  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is PI BS-NSI of  $\mathcal{K}$  if and only if it satisfies the following conditions

$$\begin{aligned} \mathcal{N}_t((\zeta_o * \theta_o) * (\eta_o * \theta_o)) &\geq \min\{\mathcal{N}_t(a_1), \mathcal{N}_t(a_2), \dots, \mathcal{N}_t(a_n)\}, \\ \mathcal{N}_i((\zeta_o * \theta_o) * (\eta_o * \theta_o)) &\geq \min\{\mathcal{N}_i(a_1), \mathcal{N}_i(a_2), \dots, \mathcal{N}_i(a_n)\}, \\ \widehat{\mathcal{N}}_f((\zeta_o * \theta_o) * (\eta_o * \theta_o)) &\preceq rmax\{\widehat{\mathcal{N}}_f(a_1), \widehat{\mathcal{N}}_f(a_2), \dots, \widehat{\mathcal{N}}_f(a_n)\} \end{aligned}$$

for all  $\zeta_o, \eta_o, a_1, a_2, \dots, a_n \in \mathcal{K}$ .with

$$(\dots\dots\dots((\zeta_o * \eta_o) * \theta_o) * a_1 * a_2 \dots\dots\dots) * a_n = 0$$

**Theorem 3.15.** A BS-NSS  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  in  $\mathcal{K}$  is a PI BS-NSI of  $\mathcal{K}$  if and only if the non-empty sets  $\mathcal{U}(\mathcal{N}_t; l), \mathcal{U}(\mathcal{N}_i; m)$  and  $\mathcal{L}(\widehat{\mathcal{N}}_f; [n_1, n_2])$  are ideals of  $\mathcal{K}$  for all  $l, m \in [0, 1]$  and  $[n_1, n_2] \in [I]$ .

*Proof.* Suppose that  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a PI BS-NSI of  $\mathcal{K}$ . Let  $l, m \in [0, 1]$  and  $[n_1, n_2] \in [I]$  be such that  $\mathcal{U}(\mathcal{N}_t; l), \mathcal{U}(\mathcal{N}_i; m)$  and  $\mathcal{L}(\widehat{\mathcal{N}}_f; [n_1, n_2])$  are non-empty. Obviously  $0 \in \mathcal{U}(\mathcal{N}_t; l), 0 \in \mathcal{U}(\mathcal{N}_i; m)$  and  $0 \in \mathcal{L}(\widehat{\mathcal{N}}_f; [n_1, n_2])$  For any  $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathcal{K}$  if  $(a_1 * a_2) * a_3, a_2 * a_3 \in \mathcal{U}(\mathcal{N}_t; l), (b_1 * b_2) * b_3, b_2 * b_3 \in \mathcal{U}(\mathcal{N}_i; m)$  and  $(c_1 * c_2) * c_3, c_2 * c_3 \in \mathcal{L}(\widehat{\mathcal{N}}_f; [n_1, n_2])$  then

$$\begin{aligned} \mathcal{N}_t(a_1 * a_3) &\geq \min\{\mathcal{N}_t((a_1 * a_2) * a_3), \mathcal{N}_t(a_2 * a_3)\} \geq \min\{l, l\} = l \\ \mathcal{N}_i(b_1 * b_3) &\geq \min\{\mathcal{N}_i((b_1 * b_2) * b_3), \mathcal{N}_i(b_2 * b_3)\} \geq \min\{m, m\} = m \\ \widehat{\mathcal{N}}_f(c_1 * c_3) &\preceq rmax\{\widehat{\mathcal{N}}_f((c_1 * c_2) * c_3), \widehat{\mathcal{N}}_f(c_2 * c_3)\} \\ &\preceq rmax\{[n_1, n_2], [n_1, n_2]\} = [n_1, n_2]. \end{aligned}$$

Therefore  $a_1 * a_3 \in \mathcal{U}(\mathcal{N}_t; l), b_1 * b_3 \in \mathcal{U}(\mathcal{N}_i; m)$  and  $c_1 * c_3 \in \mathcal{L}(\widehat{\mathcal{N}}_f; [n_1, n_2])$  Hence  $\mathcal{U}(\mathcal{N}_t; l), \mathcal{U}(\mathcal{N}_i; m)$  and  $\mathcal{L}(\widehat{\mathcal{N}}_f; [n_1, n_2])$  are Positive Implicative ideals of  $\mathcal{K}$ . Conversely, assume that the non-empty sets  $\mathcal{U}(\mathcal{N}_t; l), \mathcal{U}(\mathcal{N}_i; m)$  and  $\mathcal{L}(\widehat{\mathcal{N}}_f; [n_1, n_2])$  are Positive Implicative ideals of  $\mathcal{K}$  for all  $l, m \in [0, 1]$  and  $[n_1, n_2] \in [I]$ . Suppose that  $\mathcal{N}_t(0) < \mathcal{N}_t(a), \mathcal{N}_i(0) < \mathcal{N}_i(a)$  and  $\widehat{\mathcal{N}}_f(0) \succ \widehat{\mathcal{N}}_f(a)$  for some  $a \in \mathcal{K}$ . Then  $0 \notin \mathcal{U}(\mathcal{N}_t; \mathcal{N}_t(a)) \cap \mathcal{U}(\mathcal{N}_i; \mathcal{N}_i(a)) \cap \mathcal{L}(\widehat{\mathcal{N}}_f; \widehat{\mathcal{N}}_f(a))$  which is a

contradiction. Hence  $\mathcal{N}_t(0) \geq \mathcal{N}_t(\zeta_o)$ ,  $\mathcal{N}_i(0) \geq \mathcal{N}_i(\zeta_o)$  and  $\widehat{\mathcal{N}}_f(0) \leq \widehat{\mathcal{N}}_f(x)$  for all  $\zeta_o \in \mathcal{K}$ . Suppose that  $\widehat{\mathcal{N}}_f(a_0 * c_0) \succ rmax \{ \widehat{\mathcal{N}}_f((a_0 * b_0) * c_0), \widehat{\mathcal{N}}_f(b_0 * c_0) \}$  for some  $a_0, b_0, c_0 \in \mathcal{K}$ . Let  $\widehat{\mathcal{N}}_f((a_0 * b_0) * c_0) = [\delta_1, \delta_2]$ ,  $\widehat{\mathcal{N}}_f(b_0 * c_0) = [\delta_3, \delta_4]$  and  $\widehat{\mathcal{N}}_f(a_0 * c_0) = [n_1, n_2]$ .

