On The Symbolic 6-Plithogenic and 7-Plithogenic Rings

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Abstract

Symbolic n-plithogenic algebraic structures are considered as a direct application of fuzzy generalized systems in pure algebra, where the symbolic n-plithogenic set is used to generalize algebraic structures by adding logical generators. In this paper, we study the concept of symbolic 6-plithogenic rings and 7-plithogenic rings from an algebraic point of view, where the main substructures formed by them will be presented such as AH-ideals, AHS-isomorphisms, and AH-kernels. Also, many theorems that explain their algebraic behaviors and classifications will be proved and illustrated.

Keywords: symbolic 6-plithogenic ring; symbolic 7-plithogenic ring; AH-ideal; AH-homomorphism

1. Introduction

In the theory of algebraic rings, many generalizations related to algebraic rings come from time to time, where concepts such as AH-ideals, AH-homomorphisms, and kernels were defined and studied by many authors, see [1-5].

Symbolic n-plithogenic sets were proposed by Smarandache in [6], and then they were used widely in pure algebra. We can find a lot of generalizations of classical algebraic structures based on symbolic n-plithogenic sets such as rings, spaces, and equations [7-15].

The theory of symbolic n-plithogenic rings began in [16], and then they were generalized into 3-plithogenic, 4-plithogenic and 5-plithogenic rings [17-18].

These studies have motivated us to study this type of rings for n=6, and n=7, where we study their algebraic substructures and functions that can represent them as direct products of classical rings.

Main Results

Definition:
Let R be a ring, we define the symbolic 6-plithogenic ring $6-SP_R$ as follows:

$$6-SP_R=\left\{l_0+\sum_{i=1}^{6} l_iP_i\,;l_i\in R\right\}$$

Definition.
The addition on $6-SP_R$ is defined as follows:

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$$\left[l_0+\sum_{i=1}^{6} l_i P_i\right]+\left[k_0+\sum_{i=1}^{6} k_i P_i\right]=(l_0+k_0)+\sum_{i=1}^{6}(l_i+k_i)P_i$$

Multiplication on $6-SP_R$ is defined as follows:

$$\left[l_0+\sum_{i=1}^{6} l_i P_i\right]\times\left[k_0+\sum_{i=1}^{6} k_i P_i\right]=l_0k_0+\sum_{i=1}^{6} l_i k_j P_{max(i,j)}$$

It is clear that $(6-SP_R,+,.)$ Is a ring.
If R is commutative, the $6-SP_R$ is commutative.
If R has unity (1), then $6-SP_R$ has the same unity.

Example.

Take $R=Z_3=\{0,1,2\}$ the ring of integers module 3, the corresponding symbolic 6-plithogenic ring is:
$6-SP_{Z_3}=\{l_0+l_1P_1+l_2P_2+l_3P_3+l_4P_4+l_5P_5+l_6P_6; l_i\in Z\}$
For example:
$X=2+P_1+P_3+2P_6, Y=1+P_1+P_2+2P_3+P_4+2P_6\in 6-SP_{Z_3}$, we have:
$X+Y=2P_1+P_2+P_4+P_6$

$$X.Y=2+2P_1+2P_2+4P_3+2P_4+4P_6+P_1+P_1+P_2+2P_3+P_4+2P_6+P_3+P_3+P_3+P_4+2P_6+2P_6+2P_6+4P_6+2P_6+4P_6=2+P_1+2P_3+P_4+P_6$$

Definition.

Let $\{S_i\}$, $0\le i\le 6$ be seven ideals of R, then:
$S=S_0+\sum_{i=1}^{6} S_iP_i=\{x_0+\sum_{i=1}^{6} x_iP_i\,; x_i\in S\}$ is called an AH-ideal of $6-SP_R$.
If $S_i=S_j$ for all $i,j\in\{0,\dots,6\}$, then S is called AHS-ideal.

Definition.

Let $S=S_0+\sum_{i=1}^{6} S_iP_i\,, M=M_0+\sum_{i=1}^{6} M_iP_i$ be two AH-ideals of $6-SP_R$, we define:
1). $S\cap M=S_0\cap M_0+\sum_{i=1}^{6}(S_i\cap M_i)P_i$.
2). $S.M=S_0M_0+\sum_{i=1}^{6}(S_iM_i)P_i$
3). $6-SP_R/S=R/S_0+\sum_{i=1}^{6}(R\,/\,S_i)P_i$

Example.

Consider the symbolic 6-plithogenic ring of integers $6-SP_Z=\{l_0+\sum_{i=1}^{6} l_iP_i\,; l_i\in Z\}$
We have:
$S_0=\langle 2\rangle=2Z, S_1=\langle 3\rangle=3Z, S_2=\langle 5\rangle=5Z$ be three ideals of Z, then:
1). $S=S_0+S_0P_1+S_1P_2+S_2P_3+S_2P_4+S_1P_5+S_1P_6=\{2l_0+2l_1P_1+3l_2P_2+5l_3P_3+5l_4P_4+3l_5P_5+3l_6P_6; l_i\in Z\}$ is an AH-ideal of $6-SP_R$.
2). $K=S_0+\sum_{i=1}^{6} S_0P_i=\{2l_0+2\sum_{i=1}^{6} l_iP_i\,; l_i\in Z\}$ is an AHS-ideal.
3). $S\cap K=S_0\cap S_0+(S_0\cap S_0)P_1+(S_1\cap S_0)P_2+(S_2\cap S_0)P_3+(S_2\cap S_0)P_4+(S_1\cap S_0)P_5+(S_1\cap S_0)P_6=\langle 2\rangle+\langle 2\rangle P_1+\langle 6\rangle P_2+\langle 10\rangle P_3+\langle 10\rangle P_4+\langle 6\rangle P_5+\langle 6\rangle P_6$
4). $6-SP_Z/S=(Z/\langle 2\rangle)+(Z/\langle 2\rangle)P_1+(Z/\langle 3\rangle)P_2+(Z/\langle 5\rangle)P_3+(Z/\langle 5\rangle)P_4+(Z/\langle 3\rangle)P_5+(Z/\langle 3\rangle)P_6=Z_2+Z_2P_1+Z_3P_2+Z_5P_3+Z_5P_4+Z_3P_5+Z_3P_6$

Definition.

