International Journal of Neutrosophic Science (IJN) *Vol. 22, No. 03, PP. 36-52 2023*

On Radical of Neutrosophic Primary Submodule

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Abstract

In this paper, we introduce and study the concept of neutrosophic submodules and neutrosophic primary submodule with the help of the definition of a radical submodule, and we also study the properties of these submodules. Furthermore, homomorphic image and preimage of neutrosophic primary submodule are investigated.

Keywords: neutrosophic submodules; radical submodule; neutrosophic primary submodule.

1 introduction

One of the structures that play an important role in mathematics is the structure of fuzzy algebra, which has been used extensively in many fields such as computer science, information technology, theoretical physics, and control engineering. Since the introduction of fuzzy sets in 1965,[20] researchers have done extensive research on various concepts of abstract algebra in the space of fuzzy sets, such as Rosenfeld,[15] who in 1971 was the first to define the concept of fuzzy subgroups of a group. Many extensions of this concept have been proposed since then (especially in recent decades). In 1975, Relescu and Naegoita[12] applied the concept of fuzzy sets to the theory of modules.After presenting this definition, different types of fuzzy submodules were examined in the last two decades. In 1986, Atanassov[2] introduced intuitionistic fuzzy sets based on the degree of membership and degree of nonmembership, provided that the total membership and nonmembership should not exceed one another. In 1989, Biswas[5] applied the concept of intuitionistic fuzzy sets to group theory and studied the intuitionistic fuzzy subsets of a group. In the last few years a considerable number of papers have been done on fuzzy and intuitionistic fuzzy submodules in general, and fuzzy and intuitionistic fuzzy prime and primary submodules in particular. Hur et al.[7] introduced the notion of intuitionistic fuzzy prime ideals and intuitionistic fuzzy weakly completely prime ideals in a ring. The concept of the fuzzy prime submodule and fuzzy primary submodule was studied by Mashinchi and Zahedi in.[11] Also, intuitionistic fuzzy submodules and their properties were studied by many mathematicians.[4,14,18] The notion of neutrosophic nil radicals of neutrosophic ideals in rings is introduced, and related properties are investigated. In this paper, and in the first step we define the neutrosophic radical of an intuitionistic fuzzy submodule of an R-module M and so we present some of its properties similar to the studies in the module theory. Also and related to this definition, we will define neutrosophic primary submodule and investigate their properties. We show that for any neutrosophic primary submodule of an R-module M, with sup property its radical will be a neutrosophic prime ideal of R.

https://doi.org/10.54216/IJNS.220303

Received: April 26, 2023 Revised: July 08, 2023 Accepted: October 04, 2023

Throughout this paper, R is a commutative ring with unity and M is an unitary R-module. A proper submodule P of M is called a prime submodule provided that for any $r \in R$ and $m \in M$, $rm \in P$ implies that $m \in P$ or $r \in (P : M) = \{r \in R : rM \subseteq P\}$. We use the symbol θ for the zero element of a R-module.

Let A be a nonempty set. The neutrosophic set[19] on A is defined to be a structure

$$A := \{\langle x, \mu(x), \gamma(x), \psi(x)\rangle \mid x \in A\}, \tag{1}$$

where $\mu : A \to [0, 1]$ is a truth membership function, $\gamma : A \to [0, 1]$ is an indeterminate membership function, and $\psi : A \to [0, 1]$ is a false membership function. The neutrosophic set in (1) is simply denoted by $A = (\psi_A, \gamma_A, \psi_A)$.

Definition 1.1. [9] Let R be a ring. A neutrosophic set $A = (\mu_A, \gamma_A, \psi_A)$ of R is said to be a neutrosophic ideal of R if it satisfies

$$(\forall x, y \in R)\left(\begin{array}{l} \mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y)\}, \\ \gamma_A(x-y) \geq \min\{\gamma_A(x), \gamma_A(y)\}, \\ \psi_A(x-y) \leq \max\{\psi_A(x), \psi_A(y)\} \end{array}\right), \tag{2}$$

$$(\forall x, y \in R)\left(\begin{array}{l} \mu_A(xy) \geq \max\{\mu_A(x), \mu_A(y)\}, \\ \gamma_A(xy) \geq \max\{\gamma_A(x), \gamma_A(y)\}, \\ \psi_A(xy) \leq \min\{\psi_A(x), \psi_A(y)\} \end{array}\right). \tag{3}$$

2 Properties of neutrosophic submodule

Definition 2.1. Let M be an R-module and $A \in \mathcal{N}(M)$. Then A is called a neutrosophic submodule if

1. $\mu_A(\theta) = 1$, $\gamma_A(\theta) = 1$, $\psi_A(\theta) = 0$,
2. $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)$, $\gamma_A(x + y) \geq \gamma_A(x) \wedge \gamma_A(y)$, $\psi_A(x + y) \leq \psi_A(x) \vee \psi_A(y)$ for all $x, y \in M$
3. $\mu_A(rx) \geq \mu_A(x)$, $\gamma_A(rx) \geq \gamma_A(x)$, $\psi_A(rx) \leq \psi_A(x)$ for all $x \in M$ and $r \in R$.

We denote the class of all neutrosophic submodule of an R-module M, by $\mathcal{N}M(M)$. Notice that, when $R = M$, then $A \in \mathcal{N}M(M)$ if and only if $\mu_A(\theta) = 1$, $\gamma_A(\theta) = 1$, $\psi_A(\theta) = 0$ and $A \in \mathcal{N}I(R)$. Now, for a neutrosophic submodule, we define a neutrosophic radical.

Definition 2.2. Let M be an R-module and $A \in \mathcal{N}(M)$. The neutrosophic subset $\sqrt{A} = (\sqrt{\mu_A}, \sqrt{\gamma_A}, \sqrt{\psi_A})$ of R is called the neutrosophic radical of A and is defined as follows: $\sqrt{\mu_A}(r) = \bigvee\limits_{n\in N}\bigwedge\limits_{m\in M} \mu_A(r^n \cdot m)$, $\sqrt{\gamma_A}(r) = \bigwedge\limits_{n\in N}\bigvee\limits_{m\in M} \gamma_A(r^n \cdot m)$ and $\sqrt{\psi_A}(r) = \bigvee\limits_{n\in N}\bigwedge\limits_{m\in M} \psi_A(r^n \cdot m)$ for all $r \in R$, $m \in M$, $n \in N$. It is clear that when $M = R$, then the neutrosophic fuzzy radical of a neutrosophic ideal of R will be defined by: $\sqrt{A} = (\sqrt{\mu_A}, \sqrt{\gamma_A}, \sqrt{\psi_A})$, where $\sqrt{\mu_A}(r) = \bigvee\limits_{n\in N} \mu_A(r^n)$, $\sqrt{\gamma_A}(r) = \bigvee\limits_{n\in N} \gamma_A(r^n)$ and $\sqrt{\psi_A}(r) = \bigwedge\limits_{n\in N} \psi_A(r^n)$.