Then  $[n_1, n_2] \succ max \{ [\delta_1, \delta_2], [\delta_3, \delta_4] \} = [max \{ \delta_1, \delta_3 \}, max \{ \delta_2, \delta_4 \}]$  and so  $n_1 > max \{ \delta_1, \delta_3 \}$  and  $n_2 > max \{ \delta_2, \delta_4 \}$ . Taking

$$[\eta_1, \eta_2] = \frac{1}{2} \left[ \widehat{\mathcal{N}}_f(a_0 * c_0) + max \{ \widehat{\mathcal{N}}_f((a_0 * b_0) * c_0), \widehat{\mathcal{N}}_f(b_0 * c_0) \} \right]$$

$$= \frac{1}{2} [ [n_1, n_2] + [max \{ \delta_1, \delta_3 \}, max \{ \delta_2, \delta_4 \}] ]$$

$$= \left[ \frac{1}{2} (n_1 + max \{ \delta_1, \delta_3 \}), \frac{1}{2} (n_2 + max \{ \delta_2, \delta_4 \}) \right]$$

It follows that  $n_1 > \eta_1 = \frac{1}{2} (n_1 + max \{ \delta_1, \delta_3 \}) > max \{ \delta_1, \delta_3 \}$  and

$$n_2 > \eta_2 = \frac{1}{2} (n_2 + max \{ \delta_2, \delta_4 \}) > max \{ \delta_2, \delta_4 \}$$

Hence  $[max \{ \delta_1, \delta_3 \}, max \{ \delta_2, \delta_4 \}] \prec [\eta_1, \eta_2] \prec [n_1, n_2] = \widehat{\mathcal{N}}_f(a_0 * c_0)$  Therefore  $a_0 * c_0 \notin \mathcal{U}(\widehat{\mathcal{N}}_f; [\eta_1, \eta_2])$ .

On the other hand

$$\widehat{\mathcal{N}}_f((a_0 * b_0) * c_0) = [\delta_1, \delta_2] \leq [max \{ \delta_1, \delta_3 \}, max \{ \delta_2, \delta_4 \}] \prec [\eta_1, \eta_2]$$

$\widehat{\mathcal{N}}_f(b_0 * c_0) = [\delta_3, \delta_4] \leq [max \{ \delta_1, \delta_3 \}, max \{ \delta_2, \delta_4 \}] \prec [\eta_1, \eta_2]$  that is  $((a_0 * b_0) * c_0), (b_0 * c_0) \in \mathcal{U}(\widehat{\mathcal{N}}_f; [\eta_1, \eta_2])$ . This is a contradiction and therefore

$\widehat{\mathcal{N}}_f(a_0 * c_0) \preceq rmax \{ \widehat{\mathcal{N}}_f((a_0 * b_0) * c_0), \widehat{\mathcal{N}}_f(b_0 * c_0) \}$  for all  $a_0, b_0, c_0 \in \mathcal{K}$ . Also, if  $\mathcal{N}_i(a_0 * c_0) < min \{ \mathcal{N}_i((a_0 * b_0) * c_0), \mathcal{N}_i(b_0 * c_0) \}$  for some  $a_0, b_0, c_0 \in \mathcal{K}$ , then  $(a_0 * b_0) * c_0, (b_0 * c_0) \in \mathcal{U}(\mathcal{N}_i; m_0)$  but  $a_0 * c_0 \notin \mathcal{U}(\mathcal{N}_i; m_0)$  for  $m_0 = min \{ \mathcal{N}_i((a_0 * b_0) * c_0), \mathcal{N}_i(b_0 * c_0) \}$ . This is a contradiction, and thus  $\mathcal{N}_i(a_0 * c_0) \geq min \{ \mathcal{N}_i((a_0 * b_0) * c_0), \mathcal{N}_i(b_0 * c_0) \}$  for all  $a_0, b_0, c_0 \in \mathcal{K}$ . Similarly, we can show that  $\mathcal{N}_t(a_0 * c_0) \geq min \{ \mathcal{N}_t((a_0 * b_0) * c_0), \mathcal{N}_t(b_0 * c_0) \}$  for all  $\zeta_o, \eta_o \in \mathcal{K}$ . Consequently  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a PI BS-NSI of  $\mathcal{K}$ .  $\square$

#### 4 Commutative BS-Neutrosophic Ideals

**Definition 4.1.** Let  $\mathcal{K}$  be a BCK-algebra. A BS-NSS  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  in  $\mathcal{K}$  is called a C BS-NSI (C BS-NSI) of  $\mathcal{K}$  if it satisfies (BS-NSI 1) and

(C BS-NSI 1)  $\mathcal{N}_t(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \geq min \{ \mathcal{N}_t((\zeta_o * \eta_o) * \theta_o), \mathcal{N}_t(\theta_o) \}$

(C BS-NSI 2)  $\mathcal{N}_i(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \geq min \{ \mathcal{N}_i((\zeta_o * \eta_o) * \theta_o), \mathcal{N}_i(\theta_o) \}$

(C BS-NSI 3)  $\widehat{\mathcal{N}}_f(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \preceq rmax \{ \widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \theta_o), \widehat{\mathcal{N}}_f(\theta_o) \}$

for all  $\zeta_o, \eta_o, \theta_o \in \mathcal{K}$ .

**Example 4.2.** Consider a set  $\mathcal{K} = \{0, a, b, c, d\}$  with the binary operation ‘\*’ which is given in the Table.5 Then  $(\mathcal{K}; *, 0)$  is a BCK-algebra. Let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a BS-NSS in  $\mathcal{K}$  defined in Table.6 It is routine

Table 5: BCK-algebra

| * | 0 | a | b | c | d |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| a | a | 0 | a | 0 | 0 |
| b | b | b | 0 | b | 0 |
| c | c | a | c | 0 | c |
| d | d | d | d | d | 0 |

Table 6: Commutative BS-Neutrosophic Ideal

| $\mathcal{K}$ | $\mathcal{N}_t(\zeta_o)$ | $\mathcal{N}_i(\zeta_o)$ | $\widehat{\mathcal{N}}_f(\zeta_o)$ |
|---------------|--------------------------|--------------------------|------------------------------------|
| 0             | 0.98                     | 1.0                      | [0.11, 0.37]                       |
| a             | 0.79                     | 0.83                     | [0.23, 0.55]                       |
| b             | 0.57                     | 0.65                     | [0.25, 0.63]                       |
| c             | 0.25                     | 0.47                     | [0.47, 0.71]                       |
| d             | 0.13                     | 0.29                     | [0.61, 0.97]                       |

to verify that  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a C BS-NSI of  $\mathcal{K}$