Let R,T be two rings, and $6-SP_R$ be the corresponding symbolic 6-plithogenic ring, let
$f_i: R\to T; 0\le i\le 6$ be seven ring homomorphisms, then:
1). $f: 6-SP_R\to 6-SP_T$ such that:

$$f\left(l_0+\sum_{i=1}^{6} l_iP_i\right)=f_0(l_0)+\sum_{i=1}^{6} f_i(l_i)P_i$$

The mapping (f) is called AH-homomorphism.
2). If $f_i=f_j$ for all $0\le i,j\le 6$, then f is called AHS-homomorphism.
3). If f_i are isomorphism, then we get the concept AH-isomorphism/AHS-isomorphism.
4). $AH-ker(f)=ker(f_0)+\sum_{i=1}^{6} ker(f_i)P_i$ is called the AH-kernel.
5). $AH-Im(f)=Im(f_0)+\sum_{i=1}^{6} Im(f_i)P_i$ is called the AH-image of (f).

Example.

Take $R=Z, T=Z_{30}$, we have the following homomorphisms:
$f_0: R\to T; f_0(a)=6a(mod\ 30)$
$f_1: R\to T; f_1(b)=b(mod\ 30)$
We define $f: 6-SP_R\to 6-SP_T$ a follows:
$f(l_0+\sum_{i=1}^{6} l_iP_i)=f_0(l_0)+\sum_{i=1}^{3} f_1(l_i)P_i+\sum_{i=4}^{6} f_0(l_i)P_i\,; l_i\in Z$.
For example:
$L=3+2P_1+7P_2+5P_3+9P_4+11P_5+P_6\in 6-SP_R$
$f(L)=18+2P_1+7P_2+5P_3+24P_4+6P_5+6P_6\in 6-SP_T$

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$ker(f_0) = 5Z, ker(f_1) = 30Z$

$$AH - ker(f) = ker(f_0) + \sum_{i=1}^{3} ker(f_1)P_i + \sum_{i=4}^{6} ker(f_0)P_i = 5Z + \sum_{i=1}^{3} 30ZP_i + \sum_{i=4}^{6} 5ZP_i$$

$$6 - SP_R/AH - ker(f) = Z_5 + \sum_{i=1}^{3} Z_{30}P_i + \sum_{i=4}^{6} Z_5P_i$$

Definition.

Let $f, g: 6 - SP_R \to 6 - SP_R$ be two symbolic 6-plithogenic homomorphisms, where $f = f_0 + \sum_{i=1}^{6} f_iP_i , g = g_0 + \sum_{i=1}^{6} g_iP_i$, then $f \times g = f_0 \circ g_0 + \sum_{i=1}^{6}(f_i \circ g_i)P_i$

Theorem1.

Let $6 - SP_R$ be a symbolic 6-plithogenic ring, and let $L = L_0 + \sum_{i=1}^{6} L_iP_i$ be an AHS-ideal of $6 - SP_R$, then L is an ideal by the ordinary meaning.

Theorem2.

Let $f: 6 - SP_R \to 6 - SP_T$ be a symbolic 6-plithogenic AH-homomorphism, then:

1). $AH - ker(f)$ is an $AH - ideal$ of $6 - SP_R$.

2). $AH - Im(f)$ is an $AH - ideal$ of $6 - SP_T$.

3). If f is AH-isomorphism, then $AH - ker(f) = \{o\}$, and $AH - Im(f) = 6 - SP_T$.

Theorem3.

Let $6 - SP_R$ be a symbolic 6-plithogenic ring, and $M = M_0 + \sum_{i=1}^{6} M_iP_i , N = N_0 + \sum_{i=1}^{6} N_iP_i$ be two AH-ideals of $6 - SP_R$, then:

1). $M \cap N, M.N$ are two AH-ideals.

2). If M, N are two AHS-ideals, then $M \cap N, M.N$ are two AH-ideals.

3). If M is n AHS-ideal, then $6 - SP_R/M = R/M_0 + \sum_{i=1}^{6}(R/M_0)P_i$

Theorem4.

Let $f, g: 6 - SP_R \to 6 - SP_R$ be two AH-homomorphism, where $f = f_0 + \sum_{i=1}^{6} f_iP_i , g = g_0 + \sum_{i=1}^{6} g_iP_i$, then:

1). $f \times g$ is an AH-homomorphism.

2). If f, g are AHS-homomorphisms, then $f \times g$ is an AHS-homomorphism.

3). If f, g are AH-isomorphisms, then $f \times g$ is an AH-isomorphism.

4). If f, g are AHS-isomorphisms, then $f \times g$ is an AHS-isomorphism.

Proof of theorem1.