Now we show that for any neutrosophic submodule A of M, $\sqrt{A}$ is a neutrosophic ideal of R.

Proposition 2.3. *Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic submodule of M, then $\sqrt{A} = (\sqrt{\mu_A}, \sqrt{\gamma_A}, \sqrt{\psi_A})$ is a neutrosophic ideal of R.*

Proof. Let $r_1, r_2 \in R$, then $\sqrt{\mu_A}(r_1 + r_2)$

$$\begin{aligned} &= \bigvee_{n\in N}\bigwedge_{m\in M} \mu_A((r_1 + r_2)^n \cdot m) \\ &= \bigvee_{n\in N}\bigwedge_{m\in M} \mu_A((r_1 + r_2)^{2n} \cdot m) \vee \bigvee_{n\in N}\bigwedge_{m\in M} \mu_A((r_1 + r_2)^{2n+1} \cdot m). \end{aligned}$$

Now, we have $\bigwedge\limits_{m\in M}\mu_A((r_1+r_2)^{2n}\cdot m)$

$$\begin{aligned}
&= \bigwedge_{m\in M}((\sum_{i=0}^{2n} C_i^{2n} r_1^{2n-i} r_2^{i})\cdot m)\\
&\geq \bigwedge_{m\in M}(\mu_A(r_1^{2n}\cdot m)\wedge\mu_A(r_1^{2n-1}\cdot r_2\cdot m)\wedge\cdots\wedge\\
&\quad \mu_A(r_1^{n}\cdot r_2^{n}\cdot m)\wedge\mu_A(r_1^{n+1}\cdot r_2^{n+1}\cdot m)\wedge\cdots\wedge\mu_A(r_2^{2n}\cdot m))\\
&\geq \bigwedge_{m\in M}(\mu_A(r_1^{2n}\cdot m)\wedge\mu_A(r_1^{2n-1}\cdot m)\wedge\cdots\wedge\\
&\quad \mu_A(r_1^{n}\cdot m)\wedge\mu_A(r_1^{n+1}\cdot m)\wedge\cdots\wedge\mu_A(r_2^{2n}\cdot m))\\
&= \bigwedge_{m\in M}(\mu_A(r_1^{n}\cdot m)\wedge\mu_A(r_2^{n+1}\cdot m))\\
&\geq \bigwedge_{m\in M}(\mu_A(r_1^{n}\cdot m)\wedge\mu_A(r_2^{n}\cdot m))\\
&= (\bigwedge_{m\in M}\mu_A(r_1^{n}\cdot m))\wedge(\bigwedge_{m\in M}\mu_A(r_2^{n}\cdot m)).
\end{aligned}$$

Also, $\bigwedge\limits_{m\in M}\mu_A((r_1+r_2)^{2n+1}\cdot m)$

$$\begin{aligned}
&= \bigwedge_{m\in M}\mu_A((\sum_{i=0}^{2n+1} C_i^{2n+1} r_1^{2n+1-i} r_2^{i})\cdot m)\\
&\geq \bigwedge_{m\in M}(\mu_A(r_1^{2n+1}\cdot m)\wedge\mu_A(r_1^{2n}\cdot r_2\cdot m)\wedge\cdots\wedge\\
&\quad \mu_A(r_1^{n}\cdot r_2^{n+1}\cdot m)\wedge\cdots\wedge\mu_A(r_2^{2n+1}\cdot m))\\
&\geq \bigwedge_{m\in M}(\mu_A(r_1^{2n+1}\cdot m)\wedge\mu_A(r_1^{2n}\cdot m)\wedge\cdots\wedge\\
&\quad \mu_A(r_1^{n+1}\cdot m)\wedge\cdots\wedge\mu_A(r_2^{n+1}\cdot m)\wedge\cdots\wedge\mu_A(r_2^{2n+1}\cdot m))\\
&= \bigwedge_{m\in M}(\mu_A(r_1^{n+1}\cdot m)\wedge\mu_A(r_2^{n+1}\cdot m))\\
&\geq \bigwedge_{m\in M}(\mu_A(r_1^{n}\cdot m)\wedge\mu_A(r_2^{n}\cdot m))\\
&= \bigwedge_{m\in M}\mu_A(r_1^{n}\cdot m)\wedge\bigwedge_{m\in M}\mu_A(r_2^{n}\cdot m).
\end{aligned}$$

Therefore, $\sqrt{\mu_A}(r_1+r_2)$

$$\begin{aligned}
&= \bigvee_{n\in N}\bigwedge_{m\in M}\mu_A((r_1+r_2)^{2n}\cdot m)\vee\bigvee_{n\in N}\bigwedge_{m\in M}\mu_A((r_1+r_2)^{2n+1}\cdot m)\\
&\geq \bigvee_{n\in N}((\bigwedge_{m\in M}\mu_A(r_1^{n}\cdot m))\wedge\bigwedge_{m\in M}\mu_A(r_2^{n}\cdot m))\vee\bigvee_{n\in N}(\bigwedge_{m\in M}\mu_A(r_1^{n}\cdot m))\wedge\bigwedge_{m\in M}\mu_A(r_2^{n}\cdot m)))\\
&= \bigvee_{n\in N}(\bigwedge_{m\in M}\mu_A(r_1^{n}\cdot m)\wedge\bigwedge_{m\in M}\mu_A(r_2^{n}\cdot m))\\
&= (\bigvee_{n\in N}\bigvee_{n\in N}(\bigwedge_{m\in M}\mu_A(r_1^{n}\cdot m))\wedge(\bigvee_{n\in N}\bigwedge_{m\in M}\mu_A(r_2^{n}\cdot m))\\
&= \sqrt{\mu_A}(r_1)\wedge\sqrt{\mu_A}(r_2).
\end{aligned}$$

Also

$$\begin{aligned}
\sqrt{\gamma_A}(r_1+r_2) &= \bigvee_{n\in N}\bigwedge_{m\in M}\gamma_A((r_1+r_2)^{n}\cdot m)\\
&= \bigvee_{n\in N}\bigwedge_{m\in M}\gamma_A((r_1+r_2)^{2n}\cdot m)\vee\bigvee_{n\in N}\bigwedge_{m\in M}\gamma_A((r_1+r_2)^{2n+1}\cdot m).
\end{aligned}$$