**Theorem 4.3.** Every C BS-NSI is a BS-NSI.

*Proof.* Let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a C BS-NSI of  $\mathcal{K}$ . If we take  $\eta_o = 0$  in (C BS-NSI 1), (C BS-NSI 2), (C BS-NSI 3) then  
 $\mathcal{N}_t(\zeta_o * (0 * (0 * \zeta_o))) \geq \min \{\mathcal{N}_t((\zeta_o * 0) * \theta_o), \mathcal{N}_t(\theta_o)\}$   
 $\Rightarrow \mathcal{N}_t(\zeta_o) \geq \min \{\mathcal{N}_t(\zeta_o * \theta_o), \mathcal{N}_t(\theta_o)\},$   
 $\mathcal{N}_i(\zeta_o * (0 * (0 * \zeta_o))) \geq \min \{\mathcal{N}_i((\zeta_o * 0) * \theta_o), \mathcal{N}_i(\theta_o)\}$   
 $\Rightarrow \mathcal{N}_i(\zeta_o) \geq \min \{\mathcal{N}_i(\zeta_o * \theta_o), \mathcal{N}_i(\theta_o)\},$   
 $\widehat{\mathcal{N}}_f(\zeta_o * (0 * (0 * \zeta_o))) \preceq rmax \{\widehat{\mathcal{N}}_f((\zeta_o * 0) * \theta_o), \widehat{\mathcal{N}}_f(\theta_o)\}$   
 $\Rightarrow \widehat{\mathcal{N}}_f(\zeta_o) \preceq rmax \{\widehat{\mathcal{N}}_f(\zeta_o * \theta_o), \widehat{\mathcal{N}}_f(\theta_o)\}$  for all  $\zeta_o, \eta_o, \theta_o \in \mathcal{K}$ . Therefore  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a BS-NSI of  $\mathcal{K}$ .  $\square$

In the following example we show that the converse of Theorem 4.3 is not true.

**Example 4.4.** Consider a BCK-algebra  $\mathcal{K} = \{0, a, b, c, d\}$  with the binary operation ‘\*’ which is given in the Table.7

Table 7: BCK-algebra

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| * | 0 | a | b | c | d |
| 0 | 0 | 0 | 0 | 0 | 0 |
| a | a | 0 | a | 0 | 0 |
| b | b | b | 0 | 0 | 0 |
| c | c | c | c | 0 | c |
| d | d | d | d | c | 0 |

Let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a BS-NSS in  $\mathcal{K}$  defined in Table.8 It is routine to verify that  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is

Table 8: BS-Neutrosophic Set

|               |                          |                          |                                    |
|---------------|--------------------------|--------------------------|------------------------------------|
| $\mathcal{K}$ | $\mathcal{N}_t(\zeta_o)$ | $\mathcal{N}_i(\zeta_o)$ | $\widehat{\mathcal{N}}_f(\zeta_o)$ |
| 0             | 0.91                     | 1.0                      | [0.21, 0.53]                       |
| a             | 0.63                     | 0.53                     | [0.33, 0.67]                       |
| b             | 0.37                     | 0.15                     | [0.77, 0.93]                       |
| c             | 0.37                     | 0.15                     | [0.77, 0.93]                       |
| d             | 0.37                     | 0.15                     | [0.77, 0.93]                       |

a BS-NSI of  $\mathcal{K}$ . Since  $\mathcal{N}_t(b * (d * (d * b))) = \mathcal{N}_t(b) = 0.37 < 0.91 = \mathcal{N}_t(0) = \min \{\mathcal{N}_t((b * d) * 0), \mathcal{N}_t(0)\}$   
 $\mathcal{N}_i(b * (d * (d * b))) = \mathcal{N}_i(b) = 0.15 < 1.0 = \mathcal{N}_i(0) = \min \{\mathcal{N}_i((b * d) * 0), \mathcal{N}_i(0)\},$   
 $\widehat{\mathcal{N}}_f(b * (d * (d * b))) = \widehat{\mathcal{N}}_f(b) = [0.77, 0.93] \succ [0.21, 0.53] = \widehat{\mathcal{N}}_f(0) = rmax \{\widehat{\mathcal{N}}_f((b * d) * 0), \widehat{\mathcal{N}}_f(0)\}$   $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is not a C BS-NSI of  $\mathcal{K}$ .

**Theorem 4.5.** Let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a BS-NSS of  $\mathcal{K}$ . Then  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is C BS-NSI of  $\mathcal{K}$  if and only if it is a BS-NSI satisfies the following conditions

$$\left( \begin{array}{l} \mathcal{N}_t(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \geq \mathcal{N}_t(\zeta_o * \eta_o) \\ \mathcal{N}_i(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \geq \mathcal{N}_i(\zeta_o * \eta_o) \\ \widehat{\mathcal{N}}_f(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \preceq \widehat{\mathcal{N}}_f(\zeta_o * \eta_o) \end{array} \right) (J) \text{ for all } \zeta_o, \eta_o \in \mathcal{K}$$

*Proof.* Suppose that  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a C BS-NSI of  $\mathcal{K}$ . If we put  $\theta_o = 0$  in (C BS-NSI 1), (C BS-NSI 2) and (C BS-NSI 3) then we have (J). Conversely, suppose that  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a BS-NSI of  $\mathcal{K}$  satisfying the condition (J) Then  
 $\mathcal{N}_t(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \geq \mathcal{N}_t(\zeta_o * \eta_o) \geq \min \{\mathcal{N}_t((\zeta_o * \eta_o) * \theta_o), \mathcal{N}_t(\theta_o)\},$

$\mathcal{N}_i(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \geq \mathcal{N}_i(\zeta_o * \eta_o) \geq \min\{\mathcal{N}_i((\zeta_o * \eta_o) * \theta_o), \mathcal{N}_i(\theta_o)\}$ ,  
 $\widehat{\mathcal{N}}_f(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \preceq \widehat{\mathcal{N}}_f(\zeta_o * \eta_o)$   
 $\preceq rmax\{\widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \theta_o), \widehat{\mathcal{N}}_f(\theta_o)\}$  for all  $\zeta_o, \eta_o, \theta_o \in \mathcal{K}$ . Therefore  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a C BS-NSI of  $\mathcal{K}$ . □

**Theorem 4.6.** In a commutative BCK-algebra, every BS-NSI is a C BS-NSI.

*Proof.* Let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a BS-NSI of a commutative BCK-algebra  $\mathcal{K}$ .

$((\zeta_o * (\eta_o * (\eta_o * \zeta_o))) * ((\zeta_o * \eta_o) * \theta_o)) * \theta_o$   
 $= ((\zeta_o * (\eta_o * (\eta_o * \zeta_o))) * \theta_o) * ((\zeta_o * \eta_o) * \theta_o)$   
 $\preceq (\zeta_o * (\eta_o * (\eta_o * \zeta_o))) * (\zeta_o * \eta_o) = (\zeta_o * (\zeta_o * \eta_o)) * (\eta_o * (\eta_o * \zeta_o)) = 0$ .  
 That is,  $((\zeta_o * (\eta_o * (\eta_o * \zeta_o))) * ((\zeta_o * \eta_o) * \theta_o)) \preceq \theta_o$  for all  $\zeta_o, \eta_o, \theta_o \in \mathcal{K}$ . By Lemma 3.9 we have  
 $\mathcal{N}_t(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \geq \min\{\mathcal{N}_t((\zeta_o * \eta_o) * \theta_o), \mathcal{N}_t(\theta_o)\}$ ,  
 $\mathcal{N}_i(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \geq \min\{\mathcal{N}_i((\zeta_o * \eta_o) * \theta_o), \mathcal{N}_i(\theta_o)\}$ ,  
 $\widehat{\mathcal{N}}_f(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \preceq rmax\{\widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \theta_o), \widehat{\mathcal{N}}_f(\theta_o)\}$ .