Let $l = l_0 + \sum_{i=1}^{6} l_iP_i , \acute{l} = \acute{l}_0 + \sum_{i=1}^{6} \acute{l}_iP_i$, then:

$l - \acute{l} = l_0 - \acute{l}_0 + \sum_{i=1}^{6}(l_i - \acute{l}_i)P_i \in L.$

Let $m = m_0 + \sum_{i=1}^{6} m_iP_i \in 6 - SP_R$, then:

$m.l = m_0l_0 + \sum_{i=1}^{6} m_il_jP_{max(i,j)} \in L$, that is because:

l_0 is an ideal of R, and $m_il_j \in l_0$, which implies the proof.

Proof of theorem2.

1). Since $ker(f_i); 0 \le i \le 6$ is an ideal of R, then $AH - ker(f)$ is an AH-ideal.

2). Since $Im(f_i); 0 \le i \le 6$ is an ideal of T, then $AH - Im(f)$ is an AH-ideal of $6 - SP_T$.

3). it holds directly from the fact that (f_i) is a bijection.

Proof of theorem3.

1). We have $M_i \cap N_i, M_i.N_i$ are ideals of R, that is because M_i, N_i are ideals for all $0 \le i \le 6$, thus: $M \cap N = M_0 \cap N_0 + \sum_{i=1}^{6}(M_i \cap N_i)P_i , M.N = M_0N_0 + \sum_{i=1}^{6}(M_iN_i)P_i$ are AH-ideals of $6 - SP_R$.

2). If M, N are AHS-ideals, then:

$M = M_0 + \sum_{i=1}^{6} M_0P_i , N = N_0 + \sum_{i=1}^{6} N_0P_i$, thus:

$$M \cap N = M_0 \cap N_0 + \sum_{i=1}^{6}(M_i \cap N_i)P_i , M.N = M_0N_0 + \sum_{i=1}^{6}(M_iN_i)P_i$$

Hence $M \cap N, M.N$ are two AHS-ideals.

3). M is an AHS- ideal, hence $M = M_0 + \sum_{i=1}^{6} M_0P_i$, thus $6 - SP_R/M = R/M_0 + \sum_{i=1}^{6}(R/M_0)P_i$

Proof of theorem4.

1). $f \times g = f_0 \circ g_0 + \sum_{i=1}^{6}(f_i \circ g_i)P_i$. It is known that:

$f_i \circ g_i$ is a homomorphism for all $0 \le i \le 6$, hence $f \times g$ is an AH-homomorphism.

2). If f, g are AHS-homomorphisms, then:

$f = f_0 + \sum_{i=1}^{6} f_0P_i , g = g_0 + \sum_{i=1}^{6} g_0P_i$, so that:

$f \times g = f_0 \circ g_0 + \sum_{i=1}^{6}(f_i \circ g_i)P_i$ is an AH-homomorphism.

3) and 4) holds by a similar argument.

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The classification of symbolic 6-plithogenic rings.
Theorem5.
Let $6-SP_R$ be a symbolic 6-plithogenic ring, then $6-SP_R \cong R^7$.
Proof.
Define the mapping $f: 6-SP_R \to R^7$ such that:

$$f\left(l_0+\sum_{i=1}^{6} l_i P_i\right)=\left(l_0, \sum_{i=0}^{1} l_i, \sum_{i=0}^{2} l_i, \sum_{i=0}^{3} l_i, \sum_{i=0}^{4} l_i, \sum_{i=0}^{5} l_i, \sum_{i=0}^{6} l_i\right)$$

(f) is well defined:
Assume that $l_0+\sum_{i=1}^{6} l_i P_i=\acute{l}_0+\sum_{i=1}^{6} \acute{l}_i P_i$, then $l_i=\acute{l}_i$ for all $0 \leq i \leq 6$, hence:

$$\begin{cases} l_0=\acute{l}_0 \\ \sum_{i=0}^{1} l_i=\sum_{i=0}^{1} \acute{l}_i \\ \sum_{i=0}^{2} l_i=\sum_{i=0}^{2} \acute{l}_i \\ \sum_{i=0}^{3} l_i=\sum_{i=0}^{3} \acute{l}_i \\ \sum_{i=0}^{4} l_i=\sum_{i=0}^{4} \acute{l}_i \\ \sum_{i=0}^{5} l_i=\sum_{i=0}^{5} \acute{l}_i \\ \sum_{i=0}^{6} l_i=\sum_{i=0}^{6} \acute{l}_i \end{cases}$$

Thus $f\left(l_0+\sum_{i=1}^{6} l_i P_i\right)=f\left(\acute{l}_0+\sum_{i=1}^{6} \acute{l}_i P_i\right)$
(f) preserves addition and multiplication:
For $m=m_0+\sum_{i=1}^{6} m_i P_i, n=n_0+\sum_{i=1}^{6} n_i P_i$, we can write:

$$m+n=(m_0+n_0)+\sum_{i=1}^{6}(m_i+n_i) P_i$$

$$m.n=m_0.n_0+\sum_{i=1}^{6} m_i.n_j P_{max(i,j)}$$

$f(m+n)=(t_0, t_1, t_2, t_3, t_4, t_5, t_6)$
$f(m.n)=(s_0, s_1, s_2, s_3, s_4, s_5, s_6)$

$$\begin{cases} t_0=m_0+n_0 \\ t_1=\sum_{i=0}^{1} m_i+\sum_{i=0}^{1} n_i \\ t_2=\sum_{i=0}^{2} m_i+\sum_{i=0}^{2} n_i \\ t_3=\sum_{i=0}^{3} m_i+\sum_{i=0}^{3} n_i \\ t_4=\sum_{i=0}^{4} m_i+\sum_{i=0}^{4} n_i \\ t_5=\sum_{i=0}^{5} m_i+\sum_{i=0}^{5} n_i \\ t_6=\sum_{i=0}^{6} m_i+\sum_{i=0}^{6} n_i \end{cases}$$