Now, we have $\bigwedge\limits_{m\in M}\gamma_A((r_1+r_2)^{2n}\cdot m)$

$$\begin{aligned}
&= \bigwedge_{m\in M}((\sum_{i=0}^{2n} C_i^{2n} r_1^{2n-i} r_2^{i})\cdot m)\\
&\geq \bigwedge_{m\in M}(\gamma_A(r_1^{2n}\cdot m)\wedge\gamma_A(r_1^{2n-1}\cdot r_2\cdot m)\wedge\cdots\wedge\\
&\quad \gamma_A(r_1^{n}\cdot r_2^{n}\cdot m)\wedge\gamma_A(r_1^{n+1}\cdot r_2^{n+1}\cdot m)\wedge\cdots\wedge\gamma_A(r_2^{2n}\cdot m))\\
&\geq \bigwedge_{m\in M}(\gamma_A(r_1^{2n}\cdot m)\wedge\gamma_A(r_1^{2n-1}\cdot m)\wedge\cdots\wedge\\
&\quad \gamma_A(r_1^{n}\cdot m)\wedge\gamma_A(r_1^{n+1}\cdot m)\wedge\cdots\wedge\gamma_A(r_2^{2n}\cdot m))\\
&= \bigwedge_{m\in M}(\gamma_A(r_1^{n}\cdot m)\wedge\gamma_A(r_2^{n+1}\cdot m))\\
&\geq \bigwedge_{m\in M}(\gamma_A(r_1^{n}\cdot m)\wedge\gamma_A(r_2^{n}\cdot m))\\
&= (\bigwedge_{m\in M}\gamma_A(r_1^{n}\cdot m))\wedge(\bigwedge_{m\in M}\gamma_A(r_2^{n}\cdot m)).
\end{aligned}$$

Received: April 26, 2023 Revised: July 08, 2023 Accepted: October 04, 2023

Also, $\bigwedge\limits_{m\in M}\gamma_A((r_1+r_2)^{2n+1}\cdot m)$

$$\begin{aligned}
&= \bigwedge_{m\in M}\gamma_A((\sum_{i=0}^{2n+1}C_i^{2n+1}r_1^{2n+1-i}r_2^i)\cdot m)\\
&\geq \bigwedge_{m\in M}(\gamma_A(r_1^{2n+1}\cdot m)\wedge\gamma_A(r_1^{2n}\cdot r_2\cdot m)\wedge\cdots\wedge\\
&\quad \gamma_A(r_1^n\cdot r_2^{n+1}\cdot m)\wedge\cdots\wedge\gamma_A(r_2^{2n+1}\cdot m))\\
&\geq \bigwedge_{m\in M}(\gamma_A(r_1^{2n+1}\cdot m)\wedge\gamma_A(r_1^{2n}\cdot m)\wedge\cdots\wedge\\
&\quad \gamma_A(r_1^{n+1}\cdot m)\wedge\cdots\wedge\gamma_A(r_2^{n+1}\cdot m)\wedge\cdots\wedge\gamma_A(r_2^{2n+1}\cdot m))\\
&= \bigwedge_{m\in M}(\gamma_A(r_1^{n+1}\cdot m)\wedge\gamma_A(r_2^{n+1}\cdot m))\\
&\geq \bigwedge_{m\in M}(\gamma_A(r_1^n\cdot m)\wedge\gamma_A(r_2^n\cdot m))\\
&= \bigwedge_{m\in M}\gamma_A(r_1^n\cdot m)\wedge\bigwedge_{m\in M}\gamma_A(r_2^n\cdot m).
\end{aligned}$$

Therefore, $\sqrt{\gamma_A}(r_1+r_2)$

$$\begin{aligned}
&= \bigvee_{n\in N}\bigwedge_{m\in M}\gamma_A((r_1+r_2)^{2n}\cdot m)\vee\bigvee_{n\in N}\bigwedge_{m\in M}\gamma_A((r_1+r_2)^{2n+1}\cdot m)\\
&\geq \bigvee_{n\in N}((\bigwedge_{m\in M}\gamma_A(r_1^n\cdot m))\wedge\bigwedge_{m\in M}\gamma_A(r_2^n\cdot m))\vee\bigvee_{n\in N}(\bigwedge_{m\in M}\gamma_A(r_1^n\cdot m))\wedge\bigwedge_{m\in M}\gamma_A(r_2^n\cdot m)))\\
&= \bigvee_{n\in N}(\bigwedge_{m\in M}\gamma_A(r_1^n\cdot m)\wedge\bigwedge_{m\in M}\gamma_A(r_2^n\cdot m))\\
&= (\bigvee_{n\in N}\bigvee_{n\in N}(\bigwedge_{m\in M}\gamma_A(r_1^n\cdot m))\wedge(\bigvee_{n\in N}\bigwedge_{m\in M}\gamma_A(r_2^n\cdot m))\\
&= \sqrt{\gamma_A}(r_1)\wedge\sqrt{\gamma_A}(r_2).
\end{aligned}$$

Similarly, $\sqrt{\psi_A}(r_1+r_2)$

$$\begin{aligned}
&= \bigwedge_{n\in N}\bigvee_{m\in M}\psi_A((r_1+r_2)^n\cdot m)\\
&= \bigwedge_{n\in N}\bigvee_{m\in M}\psi_A((\sum_{i=0}^{n}C_i^n r_1^{n-i}r_2^i)\cdot m)\\
&\leq \bigwedge_{n\in N}\bigvee_{m\in M}(\psi_A(r_1^n\cdot m)\vee\psi_A(r_1^{n-1}\cdot r_2\cdot m)\vee\cdots\vee\psi_A(r_1\cdot r_2^{n-1}\cdot m)\vee\cdots\wedge\mu_A(r_2^n\cdot m)\\
&\leq \bigwedge_{n\in N}\bigvee_{m\in M}(\psi_A(r_1^n\cdot m)\vee\psi_A(r_1^{n-1}\cdot m)\vee\cdots\vee\psi_A(r_2^{n-1}\cdot m)\vee\cdots\wedge\mu_A(r_2^n\cdot m)\\
&= \bigwedge_{n\in N}\bigvee_{m\in M}(\psi_A(r_1^n\cdot m)\vee\psi_A(r_2^n\cdot m))\\
&= (\bigwedge_{n\in N}\bigvee_{m\in M}(\psi_A(r_1^n\cdot m))\vee(\bigwedge_{n\in N}\bigvee_{m\in M}(\psi_A(r_2^n\cdot m))\\
&= \sqrt{\psi_A}(r_1)\vee\sqrt{\psi_A}(r_2).
\end{aligned}$$

And

$$\begin{aligned}
\sqrt{\mu_A}(r_1r_2) &= \bigvee_{n\in N}\bigwedge_{m\in M}(\mu_A((r_1r_2)^n\cdot m)\\
&= \bigvee_{n\in N}\bigwedge_{m\in M}(\mu_A(r_1^nr_2^n)\cdot m)\\
&\geq \bigvee_{n\in N}\bigwedge_{m\in M}(\gamma_A(r_1^n\cdot m))\\
&= \sqrt{\mu_A}(r_1).\\
\sqrt{\mu_A}(r_1r_2) &= \bigvee_{n\in N}\bigwedge_{m\in M}(\mu_A((r_1r_2)^n\cdot m)\\
&= \bigvee_{n\in N}\bigwedge_{m\in M}(\mu_A(r_1^nr_2^n)\cdot m)\\
&\geq \bigvee_{n\in N}\bigwedge_{m\in M}(\gamma_A(r_2^n\cdot m))\\
&= \sqrt{\mu_A}(r_2).
\end{aligned}$$