Therefore  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a C BS-NSI of  $\mathcal{K}$ . □

**Theorem 4.7.** A BS-NSS  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  in  $\mathcal{K}$  is a C BS-NSI of  $\mathcal{K}$  if and only if the non-empty sets  $\mathcal{U}(\mathcal{N}_t; l)$ ,  $\mathcal{U}(\mathcal{N}_i; m)$  and  $\mathcal{L}(\widehat{\mathcal{N}}_f; [n_1, n_2])$  are ideals of  $\mathcal{K}$  for all  $l, m \in [0, 1]$  and  $[n_1, n_2] \in [I]$ .

*Proof.* Suppose that  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a C BS-NSI of  $\mathcal{K}$ . Let  $l, m \in [0, 1]$  and  $[n_1, n_2] \in [I]$  be such that  $\mathcal{U}(\mathcal{N}_t; l)$ ,  $\mathcal{U}(\mathcal{N}_i; m)$  and  $\mathcal{L}(\widehat{\mathcal{N}}_f; [n_1, n_2])$  are non-empty. Obviously  $0 \in \mathcal{U}(\mathcal{N}_t; l)$ ,  $0 \in \mathcal{U}(\mathcal{N}_i; m)$  and  $0 \in \mathcal{L}(\widehat{\mathcal{N}}_f; [n_1, n_2])$ . For any  $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathcal{K}$  if  $(a_1 * a_2) * a_3, a_3 \in \mathcal{U}(\mathcal{N}_t; l)$ ,  $(b_1 * b_2) * b_3, b_3 \in \mathcal{U}(\mathcal{N}_i; m)$  and  $(c_1 * c_2) * c_3, c_3 \in \mathcal{L}(\widehat{\mathcal{N}}_f; [n_1, n_2])$  then  
 $\mathcal{N}_t(a_1 * (a_2 * (a_2 * a_1))) \geq \min\{\mathcal{N}_t((a_1 * a_2) * a_3), \mathcal{N}_t(a_3)\}$   
 $\geq \min\{l, l\} = l$ ,  
 $\mathcal{N}_i(b_1 * (b_2 * (b_2 * b_1))) \geq \min\{\mathcal{N}_i((b_1 * b_2) * b_3), \mathcal{N}_i(b_3)\}$   
 $\geq \min\{m, m\} = m$ ,  
 $\widehat{\mathcal{N}}_f(c_1 * (c_2 * (c_2 * c_1))) \preceq rmax\{\widehat{\mathcal{N}}_f((c_1 * c_2) * c_3), \widehat{\mathcal{N}}_f(c_3)\}$   
 $\preceq rmax\{[n_1, n_2], [n_1, n_2]\} = [n_1, n_2]$ .  
 Therefore  $(a_1 * (a_2 * (a_2 * a_1))) \in \mathcal{U}(\mathcal{N}_t; l)$ ,  $(b_1 * (b_2 * (b_2 * b_1))) \in \mathcal{U}(\mathcal{N}_i; m)$  and  $(c_1 * (c_2 * (c_2 * c_1))) \in \mathcal{L}(\widehat{\mathcal{N}}_f; [n_1, n_2])$  Hence  $\mathcal{U}(\mathcal{N}_t; l), \mathcal{U}(\mathcal{N}_i; m)$  and  $\mathcal{L}(\widehat{\mathcal{N}}_f; [n_1, n_2])$  are Commutative ideals of  $\mathcal{K}$ .

Conversely, assume that the non-empty sets  $\mathcal{U}(\mathcal{N}_t; l)$ ,  $\mathcal{U}(\mathcal{N}_i; m)$  and  $\mathcal{L}(\widehat{\mathcal{N}}_f; [n_1, n_2])$  are Commutative ideals of  $\mathcal{K}$  for all  $l, m \in [0, 1]$  and  $[n_1, n_2] \in [I]$ . Suppose that  $\mathcal{N}_t(0) < \mathcal{N}_t(a), \mathcal{N}_i(0) < \mathcal{N}_i(a)$  and  $\widehat{\mathcal{N}}_f(0) \succ \widehat{\mathcal{N}}_f(a)$  for some  $a \in \mathcal{K}$ . Then  $0 \notin \mathcal{U}(\mathcal{N}_t; \mathcal{N}_t(a)) \cap \mathcal{U}(\mathcal{N}_i; \mathcal{N}_i(a)) \cap \mathcal{L}(\widehat{\mathcal{N}}_f; \widehat{\mathcal{N}}_f(a))$  which is a contradiction.

Hence  $\mathcal{N}_t(0) \geq \mathcal{N}_t(\zeta_o), \mathcal{N}_i(0) \geq \mathcal{N}_i(\zeta_o)$  and  $\widehat{\mathcal{N}}_f(0) \preceq \widehat{\mathcal{N}}_f(x)$  for all  $\zeta_o \in \mathcal{K}$ .

Suppose that  $\widehat{\mathcal{N}}_f(a_0 * (b_0 * (b_0 * a_0))) \succ rmax\{\widehat{\mathcal{N}}_f((a_0 * b_0) * c_0), \widehat{\mathcal{N}}_f(c_0)\}$  for some  $a_0, b_0, c_0 \in \mathcal{K}$ . Let

$\widehat{\mathcal{N}}_f((a_0 * b_0) * c_0) = [\delta_1, \delta_2], \widehat{\mathcal{N}}_f(c_0) = [\delta_3, \delta_4]$  and  $\widehat{\mathcal{N}}_f(a_0 * (b_0 * (b_0 * a_0))) = [n_1, n_2]$ . Then  
 $[n_1, n_2] \succ rmax\{[\delta_1, \delta_2], [\delta_3, \delta_4]\} = [max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}]$  and so  $n_1 > max\{\delta_1, \delta_3\}$  and  $n_2 > max\{\delta_2, \delta_4\}$ . Taking

$$[\eta_1, \eta_2] = \frac{1}{2} [\widehat{\mathcal{N}}_f(a_0 * (b_0 * (b_0 * a_0))) + rmax\{\widehat{\mathcal{N}}_f((a_0 * b_0) * c_0), \widehat{\mathcal{N}}_f(c_0)\}]$$

$$= \frac{1}{2} [[n_1, n_2] + [max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}]]$$

$$= [\frac{1}{2}(n_1 + max\{\delta_1, \delta_3\}), \frac{1}{2}(n_2 + max\{\delta_2, \delta_4\})]$$