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$$\begin{cases} s_0 = m_0 n_0 \\ s_1 = \sum_{i=0}^{1} m_i \sum_{i=0}^{1} n_i \\ s_2 = \sum_{i=0}^{2} m_i \sum_{i=0}^{2} n_i \\ s_3 = \sum_{i=0}^{3} m_i \sum_{i=0}^{3} n_i \\ s_4 = \sum_{i=0}^{4} m_i \sum_{i=0}^{4} n_i \\ s_5 = \sum_{i=0}^{5} m_i \sum_{i=0}^{5} n_i \\ s_6 = \sum_{i=0}^{6} m_i \sum_{i=0}^{6} n_i \end{cases} \Rightarrow f(m.n) = f(m)f(m)$$

Let $m \in ker(f)$, then $f(m) = (0,0,0,0,0,0,0)$, thus:

$$\begin{cases} m_0 = 0 \\ \sum_{i=0}^{j} m_i = 0; 0 \le j \le 6 \end{cases}$$

Hence $m_i = 0$ for all $0 \le i \le 6$, and $ker(f) = \{0\}$

Let $X = (l_0, l_1, l_2, l_3, l_4, l_5, l_6) \in R^7$, then there exists

$L = l_0 + (l_1 - l_0)P_1 + (l_2 - l_1)P_2 + (l_3 - l_2)P_3 + (l_4 - l_3)P_4 + (l_5 - l_4)P_5 + (l_6 - l_5)P_6 \in 6 - SP_R$ such that $f(L) = X$.

Thus $(f)$is a ring isomorphism, and $6 - SP_R \cong R^7$.

Remark.

The inverse isomorphism is $f^{-1}: R^7 \to 6 - SP_R$ such that:

$f^{-1}(l_0, l_1, \dots, l_6) = l_0 + (l_1 - l_0)P_1 + (l_2 - l_1)P_2 + (l_3 - l_2)P_3 + (l_4 - l_3)P_4 + (l_5 - l_4)P_5 + (l_6 - l_5)P_6$

Result.

If U_{6-SP_R} is the group of units of the symbolic 6-plithogenic ring $6 - SP_R$, and U_R is the group of units of R, then $U_{6-SP_R} \cong {U_R}^7$.

Example.

Let $R_1 = Z_3, R_2 = Z_5, R_3 = Z_6, R_4 = Z, R_5 = Q$, then:

$$\begin{cases} U_{6-SP_{R_1}} \cong (Z_2)^7 \\ U_{6-SP_{R_2}} \cong (Z_4)^7 \\ U_{6-SP_{R_3}} \cong (Z_2)^7 \\ U_{6-SP_{R_4}} \cong (Z_2)^7 \\ U_{6-SP_{R_5}} \cong (Q^*)^7 \end{cases}$$

Example.

Consider $6 - SP_{Z_7}$ the symbolic 6-plithogenic ring of integers modulo 7, and take:

$X = 2 + P_1 + P_2 + P_3 + P_4 + 2P_5 + 4P_6$, we will find the inverse X^{-1} by using the isomorphism f.

$f(X) = (2,3,4,5,6,1,5), [f(X)]^{-1} = (4,5,2,3,6,1,3)$, thus $X^{-1} = f^{-1}([f(X)]^{-1}) = f^{-1}(4,5,2,3,6,1,3) = 4 + (5-4)P_1 + (2-5)P_2 + (3-2)P_3 + (6-3)P_4 + (1-6)P_5 + (3-1)P_6 = 4 + P_1 - 3P_2 + P_3 + 3P_4 - 5P_5 + 2P_6 = 4 + P_1 + 4P_2 + P_3 + 3P_4 + 2P_5 + 2P_6$

We can see that $XX^{-1} = 8 + 2P_1 + 8P_2 + 2P_3 + 6P_4 + 4P_5 + 4P_6 + P_1 + 4P_2 + P_3 + 3P_4 + 2P_5 + 2P_6 + 4P_2 + P_2 + 4P_2 + P_3 + 3P_4 + 2P_5 + 2P_6 + 4P_3 + P_3 + 4P_3 + 3P_4 + 2P_5 + 2P_6 + 4P_4 + P_4 + 4P_4 + P_4 + 3P_4 + 2P_5 + 2P_6 + 8P_5 + 2P_5 + 8P_5 + 2P_5 + 6P_5 + 4P_5 + 4P_6 + 16P_6 + 4P_6 + 16P_6 + 4P_6 + 12P_6 + 8P_6 + 8P_6 + 8P_6 = 1$

Result.

The element $X = l_0 + \sum_{i=1}^{6} l_i P_i \in 6 - SP_R$ is invertible if and only if $l_0, \sum_{i=1}^{j} l_i ; 1 \le j \le 6$ are invertible in R.

Result.

For $X = l_0 + \sum_{i=1}^{6} l_i P_i \in 6 - SP_R$, then:

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$$X^n = l_0{}^n + \left(\left(\sum_{i=0}^{1} l_i\right)^n - l_0{}^n\right)P_1 + \left(\left(\sum_{i=0}^{2} l_i\right)^n - \left(\sum_{i=0}^{1} l_i\right)^n\right)P_2 + \left(\left(\sum_{i=0}^{3} l_i\right)^n - \left(\sum_{i=0}^{2} l_i\right)^n\right)P_3 + \left(\left(\sum_{i=0}^{4} l_i\right)^n - \left(\sum_{i=0}^{3} l_i\right)^n\right)P_4 + \left(\left(\sum_{i=0}^{5} l_i\right)^n - \left(\sum_{i=0}^{4} l_i\right)^n\right)P_5 + \left(\left(\sum_{i=0}^{6} l_i\right)^n - \left(\sum_{i=0}^{5} l_i\right)^n\right)P_6$$

Example.

