On Finding Cauchy – Pompeiu's Formula in The Octal Unit Disk

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Abstract

In this paper, we find the formula of Cauchy – Pompeiu's integral in the octal of unit disk $\mathbb{D}_4$ of the complex plain, by using the reflection method to determine the Integral's Cauchy – Pompeiu's operator. Also, we give many related examples about the novel formula.

Keywords: Cauchy – Pompeiu's integral; reflection; Cauchy – Pompeiu's operator

1. Introduction

The theory of effects is an important branch of semantic analysis, studying the properties of effects (linear, finite, continuous, reducing,...) On famous spaces such as the space of continuous Functions $C(D;\mathbb{C})$ in Region D, the space of integrable functions $L(D;\mathbb{C})$, and others, in this paper we will find $T_{\mathbb{D}_4}$, the Cauchy - Pompeiu integral effect in the price of one disk of the nodal plane $\mathbb{D}_4$, by modifying the Cauchy - Pompeiu integral formula in one disk.

In the literature, we find the following results:

- By H.Begeher and T.Vaitekhovich (2009) assigned the Cauchy - Pompeiu formula in half of the torus [1].
- By B.Shupeyeva (2012) set the Cauchy - Pompeiu formula in a quarter of the torus [2].
- By B.Shupeyeva (2016) assigned the Cauchy - Pompeiu formula to the upper half of the pistol with heads ± 2 and$\pm 1 + i\sqrt{3}$ [3].
- By H.Begehr (2018) by assigning the Cauchy - Pompeiu formula to the area resulting from the truncation of a circle from another circle touching it internally and non-subscribers of the center [4].

Basic definitions:

Below we mention a set of basic definitions and some notes that help us understand the scientific terms contained in the research:

Definition (1) of the regular Domain system Area [6]:

A region $D \subset \mathbb{C}$ is called a regular region if it is open and bounded and its boundary is a smooth curve or a finite meeting of smooth curves, directed counterclockwise.

We list examples of regular areas: circle-ring, half-ring, quarter-circle, semicircle, quarter-triangle, rectangle, parallelogram,...

Cauchy-Pompeiu's representation formula [7]:

Let $D \subset \mathbb{C}$ be a regular region, and let the function $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$ then for each $z \in D$ and $\zeta = \xi + i\eta$ the function $w = w(z)$ is represented by one of two formulas:

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta).\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_{D} w_{\bar{\zeta}}(\zeta).\frac{d\xi d\eta}{\zeta - z}$$

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$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta).\frac{\overline{d\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int_D w_{\zeta}(\zeta).\frac{d\xi d\eta}{\overline{\zeta - z}}$$

Note:

When $z \in \mathbb{C}\backslash\overline{D}$, the previous Cauchy-Pompeiu formulas become as follows:

$$0 = \frac{1}{2\pi i}\int_{\partial D} w(\zeta).\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\zeta}(\zeta).\frac{d\xi d\eta}{\zeta - z}$$
$$0 = -\frac{1}{2\pi i}\int_{\partial D} w(\zeta).\frac{\overline{d\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int_D w_{\zeta}(\zeta).\frac{d\xi d\eta}{\overline{\zeta - z}}$$

From it, we can write the Cauchy - Pompeiu integral formula in another Form [8], as follows:

$$\frac{1}{2\pi i}\int_{\partial D} w(\zeta).\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\zeta}(\zeta).\frac{d\xi d\eta}{\zeta - z} = \begin{cases} w(z); z \in D \\ 0; z \notin \overline{D} \end{cases}$$
$$-\frac{1}{2\pi i}\int_{\partial D} w(\zeta).\frac{\overline{d\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int_D w_{\zeta}(\zeta).\frac{d\xi d\eta}{\overline{\zeta - z}} = \begin{cases} w(z); z \in D \\ 0; z \notin \overline{D} \end{cases}$$

Definition (2) Cauchy-Pompeiu's Oparator [7]:

Let $D \subset \mathbb{C}$ be a regular region, the integral effect is called

$$Tf(z) = -\frac{1}{\pi}\int_D f(\zeta).\frac{d\xi d\eta}{\zeta - z};\ \zeta = \xi + i\eta$$

By the Cauchy – Pompeiu oparator in Region D, where $f \in L_1(D;\mathbb{C})$

Definition (3) of the Schwartz boundary value problem (The Schwartz's Problem) [5]:

Solving the limit Schwarz problem is the process of finding a function w, in the region $D \subset \mathbb{C}$ ، the equation fulfills:

$$\begin{cases} \partial_{\overline{z}} w = f & in\ D \\ Re(w) = \gamma & on\ \partial D \end{cases}$$

Where f $f \in L_1(D;\mathbb{C})$, $\gamma \in C(D;\partial\mathbb{C})$are two imposed functions.