Hence $\sqrt{\mu_A}(r_1r_2)\geq\sqrt{\mu_A}(r_1)\vee\sqrt{\mu_A}(r_2)$.
And

$$\begin{aligned}
\sqrt{\gamma_A}(r_1r_2) &= \bigvee_{n\in N}\bigwedge_{m\in M}(\gamma_A((r_1r_2)^n\cdot m)\\
&= \bigvee_{n\in N}\bigwedge_{m\in M}(\gamma_A(r_1^nr_2^n)\cdot m)\\
&\geq \bigvee_{n\in N}\bigwedge_{m\in M}(\gamma_A(r_1^n\cdot m))\\
&= \sqrt{\gamma_A}(r_1).
\end{aligned}$$

https://doi.org/10.54216/IJNS.220303

Received: April 26, 2023 Revised: July 08, 2023 Accepted: October 04, 2023

$$\begin{aligned}
\sqrt{\gamma_A}(r_1r_2) &= \bigvee_{n\in N}\bigwedge_{m\in M}(\gamma_A((r_1r_2)^n\cdot m)\\
&= \bigvee_{n\in N}\bigwedge_{m\in M}(\gamma_A(r_1^nr_2^n)\cdot m)\\
&\geq \bigvee_{n\in N}\bigwedge_{m\in M}(\gamma_A(r_2^n\cdot m))\\
&= \sqrt{\gamma_A}(r_2).
\end{aligned}$$

Hence $\sqrt{\gamma_A}(r_1r_2) \geq \sqrt{\gamma_A}(r_1) \vee \sqrt{\gamma_A}(r_2)$.

Also

$$\begin{aligned}
\sqrt{\psi_A}(r_1r_2) &= \bigwedge_{n\in N}\bigvee_{m\in M}(\psi_A((r_1r_2)^n\cdot m)\\
&= \bigwedge_{n\in N}\bigvee_{m\in M}(\psi_A(r_1^nr_2^n)\cdot m)\\
&\geq \bigwedge_{n\in N}\bigvee_{m\in M}(\psi_A(r_1^n\cdot m))\\
&= \sqrt{\psi_A}(r_1).
\end{aligned}$$

$$\begin{aligned}
\sqrt{\psi_A}(r_1r_2) &= \bigwedge_{n\in N}\bigvee_{m\in M}(\psi_A((r_1r_2)^n\cdot m)\\
&= \bigwedge_{n\in N}\bigvee_{m\in M}(\psi_A(r_1^nr_2^n)\cdot m)\\
&\geq \bigwedge_{n\in N}\bigvee_{m\in M}(\psi_A(r_2^n\cdot m))\\
&= \sqrt{\psi_A}(r_2).
\end{aligned}$$

Hence $\sqrt{\psi_A}(r_1r_2) \leq \sqrt{\psi_A}(r_1) \wedge \sqrt{\psi_A}(r_2)$.
Now let $r \in R$,

$$\begin{aligned}
\sqrt{\mu_A}(-r) &= \bigvee_{n\in N}\bigwedge_{m\in M}(\mu_A((-r)^n\cdot m)\\
&= \bigvee_{n\in N}\bigwedge_{m\in M}(\mu_A(r^n)\cdot m)\\
&= \sqrt{\mu_A}(r).
\end{aligned}$$

$$\begin{aligned}
\sqrt{\gamma_A}(-r) &= \bigvee_{n\in N}\bigwedge_{m\in M}(\gamma_A((-r)^n\cdot m)\\
&= \bigvee_{n\in N}\bigwedge_{m\in M}(\gamma_A(r^n)\cdot m)\\
&= \sqrt{\gamma_A}(r).
\end{aligned}$$

$$\begin{aligned}
\sqrt{\psi_A}(-r) &= \bigwedge_{n\in N}\bigvee_{m\in M}(\psi_A((-r)^n\cdot m)\\
&= \bigwedge_{n\in N}\bigvee_{m\in M}(\psi_A(r^n)\cdot m)\\
&= \sqrt{\psi_A}(r).
\end{aligned}$$

Therefore, $\sqrt{A}$ is a neutrosophic ideal of R. ☐

In the following proposition, we will give some properties of neutrosophic radicals.

Proposition 2.4. *Let* $A, B \in \mathcal{N}M(M)$, *then*

1. $\sqrt{\sqrt{A}} = \sqrt{A}$.
2. $A \subseteq B \Rightarrow \sqrt{A} \subseteq \sqrt{B}$.
3. $\sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$.
4. $\sqrt{\sqrt{A} + \sqrt{B}} \subseteq \sqrt{A + B}$.

Proof. (1). Since $\sqrt{\sqrt{A}} = (\sqrt{\sqrt{\mu_A}}, \sqrt{\sqrt{\gamma_A}}, \sqrt{\sqrt{\psi_A}})$, by Proposition 2.5, $\sqrt{\sqrt{A}}$ is a neutrosophic ideal of R. Then

$$\begin{aligned}
\sqrt{\sqrt{\mu_A}}(r) &= \bigvee_{n\in N}(\sqrt{\mu_A}(r^n) \\
&= \bigvee_{n\in N}(\bigvee_{n'\in N}\bigwedge_{m\in M}(\mu_A((r^n)^{n'})\cdot m)) \\
&= \bigvee_{n\in N}(\bigvee_{n'\in N}\bigwedge_{m\in M}(\mu_A((r^{nn'})\cdot m)) \\
&= \bigvee_{n''\in N}\bigwedge_{m\in M}(\mu_A((r^{n''})\cdot m)) \\
&= \sqrt{\mu_A}(r).
\end{aligned}$$

And

$$\begin{aligned}
\sqrt{\sqrt{\gamma_A}}(r) &= \bigvee_{n\in N}(\sqrt{\gamma_A}(r^n) \\
&= \bigvee_{n\in N}(\bigvee_{n'\in N}\bigwedge_{m\in M}(\gamma_A((r^n)^{n'})\cdot m)) \\
&= \bigvee_{n\in N}(\bigvee_{n'\in N}\bigwedge_{m\in M}(\gamma_A((r^{nn'})\cdot m)) \\
&= \bigvee_{n''\in N}\bigwedge_{m\in M}(\gamma_A((r^{n''})\cdot m)) \\
&= \sqrt{\gamma_A}(r).
\end{aligned}$$

Also

$$\begin{aligned}
\sqrt{\sqrt{\psi_A}}(r) &= \bigwedge_{n\in N}(\sqrt{\psi_A}(r^n) \\
&= \bigwedge_{n\in N}(\bigwedge_{n'\in N}\bigvee_{m\in M}(\psi_A((r^n)^{n'})\cdot m)) \\
&= \bigwedge_{n\in N}(\bigwedge_{n'\in N}\bigvee_{m\in M}(\psi_A((r^{nn'})\cdot m)) \\
&= \bigwedge_{n''\in N}\bigvee_{m\in M}(\psi_A((r^{n''})\cdot m)) \\
&= \sqrt{\psi_A}(r).
\end{aligned}$$