Assuming that $R = Q$ is the field of rationals, $6 - SP_Q$ be the corresponding symbolic 6-plithogenic ring, now we will find all non-isomorphic ideals in $6 - SP_Q$.

The non-isomorphic ideals of Q^7 are:

$F_1 = \{(0,0,0,0,0,0,0)\}$
$F_2 = \{(Q,0,0,0,0,0,0)\}$
$F_3 = \{(Q,Q,0,0,0,0,0)\}$
$F_4 = \{(Q,Q,Q,0,0,0,0)\}$
$F_5 = \{(Q,Q,Q,Q,0,0,0)\}$
$F_6 = \{(Q,Q,Q,Q,Q,0,0)\}$
$F_7 = \{(Q,Q,Q,Q,Q,Q,0)\}$
$F_8 = \{(Q,Q,Q,Q,Q,Q,Q)\}$

The ideals of $6 - SP_Q$ are:

$f^{-1}(F_1) = \{O\}$
$f^{-1}(F_2) = \{t_0 - t_0P_1; t_0 \in Q\}$
$f^{-1}(F_3) = \{t_0 + (t_1 - t_0)P_1 - t_1P_2; t_0, t_1 \in Q\}$
$f^{-1}(F_4) = \{t_0 + (t_1 - t_0)P_1 + (t_2 - t_1)P_2 - t_2P_3; t_0, t_1, t_2 \in Q\}$
$f^{-1}(F_5) = \{t_0 + (t_1 - t_0)P_1 + (t_2 - t_1)P_2 + (t_3 - t_2)P_3 - t_3P_4; t_0, t_1, t_2, t_3 \in Q\}$
$f^{-1}(F_6) = \{t_0 + (t_1 - t_0)P_1 + (t_2 - t_1)P_2 + (t_3 - t_2)P_3 + (t_4 - t_3)P_4 - t_4P_5; t_0, t_1, t_2, t_3, t_4 \in Q\}$
$f^{-1}(F_7) = \{t_0 + (t_1 - t_0)P_1 + (t_2 - t_1)P_2 + (t_3 - t_2)P_3 + (t_4 - t_3)P_4 + (t_5 - t_4)P_5 - t_5P_6; t_0, t_1, t_2, t_3, t_4, t_5 \in Q\}$
$f^{-1}(F_7) = 6 - SP_Q$

Definition:

Let R be a ring, we define the symbolic 7-plithogenic ring $7 - SP_R$ as follows:

$$7 - SP_R = \left\{l_0 + \sum_{i=1}^{7} l_iP_i\,; l_i \in R\right\}$$

Definition.

The addition on $7 - SP_R$ is defined as follows:

$$\left[l_0 + \sum_{i=1}^{7} l_iP_i\right] + \left[k_0 + \sum_{i=1}^{7} k_iP_i\right] = (l_0 + k_0) + \sum_{i=1}^{7}(l_i + k_i)P_i$$

Multiplication on $7 - SP_R$ is defined as follows:

$$\left[l_0 + \sum_{i=1}^{7} l_iP_i\right] \times \left[k_0 + \sum_{i=1}^{7} k_iP_i\right] = l_0k_0 + \sum_{i=1}^{7} l_ik_jP_{max(i,j)}$$

It is clear that $(7 - SP_R, +, .)$ Is a ring.
If R is commutative, the $7 - SP_R$ is commutative.
If R has unity (1), then $7 - SP_R$ has the same unity.

Example.

Take $R = Z_3 = \{0,1,2\}$ the ring of integers module 3, the corresponding symbolic 7-plithogenic ring is:
$7 - SP_{Z_3} = \{l_0 + l_1P_1 + l_2P_2 + l_3P_3 + l_4P_4 + l_5P_5 + l_6P_6 + l_7P_7; l_i \in Z\}$

Definition.

Let $\{S_i\}$, $0 \le i \le 7$ be eight ideals of R, then:
$S = S_0 + \sum_{i=1}^{7} S_iP_i = \{x_0 + \sum_{i=1}^{7} x_iP_i\,; x_i \in S\}$ is called an AH-ideal of $7 - SP_R$.
If $S_i = S_j$ for all $i, j \in \{0, \dots, 7\}$, then S is called AHS-ideal.

Definition.

Let $S = S_0 + \sum_{i=1}^{7} S_iP_i\,, M = M_0 + \sum_{i=1}^{7} M_iP_i$ be two AH-ideals of $7 - SP_R$, we define:
1). $S \cap M = S_0 \cap M_0 + \sum_{i=1}^{7}(S_i \cap M_i)P_i$.
2). $S.M = S_0M_0 + \sum_{i=1}^{7}(S_iM_i)P_i$
3). $7 - SP_R/S = R/S_0 + \sum_{i=1}^{7}(R\,/\,S_i)P_i$

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Example.