Note: the Cauchy-Pompeiu integral formula is modified from region to region, to fulfill the limit condition of the Schwartz problem.

Results and discussion:

The price of one tablet is defined as follows:

$$\mathbb{D}_4 = \{z \in \mathbb{C}; |z| < 1, 0 < \arg z < \frac{\pi}{4}\}$$

Suppose:

$\partial_1\mathbb{D}_4$ for the straight segment [0,1]

$\partial_2\mathbb{D}_4$ for the Bracket$\left[1, e^{\frac{\pi}{4}i}\right]: \tau \to e^{\tau i}$, where $\tau \in [0, \frac{\pi}{4}]$

$\partial_3\mathbb{D}_4$ for the straight segment $\left[e^{\frac{\pi}{4}i}, 0\right]$

We will use the inversion method, to modify the Cauchy-Pompeiu integral formula, and define it in region $\mathbb{D}_4$

We assume $z, \zeta \in \mathbb{D}_4$, where $z \neq \zeta$, the reflection of z relative to the line segment $\partial_3\mathbb{D}_4$ gives $i\overline{z}$, from which the reflection of z and $i\overline{z}$ relative to the truncation axis are $iz - \overline{z}$, respectively،

And the reflection of the points: $z, i\overline{z}, -\overline{z}, iz$ relative to the axis the intervals are $\overline{z}, -iz, -z, i\overline{z}$ respectively،

Taking into account how z is from $\mathbb{D}_4$, we get eight groups that are:

$$\mathbb{D}_{4,0} = \mathbb{D}_4$$
$$\mathbb{D}_{4,1} = \left\{z \in \mathbb{C}; |z| < 1, \frac{\pi}{4} < \arg z < \frac{\pi}{2}\right\} = \{i\overline{z}; z \in \mathbb{D}_4\}$$
$$\mathbb{D}_{4,2} = \left\{z \in \mathbb{C}; |z| < 1, \frac{\pi}{2} < \arg z < \frac{3\pi}{4}\right\} = \{iz; z \in \mathbb{D}_4\}$$

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$$\mathbb{D}_{4,3} = \left\{z \in \mathbb{C}; |z| < 1, \frac{3\pi}{4} < \arg z < \pi\right\} = \{\bar{z}; z \in \mathbb{D}_4\}$$
$$\mathbb{D}_{4,4} = \left\{z \in \mathbb{C}; |z| < 1, \pi < \arg z < \frac{5\pi}{4}\right\} = \{-z; z \in \mathbb{D}_4\}$$
$$\mathbb{D}_{4,5} = \left\{z \in \mathbb{C}; |z| < 1, \frac{5\pi}{4} < \arg z < \frac{3\pi}{2}\right\} = \{-i\bar{z}; z \in \mathbb{D}_4\}$$
$$\mathbb{D}_{4,6} = \left\{z \in \mathbb{C}; |z| < 1, \frac{3\pi}{2} < \arg z < \frac{7\pi}{4}\right\} = \{-iz; z \in \mathbb{D}_4\}$$
$$\mathbb{D}_{4,7} = \left\{z \in \mathbb{C}; |z| < 1, \frac{7\pi}{4} < \arg z < 2\pi\right\} = \{\bar{z}; z \in \mathbb{D}_4\}$$

It is clear that:

$$\bigcup_{k=0}^{k=7} \overline{\mathbb{D}_{4,k}} = \overline{\mathbb{D}} = \{z \in \mathbb{C}; |z| \leq 1\}$$

The reflection of the following points:

$$z, i\bar{z}, iz, -\bar{z}, -z, -i\bar{z}, -iz, \bar{z}$$

For the arc of a single Circle $\mathbb{D}$ gives in order the following points:

$$\frac{1}{\bar{z}}, \frac{i}{z}, \frac{i}{\bar{z}}, -\frac{1}{z}, -\frac{1}{\bar{z}}, -\frac{i}{z}, -\frac{i}{\bar{z}}, \frac{1}{z}$$