It follows that  $n_1 > \eta_1 = \frac{1}{2}(n_1 + max\{\delta_1, \delta_3\}) > max\{\delta_1, \delta_3\}$  and

$$n_2 > \eta_2 = \frac{1}{2}(n_2 + max\{\delta_2, \delta_4\}) > max\{\delta_2, \delta_4\}$$

Hence  $[max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}] \prec [\eta_1, \eta_2] \prec [n_1, n_2]$

$$= \widehat{\mathcal{N}}_f(a_0 * (b_0 * (b_0 * a_0))).$$

Therefore  $(a_0 * (b_0 * (b_0 * a_0))) \notin \mathcal{L}(\widehat{\mathcal{N}}_f; [\eta_1, \eta_2])$ . On the other hand

$$\widehat{\mathcal{N}}_f((a_0 * b_0) * c_0) = [\delta_1, \delta_2] \leq [\max\{\delta_1, \delta_3\}, \max\{\delta_2, \delta_4\}] \prec [\eta_1, \eta_2]$$

$$\widehat{\mathcal{N}}_f(c_0) = [\delta_3, \delta_4] \leq [\max\{\delta_1, \delta_3\}, \max\{\delta_2, \delta_4\}] \prec [\eta_1, \eta_2] \text{ that is}$$

$((a_0 * b_0) * c_0), (c_0) \in \mathcal{L}(\widehat{\mathcal{N}}_f; [\eta_1, \eta_2])$ . This is a contradiction and therefore

$$\widehat{\mathcal{N}}_f(a_0 * (b_0 * (b_0 * a_0))) \preceq rmax\{\widehat{\mathcal{N}}_f((a_0 * b_0) * c_0), \widehat{\mathcal{N}}_f(c_0)\} \text{ for all } a_0, b_0, c_0 \in \mathcal{K}.$$

Also, if  $\mathcal{N}_t(a_0 * (b_0 * (b_0 * a_0))) < \min\{\mathcal{N}_t((a_0 * b_0) * c_0), \mathcal{N}_t(c_0)\}$

for some  $a_0, b_0, c_0 \in \mathcal{K}$ , then  $(a_0 * b_0) * c_0, c_0 \in \mathcal{U}(\mathcal{N}_t; l_0)$  but

$(a_0 * (b_0 * (b_0 * a_0))) \notin \mathcal{U}(\mathcal{N}_t; l_0)$  for

$$l_0 = \min\{\mathcal{N}_t((a_0 * b_0) * c_0), \mathcal{N}_t(c_0)\}.$$

This is a contradiction, and thus

$$\mathcal{N}_t(a_0 * (b_0 * (b_0 * a_0))) \geq \min\{\mathcal{N}_t((a_0 * b_0) * c_0), \mathcal{N}_t(c_0)\}$$

for all  $a_0, b_0, c_0 \in \mathcal{K}$ . Similarly, we can show that

$$\mathcal{N}_i(a_0 * (b_0 * (b_0 * a_0))) \geq \min\{\mathcal{N}_i((a_0 * b_0) * c_0), \mathcal{N}_i(c_0)\}$$

for all  $\zeta_o, \eta_o \in \mathcal{K}$ . Consequently  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a C BS-NSI of  $\mathcal{K}$ . □

**Theorem 4.8.** Given a Commutative ideal of a BCK/BCI-algebra  $\mathcal{K}$ . Let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is BS-NSI of  $\mathcal{K}$  defined by

$$\mathcal{N}_t(\zeta_o) = \begin{cases} l & \text{if } \zeta_o \in \mathcal{J}, \\ 0 & \text{otherwise.} \end{cases} \quad \mathcal{N}_i(\zeta_o) = \begin{cases} m & \text{if } \zeta_o \in \mathcal{J}, \\ 0 & \text{otherwise.} \end{cases} \quad \text{and}$$

$$\widehat{\mathcal{N}}_f(\zeta_o) = \begin{cases} [\eta_1, \eta_2] & \text{if } \zeta_o \in \mathcal{J}, \\ [1, 1] & \text{otherwise.} \end{cases} \quad \text{Where } l, m \in (0, 1] \text{ and } [\eta_1, \eta_2] \text{ with } \eta_1 < \eta_2 \text{ and } \eta_1, \eta_2 \in [0, 1]. \text{ Then}$$

$\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a C BS-NSI of  $\mathcal{K}$  such that  $\mathcal{U}(\mathcal{N}_t; l) = \mathcal{J}, \mathcal{U}(\mathcal{N}_i; m) = \mathcal{J}$  and  $\mathcal{L}(\widehat{\mathcal{N}}_f; [\eta_1, \eta_2]) = \mathcal{J}$ .

*Proof.* Let  $\zeta_o, \eta_o, \theta_o \in \mathcal{K}$ . If  $(\zeta_o * \eta_o) * \theta_o \in \mathcal{J}$  and  $\theta_o \in \mathcal{J}$  then  $(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \in \mathcal{J}$  and so

$$\mathcal{N}_t(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) = l = \min\{\mathcal{N}_t((\zeta_o * \eta_o) * \theta_o), \mathcal{N}_t(\theta_o)\},$$

$$\mathcal{N}_i(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) = m = \min\{\mathcal{N}_i((\zeta_o * \eta_o) * \theta_o), \mathcal{N}_i(\theta_o)\},$$

$$\widehat{\mathcal{N}}_f(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) = [n_1, n_2] = rmax\{\widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \theta_o), \widehat{\mathcal{N}}_f(\theta_o)\}$$

If any one of  $(\zeta_o * \eta_o) * \theta_o$  and  $\theta_o$  is contained in  $\mathcal{J}$ , say  $(\zeta_o * \eta_o) * \theta_o \in \mathcal{J}$ ,

then  $\mathcal{N}_t((\zeta_o * \eta_o) * \theta_o) = l, \mathcal{N}_i((\zeta_o * \eta_o) * \theta_o) = m, \widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \theta_o) = [n_1, n_2], \mathcal{N}_t(\theta_o) = 0,$

$\mathcal{N}_i(\theta_o) = 0$ , and  $\widehat{\mathcal{N}}_f(\theta_o) = [1, 1]$ . Hence

$$\mathcal{N}_t(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \geq 0 = \min\{l, 0\}$$

$$= \min\{\mathcal{N}_t((\zeta_o * \eta_o) * \theta_o), \mathcal{N}_t(\theta_o)\},$$

$$\mathcal{N}_i(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \geq 0 = \min\{m, 0\}$$

$$= \min\{\mathcal{N}_i((\zeta_o * \eta_o) * \theta_o), \mathcal{N}_i(\theta_o)\},$$

$$\widehat{\mathcal{N}}_f(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \preceq [1, 1] = rmax\{[n_1, n_2], [1, 1]\}$$

$$= rmax\{\widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \theta_o), \widehat{\mathcal{N}}_f(\theta_o)\}.$$