Hence $\sqrt{\sqrt{A}} = (\sqrt{\mu_A}, \sqrt{\gamma_A}, \sqrt{\psi_A}) = \sqrt{A}$.
(3). We have $A\cap B = \langle m, \mu_A(m)\wedge\mu_B(m), \gamma(m)\wedge\gamma_B(m), \psi_A(m)\vee\psi_B(m)\rangle$. Then

$$\begin{aligned}
\sqrt{\mu_A\wedge\mu_B}(r) &= \bigvee_{n\in N}\bigwedge_{m\in M}(\mu_A\wedge\mu_B)(r^n\cdot m) \\
&= \bigvee_{n\in N}\bigwedge_{m\in M}(\mu_A(r^n\cdot m)\wedge\mu_B(r^n\cdot m)) \\
&= (\bigvee_{n\in N}\bigwedge_{m\in M}(\mu_A(r^n\cdot m))\wedge(\bigvee_{n\in N}\bigwedge_{m\in M}(\mu_B(r^n\cdot m)) \\
&= \bigvee_{n''\in N}\bigwedge_{m\in M}(\mu_A((r^{n''})\cdot m)) \\
&= \sqrt{\mu_A}(r)\wedge\sqrt{\mu_B}(r). \\
\sqrt{\gamma_A\wedge\gamma_B}(r) &= \bigvee_{n\in N}\bigwedge_{m\in M}(\gamma_A\wedge\gamma_B)(r^n\cdot m) \\
&= \bigvee_{n\in N}\bigwedge_{m\in M}(\gamma_A(r^n\cdot m)\wedge\gamma_B(r^n\cdot m)) \\
&= (\bigvee_{n\in N}\bigwedge_{m\in M}(\gamma_A(r^n\cdot m))\wedge(\bigvee_{n\in N}\bigwedge_{m\in M}(\gamma_B(r^n\cdot m)) \\
&= \bigvee_{n''\in N}\bigwedge_{m\in M}(\gamma_A((r^{n''})\cdot m)) \\
&= \sqrt{\gamma_A}(r)\wedge\sqrt{\gamma_B}(r). \\
\sqrt{\psi_A\vee\psi_B}(r) &= \bigwedge_{n\in N}\bigvee_{m\in M}(\psi_A\vee\psi_B)(r^n\cdot m) \\
&= \bigwedge_{n\in N}\bigvee_{m\in M}(\psi_A(r^n\cdot m)\vee\psi_B(r^n\cdot m)) \\
&= (\bigwedge_{n\in N}\bigvee_{m\in M}(\psi_A(r^n\cdot m))\vee(\bigwedge_{n\in N}\bigvee_{m\in M}(\psi_B(r^n\cdot m)) \\
&= \bigwedge_{n''\in N}\bigvee_{m\in M}(\psi_A((r^{n''})\cdot m)) \\
&= \sqrt{\psi_A}(r)\vee\sqrt{\psi_B}(r).
\end{aligned}$$

Therefore, $\sqrt{A\cap B} = (r, \sqrt{\mu_A}(r)\wedge\sqrt{\mu_B}(r), \sqrt{\gamma_A}(r)\wedge\sqrt{\gamma_B}(r), \sqrt{\psi_A}(r)\vee\sqrt{\psi_B}(r)) = \sqrt{A}\cap\sqrt{B}$.
(4). We have

$$\begin{aligned}
(\mu_A+\mu_B)(x) &= \bigvee_{y+z=x}(\mu_A(y)\wedge\mu_B(z)) \\
&\geq \mu_A(x)\wedge\mu_B(\theta),\ where\ \mu_B(\theta)=1 \\
&= \mu_A(x).
\end{aligned}$$

$$\begin{aligned}(\gamma_A+\gamma_B)(x) &= \bigvee_{y+z=x}(\gamma_A(y)\wedge\gamma_B(z))\\ &\geq \gamma_A(x)\wedge\gamma_B(\theta),\ where\ \gamma_B(\theta)=1\\ &= \gamma_A(x).\end{aligned}$$

and

$$\begin{aligned}(\psi_A+\psi_B)(x) &= \bigvee_{y+z=x}(\psi_A(y)\vee\psi_B(z))\\ &\geq \psi_A(x)\vee\psi_B(\theta),\ where\ \psi_B(\theta)=0\\ &= \psi_A(x).\end{aligned}$$

We have

$$\begin{aligned}(\mu_A+\mu_B)(x) &= \bigvee_{y+z=x}(\mu_A(y)\wedge\mu_B(z))\\ &\geq \mu_A(x)\wedge\mu_B(\theta),\ where\ \mu_B(\theta)=1\\ &= \mu_A(x).\\ (\gamma_A+\gamma_B)(x) &= \bigvee_{y+z=x}(\gamma_A(y)\wedge\gamma_B(z))\\ &\geq \gamma_A(x)\wedge\gamma_B(\theta),\ where\ \gamma_B(\theta)=1\\ &= \gamma_A(x),\end{aligned}$$

and

$$\begin{aligned}(\psi_A+\psi_B)(x) &= \bigwedge_{y+z=x}(\psi_A(y)\wedge\psi_B(z))\\ &\leq \psi_A(x)\wedge\psi_B(\theta),\ where\ \psi_B(\theta)=0\\ &= \psi_A(x).\end{aligned}$$

Now $\sqrt{\mu_A}\subseteq\sqrt{\mu_A+\mu_B}, \sqrt{\mu_B}\subseteq\sqrt{\mu_A+\mu_B}\Rightarrow\sqrt{\mu_A}+\sqrt{\mu_B}\subseteq\sqrt{\mu_A+\mu_B}$. Hence $\sqrt{\sqrt{\mu_A}+\sqrt{\mu_B}}\subseteq\sqrt{\sqrt{\mu_A+\mu_B}}=\sqrt{\mu_A+\mu_B}$. Similarly, $\sqrt{\sqrt{\gamma_A}+\sqrt{\gamma_B}}\subseteq\sqrt{\sqrt{\gamma_A+\gamma_B}}=\sqrt{\gamma_A+\gamma_B}$. And so $\sqrt{\psi_A}\supseteq\sqrt{\psi_A+\psi_B}, \sqrt{\psi_B}\supseteq\sqrt{\psi_A+\psi_B}\Rightarrow\sqrt{\psi_A}+\sqrt{\psi_B}\supseteq\sqrt{\psi_A+\psi_B}$. Hence $\sqrt{\sqrt{\psi_A}+\sqrt{\psi_B}}\supseteq\sqrt{\sqrt{\psi_A+\psi_B}}=\sqrt{\psi_A+\psi_B}$. Therefore $\sqrt{\sqrt{A}+\sqrt{B}}\subseteq\sqrt{A+B}$. □

Definition 2.5. If $A\in\mathcal{N}(R)$, then A is said to have the sup property if for every $\emptyset\neq Y\subseteq R$, there is a $y_0\in Y$ such that $\sup_{y\in Y}\mu_A(y)=\mu_A(y_0)$, $\sup_{y\in Y}\gamma_A(y)=\gamma_A(y_0)$, $\inf_{y\in Y}\psi_A(y)=\psi_A(y_0)$.