Taking into account how z is from$\mathbb{D}_4$, we get eight more groups that are:

$$\mathbb{S}_{4,0} = \left\{z \in \mathbb{C}; |z| > 1, 0 < \arg z < \frac{\pi}{4}\right\} = \{\frac{1}{\bar{z}}; z \in \mathbb{D}_4\}$$
$$\mathbb{S}_{4,1} = \left\{z \in \mathbb{C}; |z| > 1, \frac{\pi}{4} < \arg z < \frac{\pi}{2}\right\} = \{\frac{i}{z}; z \in \mathbb{D}_4\}$$
$$\mathbb{S}_{4,2} = \left\{z \in \mathbb{C}; |z| > 1, \frac{\pi}{2} < \arg z < \frac{3\pi}{4}\right\} = \{\frac{i}{\bar{z}}; z \in \mathbb{D}_4\}$$
$$\mathbb{S}_{4,3} = \left\{z \in \mathbb{C}; |z| > 1, \frac{3\pi}{4} < \arg z < \pi\right\} = \{-\frac{1}{z}; z \in \mathbb{D}_4\}$$
$$\mathbb{S}_{4,4} = \left\{z \in \mathbb{C}; |z| > 1, \pi < \arg z < \frac{5\pi}{4}\right\} = \{-\frac{1}{\bar{z}}; z \in \mathbb{D}_4\}$$
$$\mathbb{S}_{4,5} = \left\{z \in \mathbb{C}; |z| > 1, \frac{5\pi}{4} < \arg z < \frac{3\pi}{2}\right\} = \{-\frac{i}{z}; z \in \mathbb{D}_4\}$$
$$\mathbb{S}_{4,6} = \left\{z \in \mathbb{C}; |z| > 1, \frac{3\pi}{2} < \arg z < \frac{7\pi}{4}\right\} = \{-\frac{i}{\bar{z}}; z \in \mathbb{D}_4\}$$
$$\mathbb{S}_{4,7} = \left\{z \in \mathbb{C}; |z| > 1, \frac{7\pi}{4} < \arg z < 2\pi\right\} = \{\frac{1}{\bar{z}}; z \in \mathbb{D}_4\}$$

We note that:

$$\bigcup_{k=0}^{k=7} \overline{\mathbb{S}_{4,k}} = \mathbb{S} \backslash \mathbb{D} = \{z \in \mathbb{C}; |z| \geq 1\}, \bigcup_{k=0}^{k=7} [\overline{\mathbb{S}_{4,k}} \cup \overline{\mathbb{D}_{4,k}}] = \mathbb{C}$$

Where:

$$\overline{\mathbb{S}_{4,k}} \cup \overline{\mathbb{D}_{4,k}} = \left\{z \in \mathbb{C}; \frac{k\pi}{4} \leq \arg z \leq \frac{(k+1)\pi}{4}\right\}; k = 0,1,2, \dots, 7$$

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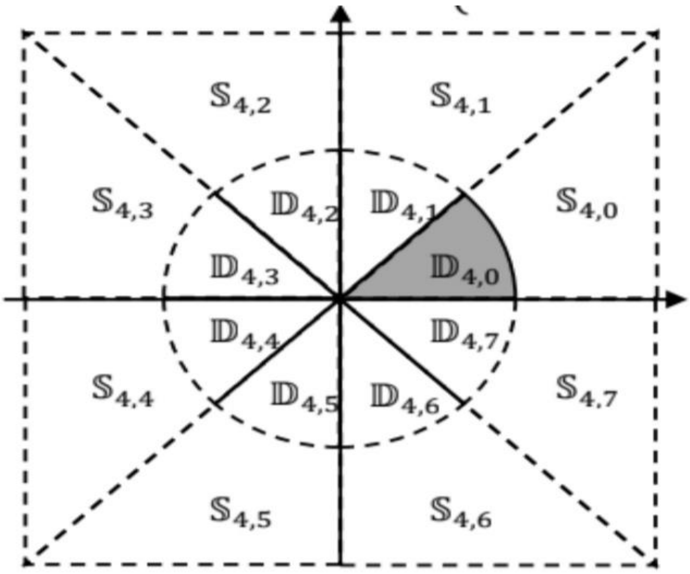