If  $(\zeta_o * \eta_o) * \theta_o \notin \mathcal{J}$  and  $\theta_o \notin \mathcal{J}$ , then

$$\mathcal{N}_t((\zeta_o * \eta_o) * \theta_o) = 0, \mathcal{N}_i((\zeta_o * \eta_o) * \theta_o) = 0, \widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \theta_o) = [1, 1],$$

$$\mathcal{N}_t(\theta_o) = 0, \mathcal{N}_i(\theta_o) = 0, \text{ and } \widehat{\mathcal{N}}_f(\theta_o) = [1, 1] \text{ it follows that}$$

$$\mathcal{N}_t(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \geq 0 = \min\{0, 0\}$$

$$= \min\{\mathcal{N}_t((\zeta_o * \eta_o) * \theta_o), \mathcal{N}_t(\theta_o)\}$$

$$\mathcal{N}_i(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \geq 0 = \min\{0, 0\}$$

$$= \min\{\mathcal{N}_i((\zeta_o * \eta_o) * \theta_o), \mathcal{N}_i(\theta_o)\}$$

$$\widehat{\mathcal{N}}_f(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \preceq [1, 1] = rmax\{[1, 1], [1, 1]\}$$

$$= rmax\{\widehat{\mathcal{N}}_f((\zeta_o * \eta_o) * \theta_o), \widehat{\mathcal{N}}_f(\theta_o)\}.$$

It is obvious that  $\mathcal{N}_t(0) \geq \mathcal{N}_t(\zeta_o), \mathcal{N}_i(0) \geq \mathcal{N}_i(\zeta_o)$  and  $\widehat{\mathcal{N}}_f(0) \preceq \widehat{\mathcal{N}}_f(\zeta_o)$  for all  $\zeta_o \in \mathcal{K}$ . Therefore  $\mathcal{N} =$

$(\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a C BS-NSI of  $\mathcal{K}$ . Obviously, we have  $\mathcal{U}(\mathcal{N}_t; l) = \mathcal{J}, \mathcal{U}(\mathcal{N}_i; m) = \mathcal{J}$  and  $\mathcal{L}(\widehat{\mathcal{N}}_f; [\eta_1, \eta_2]) = \mathcal{J}$ . □

A mapping from  $\lambda : \mathcal{K} \rightarrow \mathcal{Y}$  of a BCK/BCI-algebra is called a homomorphism if  $\lambda(\zeta_o * \eta_o) = \lambda(\zeta_o) * \lambda(\eta_o)$  for all  $\zeta_o, \eta_o \in \mathcal{K}$ . Note that if  $\lambda : \mathcal{K} \rightarrow \mathcal{Y}$  is a homomorphism, then  $\lambda(0) = 0$ . Let  $\lambda : \mathcal{K} \rightarrow \mathcal{Y}$  be a homomorphism of BCK/BCI-algebra. For any BS-NSS  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$ , we define a new BS-NSS

$\mathcal{N}^\lambda = (\mathcal{N}_t^\lambda, \mathcal{N}_i^\lambda, \widehat{\mathcal{N}}_f^\lambda)$  in  $\mathcal{K}$ , which is called the induced BS-NSS defined by  $\mathcal{N}_t^\lambda(\zeta_o) = \mathcal{N}_t(\lambda(\zeta_o))$ ,  $\mathcal{N}_i^\lambda(\zeta_o) = \mathcal{N}_i(\lambda(\zeta_o))$  and  $\widehat{\mathcal{N}}_f^\lambda(\zeta_o) = \widehat{\mathcal{N}}_f(\lambda(\zeta_o))$  for all  $\zeta_o \in \mathcal{K}$ .

**Lemma 4.9.** *Let  $\lambda : \mathcal{K} \rightarrow \mathcal{Y}$  be a homomorphism of BCK/BCI-algebras. If a BS-NSS  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  in  $\mathcal{Y}$  is a BS-NSI of  $\mathcal{Y}$ , then the induced BS-NSS  $\mathcal{N}^\lambda = (\mathcal{N}_t^\lambda, \mathcal{N}_i^\lambda, \widehat{\mathcal{N}}_f^\lambda)$  in  $\mathcal{K}$  is a BS-NSI of  $\mathcal{K}$ .*

*Proof.* For any  $\zeta_o \in \mathcal{K}$ , we have  $\mathcal{N}_t^\lambda(0) = \mathcal{N}_t(\lambda(0)) = \mathcal{N}_t(0) \geq \mathcal{N}_t(\lambda(\zeta_o)) = \mathcal{N}_t^\lambda(\zeta_o)$ ,  $\mathcal{N}_i^\lambda(0) = \mathcal{N}_i(\lambda(0)) = \mathcal{N}_i(0) \geq \mathcal{N}_i(\lambda(\zeta_o)) = \mathcal{N}_i^\lambda(\zeta_o)$ ,  $\widehat{\mathcal{N}}_f^\lambda(0) = \widehat{\mathcal{N}}_f(\lambda(0)) = \widehat{\mathcal{N}}_f(0) \leq \widehat{\mathcal{N}}_f(\lambda(\zeta_o)) = \widehat{\mathcal{N}}_f^\lambda(\zeta_o)$ .

Let  $\zeta_o, \eta_o \in \mathcal{K}$  then

$$\begin{aligned} \mathcal{N}_t^\lambda(\zeta_o) &= \mathcal{N}_t(\lambda(\zeta_o)) \geq \min\{\mathcal{N}_t(\lambda(\zeta_o) * \lambda(\eta_o)), \mathcal{N}_t(\lambda(\eta_o))\} \\ &= \min\{\mathcal{N}_t(\lambda(\zeta_o * \eta_o)), \mathcal{N}_t(\lambda(\eta_o))\} = \min\{\mathcal{N}_t^\lambda(\zeta_o * \eta_o), \mathcal{N}_t^\lambda(\eta_o)\}, \\ \mathcal{N}_i^\lambda(\zeta_o) &= \mathcal{N}_i(\lambda(\zeta_o)) \geq \min\{\mathcal{N}_i(\lambda(\zeta_o) * \lambda(\eta_o)), \mathcal{N}_i(\lambda(\eta_o))\} \\ &= \min\{\mathcal{N}_i(\lambda(\zeta_o * \eta_o)), \mathcal{N}_i(\lambda(\eta_o))\} = \min\{\mathcal{N}_i^\lambda(\zeta_o * \eta_o), \mathcal{N}_i^\lambda(\eta_o)\}, \\ \widehat{\mathcal{N}}_f^\lambda(\zeta_o) &= \widehat{\mathcal{N}}_f(\lambda(\zeta_o)) \leq \max\{\widehat{\mathcal{N}}_f(\lambda(\zeta_o) * \lambda(\eta_o)), \widehat{\mathcal{N}}_f(\lambda(\eta_o))\} \\ &= \max\{\widehat{\mathcal{N}}_f(\lambda(\zeta_o * \eta_o)), \widehat{\mathcal{N}}_f(\lambda(\eta_o))\} = \max\{\widehat{\mathcal{N}}_f^\lambda(\zeta_o * \eta_o), \widehat{\mathcal{N}}_f^\lambda(\eta_o)\}. \end{aligned}$$