Definition 2.6. Suppose that $A\in\mathcal{N}(X)$. Then the set $X_A^{(\alpha,\beta,\omega)}=\{x: x\in X$ such that $\mu_A(x)\geq\alpha,\gamma_A(x)\geq\beta,\psi_A(x)\leq\omega\}$, where $\alpha,\beta,\omega\in[0,1]$ with $\alpha+\beta+\omega\leq 1$, is called (α,β,ω)-cut set (crisp set).

Proposition 2.7. *Let $A\in\mathcal{N}M(M)$ and suppose that A has* sup *property, then $(\sqrt{M_A})^{(\alpha,\beta,\omega)}=\sqrt{M_A^{(\alpha,\beta,\omega)}}$, where $M_A^{(\alpha,\beta,\omega)}=\{x\in M:\mu_A(x)\geq\alpha,\gamma_A(x)\geq\beta,\psi_A(x)\leq\omega,\alpha+\beta+\omega\leq 1,(\alpha,\beta,\omega)\in Im\mu_A\times Im\gamma_A\times Im\psi_A\}$.*

Proof. Let A be a neutrosophic submodule of M with sup property, then $(\sqrt{M_A})^{(\alpha,\beta,\omega)}$

$$\begin{aligned}&= \{r\in R:\sqrt{\mu_A}(r)\geq\alpha,\sqrt{\gamma_A}(r)\geq\beta,\sqrt{\psi_A}(r)\leq\omega\}\\ &= \{r\in R:\bigvee_{n\in N}\bigwedge_{m\in M}\mu_A(r^n\cdot m)\geq\alpha,\bigvee_{n\in N}\bigwedge_{m\in M}\gamma_A(r^n\cdot m)\geq\beta,\bigwedge_{n\in N}\bigvee_{m\in M}\psi_A(r^n\cdot m)\leq\omega\}\\ &= \{r\in R:\bigwedge_{m\in M}\mu_A(r^{n_1}\cdot m)\geq\alpha,\bigwedge_{m\in M}\gamma_A(r^{n_2}\cdot m)\geq\beta,\bigvee_{m\in M}\psi_A(r^{n_3}\cdot m)\leq\omega,n_1,n_2,n_3\in N\}\\ &= \{r\in R:\mu_A(r^{n_1}r^{n_2}\cdot m)\geq\alpha,\gamma_A(r^{n_1}r^{n_2}\cdot m)\geq\beta,\psi_A(r^{n_1}r^{n_2}\cdot m)\leq\omega\ \forall m\in M\}\\ &= \{r\in R:n''=n_1+n_2,\mu_A(r^{n''}\cdot m)\geq\alpha,\gamma_A(r^{n''}\cdot m)\geq\beta,\psi_A(r^{n''}\cdot m)\leq\omega\ \forall m\in M\}\\ &= \{r\in R:r^{n''}\cdot m\in M^{(\alpha,\beta,\omega)}\ \forall m\in M\ for\ some\ n''\in N\}\\ &= \{r\in R:r^{n''}\cdot M\subseteq M^{(\alpha,\beta,\omega)}\ for\ some\ n''\in N\}\\ &= \sqrt{M_A^{(\alpha,\beta,\omega)}},\end{aligned}$$

□

Definition 2.8. Let $A=(\mu_A,\gamma_A,\psi_A)$ be a neutrosophic submodule of M. Then the neutrosophic subset $\overline{A}=(\mu_{\overline{A}},\gamma_{\overline{A}},\psi_{\overline{A}})$, is defined $\mu_{\overline{A}}(r)=\bigwedge_{m\in M}\mu_A(r\cdot m)$, $\gamma_{\overline{A}}(r)=\bigwedge_{m\in M}\gamma_A(r\cdot m)$, $\psi_{\overline{A}}(r)=\bigvee_{m\in M}\psi_A(r\cdot m)$, where $\overline{A}=(A,M)$.

Proposition 2.9. *Let A be a neutrosophic submodule of M. Then, $\overline{A}$ is a neutrosophic ideal of R.*

Proof. Let $r_1, r_2 \in R$. then

$$\begin{aligned}
\mu_A(r_1 - r_2) &= \bigwedge_{m\in M} \mu_A((r_1 - r_2)\cdot m) \\
&= \bigwedge_{m\in M} \mu_A(r_1 \cdot m - r_2 \cdot m) \\
&\geq \bigwedge_{m\in M} (\mu_A(r_1 \cdot m) \wedge \mu_A(r_2 \cdot m)) \\
&= (\bigwedge_{m\in M} \mu_A(r_1 \cdot m)) \wedge (\bigwedge_{m\in M} \mu_A(r_2 \cdot m)) \\
&= \mu_{\overline{A}}(r_1) \wedge \mu_{\overline{A}}(r_2).
\end{aligned}$$

$$\begin{aligned}
\gamma_A(r_1 - r_2) &= \bigwedge_{m\in M} \gamma_A((r_1 - r_2)\cdot m) \\
&= \bigwedge_{m\in M} \gamma_A(r_1 \cdot m - r_2 \cdot m) \\
&\geq \bigwedge_{m\in M} (\gamma_A(r_1 \cdot m) \wedge \gamma_A(r_2 \cdot m)) \\
&= (\bigwedge_{m\in M} \gamma_A(r_1 \cdot m)) \wedge (\bigwedge_{m\in M} \gamma_A(r_2 \cdot m)) \\
&= \gamma_{\overline{A}}(r_1) \wedge \gamma_{\overline{A}}(r_2).
\end{aligned}$$

$$\begin{aligned}
\psi_A(r_1 - r_2) &= \bigvee_{m\in M} \psi_A((r_1 - r_2)\cdot m) \\
&= \bigvee_{m\in M} \psi_A(r_1 \cdot m - r_2 \cdot m) \\
&\geq \bigvee_{m\in M} (\psi_A(r_1 \cdot m) \vee \psi_A(r_2 \cdot m)) \\
&= (\bigvee_{m\in M} \psi_A(r_1 \cdot m)) \vee (\bigvee_{m\in M} \psi_A(r_2 \cdot m)) \\
&= \psi_{\overline{A}}(r_1) \vee \psi_{\overline{A}}(r_2).
\end{aligned}$$