Theorem 1: every function $w \in C^1(\mathbb{D}_4;\mathbb{C}) \cap C(\overline{\mathbb{D}_4};\mathbb{C})$ is represented as follows:

$$w(z) = \frac{2}{\pi i}\int_{0<\arg\zeta<\frac{\pi}{4}}^{|\zeta|=1}\left\{\mathrm{Re}[w(\zeta)].\left[\frac{\zeta^4+z^4}{\zeta^4-z^4}+\frac{\zeta^4 z^4+1}{z^4\zeta^4-1}\right]\right\}\frac{d\zeta}{\zeta}+\frac{4}{\pi}\int_{0<\arg\zeta<\frac{\pi}{4}}^{|\zeta|=1}\mathrm{Im}[w(\zeta)]\frac{d\zeta}{\zeta}$$
$$+\frac{4}{\pi i}\int_1^0 \mathrm{Re}\left[w\left(t.e^{\frac{4}{\pi}i}\right)\right].\left[\frac{t^3}{t^4+z^4}+\frac{t^3 z^4}{z^4t^4+1}\right]dt$$
$$+\frac{4}{\pi i}\int_0^1 \mathrm{Re}[w(t)].\left[\frac{t^3}{t^4-z^4}+\frac{t^3 z^4}{z^4t^4-1}\right]dt$$
$$-\frac{1}{\pi}\int_{\mathbb{D}_4}\{w_\zeta(\zeta).\left[\frac{4\zeta^3}{\zeta^4-z^4}+\frac{4\zeta^3z^4}{z^4\zeta^4-1}\right]-\overline{w_\zeta(\zeta)}.\left[\frac{4\bar\zeta^3}{\bar\zeta^4-z^4}+\frac{4\bar\zeta^3z^4}{z^4\bar\zeta^4-1}\right]\}d\xi d\eta$$

Proof:

We apply the Cauchy-Pompeiu integral formula to the following points:

$$z, i\bar z, iz, -\bar z, -z, -i\bar z, -iz, \bar z, \frac{1}{\bar z}, \frac{i}{\bar z}, \frac{i}{\bar z}, -\frac{1}{z}, -\frac{1}{\bar z}, -\frac{i}{z}, -\frac{i}{\bar z}, \frac{1}{z}$$

Noting that all the previous points lie outside $\mathbb{D}_4$, except for z, they belong to $\mathbb{D}_4$, we find:

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{d\zeta}{\zeta-z}-\frac{1}{\pi}\int_{\mathbb{D}_4} w_\zeta(\zeta).\frac{d\xi d\eta}{\zeta-z}\ \ldots.(1.1)$$
$$0 = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{d\zeta}{\zeta-\bar z}-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar\zeta}(\zeta).\frac{d\xi d\eta}{\zeta-\bar z}\ \ldots.(1.2)$$
$$0 = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{d\zeta}{\zeta+\bar z}-\frac{1}{\pi}\int_{\mathbb{D}_4} w_\zeta(\zeta).\frac{d\xi d\eta}{\zeta+z}\ \ldots.(1.3)$$
$$0 = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{d\zeta}{\zeta+\bar z}-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar\zeta}(\zeta).\frac{d\xi d\eta}{\zeta+\bar z}\ \ldots.(1.4)$$
$$0 = \frac{1}{2\pi i}\int_{\partial D} w(\zeta).\frac{d\zeta}{\zeta-iz}-\frac{1}{\pi}\int_{D} w_{\bar\zeta}(\zeta).\frac{d\xi d\eta}{\zeta-iz}\ \ldots.(1.5)$$
$$0 = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{d\zeta}{\zeta-i\bar z}-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar\zeta}(\zeta).\frac{d\xi d\eta}{\zeta-i\bar z}\ \ldots.(1.6)$$
$$0 = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{d\zeta}{\zeta+iz}-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar\zeta}(\zeta).\frac{d\xi d\eta}{\zeta+iz}\ \ldots.(1.7)$$
$$0 = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{d\zeta}{\zeta+i\bar z}-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar\zeta}(\zeta).\frac{d\xi d\eta}{\zeta+i\bar z}\ \ldots.(1.8)$$
$$0 = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{d\zeta}{\zeta-\frac{1}{z}}-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar\zeta}(\zeta).\frac{d\xi d\eta}{\zeta-\frac{1}{z}}\ \ldots.(1.9)$$
$$0 = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{d\zeta}{\zeta-\frac{1}{\bar z}}-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar\zeta}(\zeta).\frac{d\xi d\eta}{\zeta-\frac{1}{\bar z}}\ \ldots.(1.10)$$