Therefore  $\mathcal{N}^\lambda = (\mathcal{N}_t^\lambda, \mathcal{N}_i^\lambda, \widehat{\mathcal{N}}_f^\lambda)$  is a BS-NSI of  $\mathcal{K}$ . □

**Lemma 4.10.** *Let  $\lambda : \mathcal{K} \rightarrow \mathcal{Y}$  be a homomorphism of BCK/BCI-algebras. If a BS-NSS  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  in  $\mathcal{Y}$  is a C BS-NSI of  $\mathcal{Y}$ , then the induced BS-NSS  $\mathcal{N}^\lambda = (\mathcal{N}_t^\lambda, \mathcal{N}_i^\lambda, \widehat{\mathcal{N}}_f^\lambda)$  in  $\mathcal{K}$  is a C BS-NSI of  $\mathcal{K}$ .*

*Proof.* Suppose that  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a C BS-NSI of  $\mathcal{Y}$ . Then  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a BS-NSI of  $\mathcal{Y}$  by Theorem 4.3 and so  $\mathcal{N}^\lambda = (\mathcal{N}_t^\lambda, \mathcal{N}_i^\lambda, \widehat{\mathcal{N}}_f^\lambda)$  is a BS-NSI of  $\mathcal{Y}$  by Lemma 4.9. For any  $\zeta_o, \eta_o \in \mathcal{K}$  we have

$$\begin{aligned} \mathcal{N}_t^\lambda(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) &= \mathcal{N}_t(\lambda(\zeta_o * (\eta_o * (\eta_o * \zeta_o)))) \\ &= \mathcal{N}_t((\lambda(\zeta_o) * (\lambda(\eta_o) * (\lambda(\eta_o) * \lambda(\zeta_o)))))) \\ &\geq \mathcal{N}_t(\lambda(\zeta_o) * \lambda(\eta_o)) = \mathcal{N}_t(\lambda(\zeta_o * \eta_o)) = \mathcal{N}_t^\lambda(\zeta_o * \eta_o), \\ \mathcal{N}_i^\lambda(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) &= \mathcal{N}_i(\lambda(\zeta_o * (\eta_o * (\eta_o * \zeta_o)))) \\ &= \mathcal{N}_i((\lambda(\zeta_o) * (\lambda(\eta_o) * (\lambda(\eta_o) * \lambda(\zeta_o)))))) \\ &\geq \mathcal{N}_i(\lambda(\zeta_o) * \lambda(\eta_o)) = \mathcal{N}_i(\lambda(\zeta_o * \eta_o)) = \mathcal{N}_i^\lambda(\zeta_o * \eta_o), \\ \widehat{\mathcal{N}}_f^\lambda(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) &= \widehat{\mathcal{N}}_f(\lambda(\zeta_o * (\eta_o * (\eta_o * \zeta_o)))) \\ &= \widehat{\mathcal{N}}_f((\lambda(\zeta_o) * (\lambda(\eta_o) * (\lambda(\eta_o) * \lambda(\zeta_o)))))) \\ &\leq \widehat{\mathcal{N}}_f(\lambda(\zeta_o) * \lambda(\eta_o)) = \widehat{\mathcal{N}}_f(\lambda(\zeta_o * \eta_o)) = \widehat{\mathcal{N}}_f^\lambda(\zeta_o * \eta_o). \end{aligned}$$

Therefore  $\mathcal{N}^\lambda = (\mathcal{N}_t^\lambda, \mathcal{N}_i^\lambda, \widehat{\mathcal{N}}_f^\lambda)$  is a C BS-NSI of  $\mathcal{K}$  by Theorem 4.5 □

**Lemma 4.11.** *Let  $\lambda : \mathcal{K} \rightarrow \mathcal{Y}$  be an onto homomorphism of BCK/BCI-algebras and let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a BS-NSS in  $\mathcal{Y}$ . If the induced BS-NSS  $\mathcal{N}^\lambda = (\mathcal{N}_t^\lambda, \mathcal{N}_i^\lambda, \widehat{\mathcal{N}}_f^\lambda)$  in  $\mathcal{K}$  is a BS-NSI of  $\mathcal{K}$  then  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a BS-NSI of  $\mathcal{Y}$ .*

*Proof.* Suppose that the induced BS-NSS  $\mathcal{N}^\lambda = (\mathcal{N}_t^\lambda, \mathcal{N}_i^\lambda, \widehat{\mathcal{N}}_f^\lambda)$  in  $\mathcal{K}$  is a BS-NSI of  $\mathcal{K}$ . For any  $a \in \mathcal{Y}$ , there exists  $\zeta_o \in \mathcal{K}$  such that  $\lambda(\zeta_o) = a$ . Thus

$$\begin{aligned} \mathcal{N}_t(0) &= \mathcal{N}_t(\lambda(0)) = \mathcal{N}_t^\lambda(0) \geq \mathcal{N}_t^\lambda(\zeta_o) = \mathcal{N}_t(\lambda(\zeta_o)) = \mathcal{N}_t(a), \\ \mathcal{N}_i(0) &= \mathcal{N}_i(\lambda(0)) = \mathcal{N}_i^\lambda(0) \geq \mathcal{N}_i^\lambda(\zeta_o) = \mathcal{N}_i(\lambda(\zeta_o)) = \mathcal{N}_i(a), \\ \widehat{\mathcal{N}}_f(0) &= \widehat{\mathcal{N}}_f(\lambda(0)) = \widehat{\mathcal{N}}_f^\lambda(0) \leq \widehat{\mathcal{N}}_f^\lambda(\zeta_o) = \widehat{\mathcal{N}}_f(\lambda(\zeta_o)) = \widehat{\mathcal{N}}_f(a). \end{aligned}$$