Now suppose that $r_1, r_2 \in R$. Then we have

$$\begin{aligned}
\mu_{\overline{A}}(r_1 r_2) &= \bigwedge_{m\in M} \mu_A((r_1 r_2)\cdot m) \\
&\geq \bigwedge_{m\in M} \mu_A(r_1 \cdot m) \\
&= \mu_{\overline{A}}(r_1).
\end{aligned}$$

$$\begin{aligned}
\gamma_{\overline{A}}(r_1 r_2) &= \bigwedge_{m\in M} \gamma_A((r_1 r_2)\cdot m) \\
&\geq \bigwedge_{m\in M} \gamma_A(r_1 \cdot m) \\
&= \gamma_{\overline{A}}(r_1).
\end{aligned}$$

$$\begin{aligned}
\psi_{\overline{A}}(r_1 r_2) &= \bigvee_{m\in M} \psi_A((r_1 r_2)\cdot m) \\
&\geq \bigvee_{m\in M} \psi_A(r_1 \cdot m) \\
&= \psi_{\overline{A}}(r_1).
\end{aligned}$$

Hence $\mu_{\overline{A}}(r_1 r_2) \geq \mu_{\overline{A}}(r_1) \vee \mu_{\overline{A}}(r_2)$, $\gamma_{\overline{A}}(r_1 r_2) \geq \gamma_{\overline{A}}(r_1) \vee \gamma_{\overline{A}}(r_2)$ and $\psi_{\overline{A}}(r_1 r_2) \leq \psi_{\overline{A}}(r_1) \wedge \psi_{\overline{A}}(r_2)$. Therefore, $\overline{A}$ is a neutrosophic ideal of R. ☐

Remark 2.10. If $A \in \mathcal{N}M(M)$, then $\sqrt{\overline{A}} = \sqrt{A}$.

Proof. We have

$$\begin{aligned}
\sqrt{\overline{A}} &= \{r, \sqrt{\mu_{\overline{A}}}(r), \sqrt{\gamma_{\overline{A}}}(r), \sqrt{\psi_{\overline{A}}}(r)\} \\
&= \{r, \bigvee_{m\in M} \mu_{\overline{A}}(r^n), \bigvee_{m\in M} \gamma_{\overline{A}}(r^n), \bigwedge_{m\in M} \psi_{\overline{A}}(r^n)\} \\
&= \{r, \bigvee_{m\in M} \bigwedge_{m\in M} \mu_{\overline{A}}(r^n \cdot m), \bigvee_{m\in M} \bigwedge_{m\in M} \gamma_{\overline{A}}(r^n \cdot m), \bigwedge_{m\in M} \bigvee_{m\in M} \psi_{\overline{A}}(r^n \cdot m)\} \\
&= \{r, \sqrt{\mu_A}(r), \sqrt{\gamma_A}(r), \sqrt{\psi_A}(r)\} \\
&= \sqrt{A}.
\end{aligned}$$

☐

Proposition 2.11. *Let* M *be an* R*-module and* $A \in \mathcal{N}M(M)$*. Then* $M_A^{(\alpha,\beta,\omega)} = \overline{M_A^{(\alpha,\beta,\omega)}}$*, where* $M_A^{(\alpha,\beta,\omega)} = \{x \in M : \mu_A(x) \geq \alpha, \gamma_A(x) \geq \beta, \psi_A(x) \leq \omega\}$.

Proof. Consider $\overline{M_A^{(\alpha,\beta,\omega)}}$

$$\begin{aligned}
&= (M_A^{(\alpha,\beta,\omega)} : M) \\
&= Ann \frac{M}{M_A^{(\alpha,\beta,\omega)}} \\
&= \{r \in R : r \cdot (m + M_A^{(\alpha,\beta,\omega)}) = M_A^{(\alpha,\beta,\omega)} \; \forall m \in M\} \\
&= \{r \in R : r \cdot m + M_A^{(\alpha,\beta,\omega)} = M_A^{(\alpha,\beta,\omega)} \; \forall m \in M\} \\
&= \{r \in R : r \cdot m \in M_A^{(\alpha,\beta,\omega)} \; \forall m \in M\} \\
&= \{r \in R : \mu_A(r \cdot m) \geq \alpha, \gamma_A(r \cdot m) \geq \beta, \psi_A(r \cdot m) \leq \omega \; \forall m \in M\} \\
&= \{r \in R : \bigwedge_{m \in M} \mu_A(r \cdot m) \geq \alpha, \bigwedge_{m \in M} \gamma_A(r \cdot m) \geq \beta, \bigvee_{m \in M} \psi_A(r \cdot m) \leq \omega \; \forall m \in M\} \\
&= \{r \in R : \mu_A(r) \geq \alpha, \gamma_A(r) \geq \beta, \psi_A(r) \leq \omega\} \\
&= M_A^{(\alpha,\beta,\omega)}.
\end{aligned}$$

□

Definition 2.12. (see [14]). Let M and N be modules over the same ring R and let f be a mapping from M to N. Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic submodule of M and $B = (\mu_B, \gamma_B, \psi_B)$ be a neutrosophic submodule of N. Then, the image of the neutrosophic submodule A of M, $f(A) = (f(\mu_A), f(\gamma_A), f(\psi_A)) \in N$ can be defined for all $y \in N$ as follows:

$$f(\mu_A)(y) = \begin{cases} \vee\{\mu_A(x) : x \in R, f(x) = y\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & otherwise. \end{cases}$$

$$f(\gamma_A)(y) = \begin{cases} \vee\{\gamma_A(x) : x \in R, f(x) = y\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & otherwise. \end{cases}$$

$$f(\psi_A)(y) = \begin{cases} \wedge\{\psi_A(x) : x \in R, f(x) = y\} & \text{if } f^{-1}(y) \neq \emptyset \\ 1 & otherwise. \end{cases}$$

and inverse image of a neutrosophic submodule B of N can be defined as $x \in M$, $f^{-1}(B) = B(f(x))$.

Theorem 2.13. *Let* $f : M \to M'$ *be an epimorphism of* R*-modules. If* $A \in \mathcal{N}M(M)$*, then* $\overline{A} \subseteq \overline{f(A)}$*.*