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$$0=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{d\zeta}{\zeta+\frac{1}{z}}-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\frac{d\xi d\eta}{\zeta+\frac{1}{z}} \dots.(1.11)$$

$$0=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{d\zeta}{\zeta+\frac{1}{\bar{z}}}-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\frac{d\xi d\eta}{\zeta+\frac{1}{\bar{z}}} \dots.(1.12)$$

$$0=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{d\zeta}{\zeta-\frac{i}{z}}-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\frac{d\xi d\eta}{\zeta-\frac{i}{z}} \dots.(1.13)$$

$$0=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{d\zeta}{\zeta-\frac{i}{\bar{z}}}-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\frac{d\xi d\eta}{\zeta-\frac{i}{\bar{z}}} \dots.(1.14)$$

$$0=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{d\zeta}{\zeta+\frac{i}{z}}-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\frac{d\xi d\eta}{\zeta+\frac{i}{z}} \dots.(1.15)$$

$$0=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{d\zeta}{\zeta+\frac{i}{\bar{z}}}-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\frac{d\xi d\eta}{\zeta+\frac{i}{\bar{z}}} \dots.(1.16)$$

Adding (1.1) and (1.3) on the one hand, and adding (1.2) and (1.4) on the other, we find:

$$w(z)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{2\zeta}{\zeta^2-z^2}d\zeta-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\frac{2\zeta}{\zeta^2-z^2}d\xi d\eta \dots (1.1')$$

$$0=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{2\zeta}{\zeta^2-\bar{z}^2}d\zeta-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\frac{2\zeta}{\zeta^2-\bar{z}^2}d\xi d\eta \dots (1.2')$$

Adding (1.5) and (1.7) on the one hand, and adding (1.6) and (1.8) on the other, we find:

$$0=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{2\zeta}{\zeta^2+z^2}d\zeta-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\frac{2\zeta}{\zeta^2+z^2}d\xi d\eta \dots (1.3')$$

$$0=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{2\zeta}{\zeta^2+\bar{z}^2}d\zeta-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\frac{2\zeta}{\zeta^2+\bar{z}^2}d\xi d\eta \dots (1.4')$$

Adding (1.9) and (1.11) on the one hand, and adding (1.10) and (1.12) on the other hand, we find:

$$0=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{2\zeta z^2}{z^2\zeta^2-1}d\zeta-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\frac{2\zeta z^2}{z^2\zeta^2-1}d\xi d\eta \dots (1.5')$$

$$0=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{2\zeta\bar{z}^2}{\bar{z}^2\zeta^2-1}d\zeta-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\frac{2\zeta\bar{z}^2}{\bar{z}^2\zeta^2-1}d\xi d\eta \dots (1.6')$$

Adding (1.13) and (1.15) on the one hand, and adding (1.14) and (1.16) on the other hand, we find:

$$0=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{2\zeta z^2}{z^2\zeta^2+1}d\zeta-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\frac{2\zeta z^2}{z^2\zeta^2+1}d\xi d\eta \dots (1.7')$$

$$0=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{2\zeta\bar{z}^2}{\bar{z}^2\zeta^2+1}d\zeta-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\frac{2\zeta\bar{z}^2}{\bar{z}^2\zeta^2+1}d\xi d\eta \dots (1.8')$$

Adding (1.1′) and (1.3′) on the one hand, and adding (1.2′) and (1.4′) on the other, we find:

$$w(z)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{4\zeta^3}{\zeta^4-z^4}d\zeta-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\frac{4\zeta^3}{\zeta^4-z^4}d\xi d\eta \dots (1.1'')$$

$$0=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{4\zeta^3}{\zeta^4-\bar{z}^4}d\zeta-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\frac{4\zeta^3}{\zeta^4-\bar{z}^4}d\xi d\eta \dots (1.2'')$$