Let  $a, b \in \mathcal{Y}$ . Then  $\lambda(\zeta_o) = a, \lambda(\eta_o) = b$  for some  $\zeta_o, \eta_o \in \mathcal{K}$ . Hence

$$\begin{aligned} \mathcal{N}_t(a) &= \mathcal{N}_t(\lambda(\zeta_o)) = \mathcal{N}_t^\lambda(\zeta_o) \geq \min\{\mathcal{N}_t^\lambda(\zeta_o * \eta_o), \mathcal{N}_t^\lambda(\eta_o)\} \\ &= \min\{\mathcal{N}_t(\lambda(\zeta_o * \eta_o)), \mathcal{N}_t(\lambda(\eta_o))\} \\ &= \min\{\mathcal{N}_t(\lambda(\zeta_o) * \lambda(\eta_o)), \mathcal{N}_t(\lambda(\eta_o))\} = \min\{\mathcal{N}_t(a * b), \mathcal{N}_t(b)\}, \end{aligned}$$

$$\begin{aligned}
 \mathcal{N}_i(a) &= \mathcal{N}_i(\lambda(\zeta_o)) = \mathcal{N}_i^\lambda(\zeta_o) \geq \min \{ \mathcal{N}_i^\lambda(\zeta_o * \eta_o), \mathcal{N}_i^\lambda(\eta_o) \} \\
 &= \min \{ \mathcal{N}_i(\lambda(x * \eta_o)), \mathcal{N}_i(\lambda(\eta_o)) \} \\
 &= \min \{ \mathcal{N}_i(\lambda(\zeta_o) * \lambda(\eta_o)), \mathcal{N}_i(\lambda(\eta_o)) \} = \min \{ \mathcal{N}_i(a * b), \mathcal{N}_i(b) \}, \\
 \widehat{\mathcal{N}}_f(a) &= \widehat{\mathcal{N}}_f(\lambda(\zeta_o)) = \widehat{\mathcal{N}}_f^\lambda(\zeta_o) \leq \text{rmax} \{ \widehat{\mathcal{N}}_f^\lambda(\zeta_o * \eta_o), \widehat{\mathcal{N}}_f^\lambda(\eta_o) \} \\
 &= \text{rmax} \{ \widehat{\mathcal{N}}_f(\lambda(\zeta_o * \eta_o)), \widehat{\mathcal{N}}_f(\lambda(\eta_o)) \} \\
 &= \text{rmax} \{ \widehat{\mathcal{N}}_f(\lambda(\zeta_o) * \lambda(\eta_o)), \widehat{\mathcal{N}}_f(\lambda(\eta_o)) \} = \text{rmax} \{ \widehat{\mathcal{N}}_f(a * b), \widehat{\mathcal{N}}_f(b) \}
 \end{aligned}$$

Therefore  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a BS-NSI of  $\mathcal{Y}$ . □

**Theorem 4.12.** Let  $\lambda : \mathcal{K} \rightarrow \mathcal{Y}$  be an onto homomorphism of BCK/BCI-algebras and let  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  be a BS-NSS in  $\mathcal{Y}$ . If the induced BS-NSS  $\mathcal{N}^\lambda = (\mathcal{N}_t^\lambda, \mathcal{N}_i^\lambda, \widehat{\mathcal{N}}_f^\lambda)$  in  $\mathcal{K}$  is a C BS-NSI of  $\mathcal{K}$  then  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a C BS-NSI of  $\mathcal{Y}$ .

*Proof.* Suppose that the induced BS-NSS  $\mathcal{N}^\lambda = (\mathcal{N}_t^\lambda, \mathcal{N}_i^\lambda, \widehat{\mathcal{N}}_f^\lambda)$  in  $\mathcal{K}$  is a C BS-NSI of  $\mathcal{K}$ . Then  $\mathcal{N}^\lambda = (\mathcal{N}_t^\lambda, \mathcal{N}_i^\lambda, \widehat{\mathcal{N}}_f^\lambda)$  is a BS-NSI of  $\mathcal{Y}$  by Theorem 4.3 and so  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a BS-NSI of  $\mathcal{Y}$  by Lemma 4.11. For any  $a, b, c \in \mathcal{Y}$ , there exists  $\zeta_o, \eta_o, \theta_o \in \mathcal{K}$  such that  $\lambda(\zeta_o) = a, \lambda(\eta_o) = b$  and  $\lambda(\theta_o) = c$ . It follows that

$$\begin{aligned}
 \mathcal{N}_t(a * (b * (b * a))) &= \mathcal{N}_t((\lambda(\zeta_o) * (\lambda(\eta_o) * (\lambda(\eta_o) * \lambda(\zeta_o)))))) \\
 &= \mathcal{N}_t(\lambda(\zeta_o * (\eta_o * (\eta_o * \zeta_o)))) = \mathcal{N}_t^\lambda(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \\
 &\geq \mathcal{N}_t^\lambda(\zeta_o * \eta_o) = \mathcal{N}_t(\lambda(\zeta_o * \eta_o)) = \mathcal{N}_t(\lambda(\zeta_o) * \lambda(\eta_o)) = \mathcal{N}_t(a * b), \\
 \mathcal{N}_i(a * (b * (b * a))) &= \mathcal{N}_i((\lambda(\zeta_o) * (\lambda(\eta_o) * (\lambda(\eta_o) * \lambda(\zeta_o)))))) \\
 &= \mathcal{N}_i(\lambda(\zeta_o * (\eta_o * (\eta_o * \zeta_o)))) = \mathcal{N}_i^\lambda(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \\
 &\geq \mathcal{N}_i^\lambda(\zeta_o * \eta_o) = \mathcal{N}_i(\lambda(\zeta_o * \eta_o)) = \mathcal{N}_i(\lambda(\zeta_o) * \lambda(\eta_o)) = \mathcal{N}_i(a * b), \\
 \widehat{\mathcal{N}}_f(a * (b * (b * a))) &= \widehat{\mathcal{N}}_f((\lambda(\zeta_o) * (\lambda(\eta_o) * (\lambda(\eta_o) * \lambda(\zeta_o)))))) \\
 &= \widehat{\mathcal{N}}_f(\lambda(\zeta_o * (\eta_o * (\eta_o * \zeta_o)))) = \widehat{\mathcal{N}}_f^\lambda(\zeta_o * (\eta_o * (\eta_o * \zeta_o))) \\
 &\leq \widehat{\mathcal{N}}_f^\lambda(\zeta_o * \eta_o) = \widehat{\mathcal{N}}_f(\lambda(\zeta_o * \eta_o)) = \widehat{\mathcal{N}}_f(\lambda(\zeta_o) * \lambda(\eta_o)) = \widehat{\mathcal{N}}_f(a * b).
 \end{aligned}$$

Therefore  $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$  is a BS-NSI of  $\mathcal{Y}$  by Theorem 4.5 □

## 5 Conclusion

The application of the BS-neutrosophic set to BCK-algebra and the introduction of positive implicative BS-neutrosophic ideals and commutative BS-neutrosophic ideals represent substantial contributions to both neutrosophic set theory and BCK-algebra. Incorporating uncertainty management through interval-evaluated membership functions opens up new possibilities for dealing with complex and ambiguous situations, ultimately advancing our knowledge and utilization of neutrosophic sets in various mathematical contexts. As researchers explore this exciting area of research, we can look forward to further development and practical use of BS-neutrosophic sets in various fields.

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