Consider the symbolic 7-plithogenic ring of integers $7-SP_Z=\{l_0+\sum_{i=1}^{7} l_iP_i\,;l_i\in Z\}$

We have:

$S_0=\langle 2\rangle=2Z, S_1=\langle 3\rangle=3Z, S_2=\langle 5\rangle=5Z$ be three ideals of Z, then:

1). $S=S_0+S_0P_1+S_1P_2+S_2P_3+S_2P_4+S_1P_5+S_1P_6+S_1P_7=\{2l_0+2l_1P_1+3l_2P_2+5l_3P_3+5l_4P_4+3l_5P_5+3l_6P_6+3l_7P_7; l_i\in Z\}$ is an AH-ideal of $7-SP_R$.

2). $K=S_0+\sum_{i=1}^{7} S_0P_i=\{2l_0+2\sum_{i=1}^{7} l_iP_i\,;l_i\in Z\}$ is an AHS-ideal.

3). $S\cap K=S_0\cap S_0+(S_0\cap S_0)P_1+(S_1\cap S_0)P_2+(S_2\cap S_0)P_3+(S_2\cap S_0)P_4+(S_1\cap S_0)P_5+(S_1\cap S_0)P_6+(S_1\cap S_0)P_7=\langle 2\rangle+\langle 2\rangle P_1+\langle 6\rangle P_2+\langle 10\rangle P_3+\langle 10\rangle P_4+\langle 6\rangle P_5+\langle 6\rangle P_6+\langle 6\rangle P_7$

4). $7-SP_Z/S=(Z/\langle 2\rangle)+(Z/\langle 2\rangle)P_1+(Z/\langle 3\rangle)P_2+(Z/\langle 5\rangle)P_3+(Z/\langle 5\rangle)P_4+(Z/\langle 3\rangle)P_5+(Z/\langle 3\rangle)P_6+(Z/\langle 3\rangle)P_7=Z_2+Z_2P_1+Z_3P_2+Z_5P_3+Z_5P_4+Z_3P_5+Z_3P_6+Z_3P_7$

Definition.

Let R,T be two rings, and $7-SP_R$ be the corresponding symbolic 7-plithogenic ring, let $f_i: R\to T; 0\le i\le 7$ be eight ring homomorphisms, then:

1). $f: 7-SP_R\to 7-SP_T$ such that:

$$f\left(l_0+\sum_{i=1}^{7} l_iP_i\right)=f_0(l_0)+\sum_{i=1}^{7} f_i(l_i)P_i$$

The mapping (f) is called AH-homomorphism.

2). If $f_i=f_j$ for all $0\le i,j\le 7$, then f is called AHS-homomorphism.

3). If f_i are isomorphism, then we get the concept AH-isomorphism/AHS-isomorphism.

4). $AH-ker(f)=ker(f_0)+\sum_{i=1}^{7} ker(f_i)P_i$ is called the AH-kernel.

5). $AH-Im(f)=Im(f_0)+\sum_{i=1}^{7} Im(f_i)P_i$ is called the AH-image of (f).

Example.

Take $R=Z, T=Z_{30}$, we have the following homomorphisms:

$f_0: R\to T; f_0(a)=6a(mod\ 30)$

$f_1: R\to T; f_1(b)=b(mod\ 30)$

We define $f: 6-SP_R\to 6-SP_T$ a follows:

$f(l_0+\sum_{i=1}^{6} l_iP_i)=f_0(l_0)+\sum_{i=1}^{3} f_1(l_i)P_i+\sum_{i=4}^{7} f_0(l_i)P_i\,;l_i\in Z.$

For example:

$L=3+2P_1+7P_2+5P_3+9P_4+11P_5+P_6+P_7\in 7-SP_R$

$f(L)=18+2P_1+7P_2+5P_3+24P_4+6P_5+6P_6+6P_7\in 7-SP_T$

$ker(f_0)=5Z, ker(f_1)=30Z$

$$AH-ker(f)=ker(f_0)+\sum_{i=1}^{3} ker(f_1)P_i+\sum_{i=4}^{7} ker(f_0)P_i=5Z+\sum_{i=1}^{3} 30ZP_i+\sum_{i=4}^{7} 5ZP_i$$

$$7-SP_R/AH-ker(f)=Z_5+\sum_{i=1}^{3} Z_{30}P_i+\sum_{i=4}^{7} Z_5P_i$$

Definition.

Let $f,g: 7-SP_R\to 7-SP_R$ be two symbolic 7-plithogenic homomorphisms, where $f=f_0+\sum_{i=1}^{7} f_iP_i\,, g=g_0+\sum_{i=1}^{7} g_iP_i$, then $f\times g=f_0\circ g_0+\sum_{i=1}^{7}(f_i\circ g_i)P_i$

Theorem1.

Let $7-SP_R$ be a symbolic 6-plithogenic ring, and let $L=L_0+\sum_{i=1}^{7} L_iP_i$ be an AHS-ideal of $7-SP_R$, then L is an ideal by the ordinary meaning.

Theorem2.

Let $f: 7-SP_R\to 7-SP_T$ be a symbolic 7-plithogenic AH-homomorphism, then:

1). $AH-ker(f)$ is an $AH-ideal$ of $7-SP_R$.

2). $AH-Im(f)$ is an $AH-ideal$ of $7-SP_T$.

3). If f is AH-isomorphism, then $AH-ker(f)=\{o\}$, and $AH-Im(f)=7-SP_T$.

Theorem3.

Let $7-SP_R$ be a symbolic 7-plithogenic ring, and $M=M_0+\sum_{i=1}^{7} M_iP_i\,, N=N_0+\sum_{i=1}^{7} N_iP_i$ be two AH-ideals of $7-SP_R$, then:

1). $M\cap N, M.N$ are two AH-ideals.

2). If M,N are two AHS-ideals, then $M\cap N, M.N$ are two AH-ideals.