Adding (1.5′) and (1.7′) on the one hand, and adding (1.6′) and (1.8′) on the other hand, we find:

$$0=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{4\zeta^3 z^4}{z^4\zeta^4-1}d\zeta-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\frac{4\zeta^3 z^4}{z^4\zeta^4-1}d\xi d\eta \dots (1.3'')$$

$$0=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\frac{4\zeta^3\bar{z}^4}{\bar{z}^4\zeta^4-1}d\zeta-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\frac{4\zeta^3\bar{z}^4}{\bar{z}^4\zeta^4-1}d\xi d\eta \dots (1.4'')$$

Adding (1.1″) and (1.3″) on the one hand, and adding (1.2″) and (1.4″) on the other, we find:

$$w(z)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\left[\frac{4\zeta^3}{\zeta^4-z^4}+\frac{4\zeta^3 z^4}{z^4\zeta^4-1}\right]d\zeta-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\left[\frac{4\zeta^3}{\zeta^4-z^4}+\frac{4\zeta^3 z^4}{z^4\zeta^4-1}\right]d\xi d\eta \dots (1.1''')$$

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$$0=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_4} w(\zeta).\left[\frac{4\zeta^3}{\zeta^4-\bar{z}^4}+\frac{4\zeta^3\bar{z}^4}{\bar{z}^4\zeta^4-1}\right]d\zeta-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\left[\frac{4\zeta^3}{\zeta^4-\bar{z}^4}+\frac{4\zeta^3\bar{z}^4}{\bar{z}^4\zeta^4-1}\right]d\xi d\eta \dots(1.2''')$$

We can formulate (1.1''') as follows:

$$w(z)=\frac{1}{2\pi i}\int_{0<\arg\zeta<\frac{\pi}{4}}^{|\zeta|=1} w(\zeta).\left[\frac{4\zeta^3}{\zeta^4-z^4}+\frac{4\zeta^3 z^4}{z^4\zeta^4-1}\right]d\zeta+\frac{1}{2\pi i}\int_1^0 w\left(t.e^{\frac{\pi}{4}i}\right).\left[\frac{4t^3}{t^4+z^4}+\frac{4t^3z^4}{z^4t^4+1}\right]dt$$
$$+\frac{1}{2\pi i}\int_0^1 w(t).\left[\frac{4t^3}{t^4-z^4}+\frac{4t^3z^4}{z^4t^4-1}\right]dt-\frac{1}{\pi}\int_{\mathbb{D}_4} w_{\bar{\zeta}}(\zeta).\left[\frac{4\zeta^3}{\zeta^4-z^4}+\frac{4\zeta^3z^4}{z^4\zeta^4-1}\right]d\xi d\eta \dots(*)$$

Taking the two terminal facilities (1.2''') and reformulating the output we get:

$$0=\frac{1}{2\pi i}\int_{0<\arg\zeta<\frac{\pi}{4}}^{|\zeta|=1}\overline{w(\zeta)}.\left[\frac{4}{1-\zeta^4z^4}+\frac{4z^4}{z^4-\zeta^4}\right]\frac{d\zeta}{\zeta}-\frac{1}{2\pi i}\int_1^0\overline{w\left(t.e^{\frac{\pi}{4}i}\right)}.\left[\frac{4t^3}{t^4+z^4}+\frac{4t^3z^4}{z^4t^4+1}\right]dt$$
$$-\frac{1}{2\pi i}\int_0^1\overline{w(t)}.\left[\frac{4t^3}{t^4-z^4}+\frac{4t^3z^4}{z^4t^4-1}\right]dt-\frac{1}{\pi}\int_{\mathbb{D}_4}\overline{w_{\bar{\zeta}}(\zeta)}.\left[\frac{4\bar{\zeta}^3}{\bar{\zeta}^4-z^4}+\frac{4\bar{\zeta}^3z^4}{z^4\bar{\zeta}^4-1}\right]d\xi d\eta \dots(**)$$

Where:

$$|\zeta|=1\Rightarrow\zeta.\bar{\zeta}=1, d\bar{\zeta}=-\frac{d\zeta}{\zeta^2}$$
$$\mathrm{Im}(\zeta)=0\Rightarrow\zeta=\bar{\zeta}=t$$

Subtracting (* *) from (*) we get:

$$w(z)=\frac{1}{2\pi i}\int_{0<\arg\zeta<\frac{\pi}{4}}^{|\zeta|=1}\left\{w(\zeta).\left[\frac{4\zeta^3}{\zeta^4-z^4}+\frac{4\zeta^3z^4}{z^4\zeta^4-1}\right]-\overline{w(\zeta)}.\left[\frac{4}{1-\zeta^4z^4}+\frac{4z^4}{z^4-\zeta^4}\right]\right\}\frac{d\zeta}{\zeta}$$
$$+\frac{1}{2\pi i}\int_1^0\left\{w\left(t.e^{\frac{\pi}{4}i}\right).\left[\frac{4t^3}{t^4+z^4}+\frac{4t^3z^4}{z^4t^4+1}\right]+\overline{w\left(t.e^{\frac{\pi}{4}i}\right)}.\left[\frac{4t^3}{t^4+z^4}+\frac{4t^3z^4}{z^4t^4+1}\right]\right\}dt$$
$$+\frac{1}{2\pi i}\int_0^1\left\{w(t).\left[\frac{4t^3}{t^4-z^4}+\frac{4t^3z^4}{z^4t^4-1}\right]+\overline{w(t)}.\left[\frac{4t^3}{t^4-z^4}+\frac{4t^3z^4}{z^4t^4-1}\right]\right\}dt$$
$$-\frac{1}{\pi}\int_{\mathbb{D}_4}\left\{w_{\bar{\zeta}}(\zeta).\left[\frac{4\zeta^3}{\zeta^4-z^4}+\frac{4\zeta^3z^4}{z^4\zeta^4-1}\right]-\overline{w_{\bar{\zeta}}(\zeta)}.\left[\frac{4\bar{\zeta}^3}{\bar{\zeta}^4-z^4}+\frac{4\bar{\zeta}^3z^4}{z^4\bar{\zeta}^4-1}\right]\right\}d\xi d\eta$$

We put $w(\zeta)=\mathrm{Re}[w(\zeta)]+i\mathrm{Im}[w(\zeta)]$ in the previous equality, we find:

$$w(z)=\frac{2}{\pi i}\int_{0<\arg\zeta<\frac{\pi}{4}}^{|\zeta|=1}\left\{\mathrm{Re}[w(\zeta)].\left[\frac{\zeta^4+z^4}{\zeta^4-z^4}+\frac{\zeta^4z^4+1}{z^4\zeta^4-1}\right]\right\}\frac{d\zeta}{\zeta}+\frac{4}{\pi i}\int_1^0\mathrm{Re}\left[w\left(t.e^{\frac{\pi}{4}i}\right)\right].\left[\frac{t^3}{t^4+z^4}+\frac{t^3z^4}{z^4t^4+1}\right]dt$$
$$+\frac{4}{\pi i}\int_0^1\mathrm{Re}[\mathrm{w}(t)].\left[\frac{t^3}{t^4-z^4}+\frac{t^3z^4}{z^4t^4-1}\right]dt$$
$$-\frac{1}{\pi}\int_{\mathbb{D}_4}\left\{w_{\bar{\zeta}}(\zeta).\left[\frac{4\zeta^3}{\zeta^4-z^4}+\frac{4\zeta^3z^4}{z^4\zeta^4-1}\right]-\overline{w_{\bar{\zeta}}(\zeta)}.\left[\frac{4\bar{\zeta}^3}{\bar{\zeta}^4-z^4}+\frac{4\bar{\zeta}^3z^4}{z^4\bar{\zeta}^4-1}\right]\right\}d\xi d\eta$$

Which is required.

From the previous theorem, we can assign the modified Cauchy - Pompeiu integral effect in the region $\mathbb{D}_4$, in the following form:

$$T_{\mathbb{D}_4}f(z)=-\frac{1}{\pi}\int_{\mathbb{D}_4}\left\{f(\zeta).\left[\frac{4\zeta^3}{\zeta^4-z^4}+\frac{4\zeta^3z^4}{z^4\zeta^4-1}\right]-\overline{f(\zeta)}.\left[\frac{4\bar{\zeta}^3}{\bar{\zeta}^4-z^4}+\frac{4\bar{\zeta}^3z^4}{z^4\bar{\zeta}^4-1}\right]\right\}d\xi d\eta$$

2. Conclusion

In this research we came to:

- Modify the Cauchy - Pompeiu integral formula to correspond to the price of one tablet.
- Set the Cauchy - Pompeiu integrative effect at the price of one $\mathbb{D}_4$ disk.

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