3). If M is n AHS-ideal, then $7-SP_R/M=R/M_0+\sum_{i=1}^{7}(R/M_0)P_i$

Theorem4.

Let $f,g: 7-SP_R\to 7-SP_R$ be two AH-homomorphism, where $f=f_0+\sum_{i=1}^{7} f_iP_i\,, g=g_0+\sum_{i=1}^{7} g_iP_i$, then:

1). $f\times g$ is an AH-homomorphism.

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2). If f, g are AHS-homomorphisms, then $f \times g$ is an AHS-homomorphism.
3). If f, g are AH-isomorphisms, then $f \times g$ is an AH-isomorphism.
4). If f, g are AHS-isomorphisms, then $f \times g$ is an AHS-isomorphism.

Proof of theorem1.

Let $l = l_0 + \sum_{i=1}^{7} l_i P_i, \acute{l} = \acute{l}_0 + \sum_{i=1}^{7} \acute{l}_i P_i$, then:
$l - \acute{l} = l_0 - \acute{l}_0 + \sum_{i=1}^{7} (l_i - \acute{l}_i) P_i \in L$.
Let $m = m_0 + \sum_{i=1}^{7} m_i P_i \in 7 - SP_R$, then:
$m.l = m_0 l_0 + \sum_{i=1}^{7} m_i l_j P_{max(i,j)} \in L$, that is because:
l_0 is an ideal of R, and $m_i l_j \in l_0$, which implies the proof.

Proof of theorem2.

1). Since $ker(f_i); 0 \le i \le 7$ is an ideal of R, then $AH - ker(f)$ is an AH-ideal.
2). Since $Im(f_i); 0 \le i \le 7$ is an ideal of T, then $AH - Im(f)$ is an AH-ideal of $7 - SP_T$.
3). it holds directly from the fact that (f_i) is a bijection.

Proof of theorem3.

1). We have $M_i \cap N_i, M_i.N_i$ are ideals of R, that is because M_i, N_i are ideals for all $0 \le i \le 7$, thus:
$M \cap N = M_0 \cap N_0 + \sum_{i=1}^{7} (M_i \cap N_i) P_i, M.N = M_0 N_0 + \sum_{i=1}^{7} (M_i N_i) P_i$ are AH-ideals of $7 - SP_R$.
2). If M, N are AHS-ideals, then:
$M = M_0 + \sum_{i=1}^{7} M_0 P_i, N = N_0 + \sum_{i=1}^{7} N_0 P_i$, thus:

$$M \cap N = M_0 \cap N_0 + \sum_{i=1}^{7} (M_i \cap N_i) P_i, M.N = M_0 N_0 + \sum_{i=1}^{7} (M_i N_i) P_i$$

Hence $M \cap N, M.N$ are two AHS-ideals.
3). M is an AHS- ideal, hence $M = M_0 + \sum_{i=1}^{7} M_0 P_i$, thus $7 - SP_R/M = R/M_0 + \sum_{i=1}^{7} (R/M_0) P_i$

Proof of theorem4.

1). $f \times g = f_0 \circ g_0 + \sum_{i=1}^{7} (f_i \circ g_i) P_i$. It is known that:
$f_i \circ g_i$ is a homomorphism for all $0 \le i \le 7$, hence $f \times g$ is an AH-homomorphism.
2). If f, g are AHS-homomorphisms, then:
$f = f_0 + \sum_{i=1}^{7} f_0 P_i, g = g_0 + \sum_{i=1}^{7} g_0 P_i$, so that:
$f \times g = f_0 \circ g_0 + \sum_{i=1}^{7} (f_i \circ g_i) P_i$ is an AH-homomorphism.
3) and 4) holds by a similar argument.

The classification of symbolic 7-plithogenic rings.

Theorem5.

Let $7 - SP_R$ be a symbolic 7-plithogenic ring, then $7 - SP_R \cong R^8$.

Proof.

Define the mapping $f: 7 - SP_R \to R^8$ such that:

$$f\left(l_0 + \sum_{i=1}^{7} l_i P_i\right) = \left(l_0, \sum_{i=0}^{1} l_i, \sum_{i=0}^{2} l_i, \sum_{i=0}^{3} l_i, \sum_{i=0}^{4} l_i, \sum_{i=0}^{5} l_i, \sum_{i=0}^{6} l_i, \sum_{i=0}^{7} l_i\right)$$

(f) is well defined:
Assume that $l_0 + \sum_{i=1}^{7} l_i P_i = \acute{l}_0 + \sum_{i=1}^{7} \acute{l}_i P_i$, then $l_i = \acute{l}_i$ for all $0 \le i \le 7$, hence:

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$$\begin{cases} l_0 = \acute{l}_0 \\ \sum_{i=0}^{1} l_i = \sum_{i=0}^{1} \acute{l}_i \\ \sum_{i=0}^{2} l_i = \sum_{i=0}^{2} \acute{l}_i \\ \sum_{i=0}^{3} l_i = \sum_{i=0}^{3} \acute{l}_i \\ \sum_{i=0}^{4} l_i = \sum_{i=0}^{4} \acute{l}_i \\ \sum_{i=0}^{5} l_i = \sum_{i=0}^{5} \acute{l}_i \\ \sum_{i=0}^{6} l_i = \sum_{i=0}^{6} \acute{l}_i \\ \sum_{i=0}^{7} l_i = \sum_{i=0}^{7} \acute{l}_i \end{cases}$$

Thus $f(l_0 + \sum_{i=1}^{7} l_i P_i) = f\left(\acute{l}_0 + \sum_{i=1}^{7} \acute{l}_i P_i\right)$

(f) preserves addition and multiplication:

For $m = m_0 + \sum_{i=1}^{7} m_i P_i$, $n = n_0 + \sum_{i=1}^{7} n_i P_i$, we can write:

$$m + n = (m_0 + n_0) + \sum_{i=1}^{7} (m_i + n_i) P_i$$

$$m.n = m_0.n_0 + \sum_{i=1}^{7} m_i.n_j P_{max(i,j)}$$

$f(m + n) = (t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7)$

$f(m.n) = (s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7)$

$$\begin{cases} t_0 = m_0 + n_0 \\ t_1 = \sum_{i=0}^{1} m_i + \sum_{i=0}^{1} n_i \\ t_2 = \sum_{i=0}^{2} m_i + \sum_{i=0}^{2} n_i \\ t_3 = \sum_{i=0}^{3} m_i + \sum_{i=0}^{3} n_i \\ t_4 = \sum_{i=0}^{4} m_i + \sum_{i=0}^{4} n_i \\ t_5 = \sum_{i=0}^{5} m_i + \sum_{i=0}^{5} n_i \\ t_6 = \sum_{i=0}^{6} m_i + \sum_{i=0}^{6} n_i \\ t_6 = \sum_{i=0}^{7} m_i + \sum_{i=0}^{7} n_i \end{cases}$$

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$$\begin{cases} s_0 = m_0 n_0 \\ s_1 = \sum_{i=0}^{1} m_i \sum_{i=0}^{1} n_i \\ s_2 = \sum_{i=0}^{2} m_i \sum_{i=0}^{2} n_i \\ s_3 = \sum_{i=0}^{3} m_i \sum_{i=0}^{3} n_i \\ s_4 = \sum_{i=0}^{4} m_i \sum_{i=0}^{4} n_i \\ s_5 = \sum_{i=0}^{5} m_i \sum_{i=0}^{5} n_i \\ s_6 = \sum_{i=0}^{6} m_i \sum_{i=0}^{6} n_i \\ s_6 = \sum_{i=0}^{7} m_i \sum_{i=0}^{7} n_i \end{cases} \Rightarrow f(m.n) = f(m)f(m)$$

Let $m \in ker(f)$, then $f(m) = (0,0,0,0,0,0,0,0)$, thus:

$$\begin{cases} m_0 = 0 \\ \sum_{i=0}^{j} m_i = 0; 0 \leq j \leq 7 \end{cases}$$

Hence $m_i = 0$ for all $0 \leq i \leq 7$, and $ker(f) = \{0\}$

Let $X = (l_0, l_1, l_2, l_3, l_4, l_5, l_6, l_7) \in R^8$, then there exists $L = l_0 + (l_1 - l_0)P_1 + (l_2 - l_1)P_2 + (l_3 - l_2)P_3 + (l_4 - l_3)P_4 + (l_5 - l_4)P_5 + (l_6 - l_5)P_6 + (l_7 - l_6)P_7 \in 7 - SP_R$ such that $f(L) = X$.

Thus $(f)$is a ring isomorphism, and $7 - SP_R \cong R^8$.

Remark.

The inverse isomorphism is $f^{-1}: R^8 \to 7 - SP_R$ such that:
$f^{-1}(l_0, l_1, \dots, l_6) = l_0 + (l_1 - l_0)P_1 + (l_2 - l_1)P_2 + (l_3 - l_2)P_3 + (l_4 - l_3)P_4 + (l_5 - l_4)P_5 + (l_6 - l_5)P_6 + (l_7 - l_6)P_7$.

Result.

If U_{7-SP_R} is the group of units of the symbolic 7-plithogenic ring $7 - SP_R$, and U_R is the group of units of R, then $U_{7-SP_R} \cong {U_R}^8$.

Result.

The element $X = l_0 + \sum_{i=1}^{7} l_i P_i \in 7 - SP_R$ is invertible if and only if $l_0, \sum_{i=1}^{j} l_i\,; 1 \leq j \leq 7$ are invertible in R.

Result.

For $X = l_0 + \sum_{i=1}^{7} l_i P_i \in 7 - SP_R$, then:

$$X^n = {l_0}^n + \left(\left(\sum_{i=0}^{1} l_i\right)^n - {l_0}^n\right)P_1 + \left(\left(\sum_{i=0}^{2} l_i\right)^n - \left(\sum_{i=0}^{1} l_i\right)^n\right)P_2 + \left(\left(\sum_{i=0}^{3} l_i\right)^n - \left(\sum_{i=0}^{2} l_i\right)^n\right)P_3 + \left(\left(\sum_{i=0}^{4} l_i\right)^n - \left(\sum_{i=0}^{3} l_i\right)^n\right)P_4 + \left(\left(\sum_{i=0}^{5} l_i\right)^n - \left(\sum_{i=0}^{4} l_i\right)^n\right)P_5 + \left(\left(\sum_{i=0}^{6} l_i\right)^n - \left(\sum_{i=0}^{5} l_i\right)^n\right)P_6 + \left(\left(\sum_{i=0}^{7} l_i\right)^n - \left(\sum_{i=0}^{6} l_i\right)^n\right)P_7$$

2. Conclusion

In this paper we have presented the concept of algebraic symbolic 6-plithogenic ring and 7-plithogenic ring. Also, we have studied some of their special substructures such as AH-ideals, AH-kernels, and AH-isomorphisms. On the other hand, we have illustrated many examples to clarify the validity of our work.

As a future research direction, we aim to study 8-plithogenic rings and their algebraic classification with respect to homomorphisms.

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