Proof. First we show that $f(A)$ is a neutrosophi submodule of M'. let $u, v \in R$, then there exist $x, y \in R$ such that $f(x) = u$, $f(y) = v$; hence $f(x - y) = u - v$. Now we have

$$\begin{aligned}
f(\mu_A)(u-v) &= \bigvee_{z \in f^{-1}(u-v)} \mu_A(z) \\
&\geq \bigvee_{x-y \in f^{-1}(u-v)} \mu_A(x-y) \\
&\geq \bigvee_{x \in f^{-1}(u), y \in f^{-1}(v)} (\mu_A(x) \wedge \mu_A(y)) \\
&= (\bigvee_{x \in f^{-1}(u)} (\mu_A(x))) \wedge (\bigvee_{y \in f^{-1}(v)} (\mu_A(y))) \\
&= f(\mu_A)(u) \wedge f(\mu_A)(v).
\end{aligned}$$

$$\begin{aligned}
f(\gamma_A)(u-v) &= \bigvee_{z \in f^{-1}(u-v)} \gamma_A(z) \\
&\geq \bigvee_{x-y \in f^{-1}(u-v)} \gamma_A(x-y) \\
&\geq \bigvee_{x \in f^{-1}(u), y \in f^{-1}(v)} (\gamma_A(x) \wedge \gamma_A(y)) \\
&= (\bigvee_{x \in f^{-1}(u)} (\gamma_A(x))) \wedge (\bigvee_{y \in f^{-1}(v)} (\gamma_A(y))) \\
&= f(\gamma_A)(u) \wedge f(\gamma_A)(v).
\end{aligned}$$

https://doi.org/10.54216/IJNS.220303

Received: April 26, 2023 Revised: July 08, 2023 Accepted: October 04, 2023

$$\begin{aligned}
f(\psi_A)(u-v) &= \bigwedge_{z\in f^{-1}(u-v)} \psi_A(z)\\
&\leq \bigwedge_{x-y\in f^{-1}(u-v)} \psi_A(x-y)\\
&\leq \bigwedge_{x\in f^{-1}(u),y\in f^{-1}(v)} (\psi_A(x)\vee\psi_A(y))\\
&= (\bigwedge_{x\in f^{-1}(u)} (\psi_A(x)))\vee(\bigwedge_{y\in f^{-1}(v)} (\psi_A(y)))\\
&= f(\psi_A)(u)\vee f(\psi_A)(v).
\end{aligned}$$

$$\begin{aligned}
f(\mu_A)(r\cdot m) &= \bigvee_{m'\in f^{-1}(r\cdot m)} \mu_A(m')\\
&\geq \bigvee_{m'\in f^{-1}(m)} \mu_A(r\cdot m')\\
&\geq \bigvee_{m'\in f^{-1}(m)} (\mu_A(m')\\
&= f(\mu_A)(m).
\end{aligned}$$

$$\begin{aligned}
f(\gamma_A)(r\cdot m) &= \bigvee_{m'\in f^{-1}(r\cdot m)} \gamma_A(m')\\
&\geq \bigvee_{m'\in f^{-1}(m)} \gamma_A(r\cdot m')\\
&\geq \bigvee_{m'\in f^{-1}(m)} (\gamma_A(m')\\
&= f(\gamma_A)(m).
\end{aligned}$$

$$\begin{aligned}
f(\psi_A)(r\cdot m) &= \bigwedge_{m'\in f^{-1}(r\cdot m)} \psi_A(m')\\
&\leq \bigwedge_{m'\in f^{-1}(m)} \psi_A(r\cdot m')\\
&\leq \bigwedge_{m'\in f^{-1}(m)} (\psi_A(m')\\
&= f(\psi_A)(m).
\end{aligned}$$

$$f(\mu_A)(0_{M'}) = \bigvee_{m\in f^{-1}(0_{M'})} \mu_A(m) = \mu_A(0_M),$$

$$f(\gamma_A)(0_{M'}) = \bigvee_{m\in f^{-1}(0_{M'})} \gamma_A(m) = \gamma_A(0_M).$$

$$f(\psi_A)(0_{M'}) = \bigwedge_{m\in f^{-1}(0_{M'})} \psi_A(m) = \psi_A(0_M).$$

Hence $f(A)\in\mathcal{N}M(M')$.
Let $r\in R$. Then

$$\begin{aligned}
\overline{f(\mu_A)}(r) &= (f(\mu_A):f(M))(r)\\
&= \bigvee_{m'\in f(M)} (r\cdot m')\\
&= \bigvee_{m'\in f(M)} \bigwedge_{m\in f^{-1}(r\cdot m)} \mu_A(m)\\
&\geq \bigvee_{m'\in f(M)} \bigwedge_{m\in f^{-1}(r\cdot m)} \mu_A(r\cdot m)\\
&\geq \bigvee_{m'\in f(M),m\in f^{-1}(m')} \mu_A(r\cdot m)\\
&= \bigvee_{m'\in f(M),m\in\cup f^{-1}(m')} \mu_A(r\cdot m)\\
&= \bigvee_{m\in M} \mu_A(r\cdot m)\\
&= \mu_A(r),
\end{aligned}$$

https://doi.org/10.54216/IJNS.220303
Received: April 26, 2023 Revised: July 08, 2023 Accepted: October 04, 2023

$$\begin{aligned}
\overline{f(\gamma_A)}(r) &= (f(\gamma_A) : f(M))(r) \\
&= \bigvee_{m' \in f(M)} (r \cdot m') \\
&= \bigvee_{m' \in f(M)} \bigwedge_{m \in f^{-1}(r \cdot m)} \gamma_A(m) \\
&\geq \bigvee_{m' \in f(M)} \bigwedge_{m \in f^{-1}(r \cdot m)} \gamma_A(r \cdot m) \\
&\geq \bigvee_{m' \in f(M), m \in f^{-1}(m')} \gamma_A(r \cdot m) \\
&= \bigvee_{m' \in f(M), m \in \cup f^{-1}(m')} \gamma_A(r \cdot m) \\
&= \bigvee_{m \in M} \gamma_A(r \cdot m) \\
&= \gamma_A(r)
\end{aligned}$$

and

$$\begin{aligned}
\overline{f(\psi_A)}(r) &= (f(\psi_A) : f(M))(r) \\
&= \bigwedge_{m' \in f(M)} (r \cdot m') \\
&= \bigwedge_{m' \in f(M)} \bigvee_{m \in f^{-1}(r \cdot m)} \psi_A(m) \\
&\geq \bigwedge_{m' \in f(M)} \bigvee_{m \in f^{-1}(r \cdot m)} \psi_A(r \cdot m) \\
&\geq \bigwedge_{m' \in f(M), m \in f^{-1}(m')} \psi_A(r \cdot m) \\
&= \bigwedge_{m' \in f(M), m \in \cup f^{-1}(m')} \psi_A(r \cdot m) \\
&= \bigwedge_{m \in M} \psi_A(r \cdot m) \\
&= \psi_A(r).
\end{aligned}$$

Hence $\overline{A} \subseteq \overline{f(A)} = (\overline{f(\mu_A)}, \overline{f(\gamma_A)}, \overline{f(\psi_A)})$. ☐

Theorem 2.14. *Let $f : M \to M'$ be a homomorphism of R-modules. If $B \in \mathcal{N}M(M')$, then $\overline{B} \subseteq \overline{f^{-1}(B)}$.*

Proof. Let $r \in R$. Then

$$\begin{aligned}
\overline{f^{-1}(\mu_B)}(r) &= (f^{-1}(\mu_B) : f^{-1}(M'))(r) \\
&= \bigvee_{m \in f^{-1}(M')} f^{-1}(\mu_B)(r \cdot m') \\
&= \bigvee_{m \in f^{-1}(M')} \mu_B(f(r \cdot m)) \\
&= \bigvee_{m \in f^{-1}(M')} \mu_B(r \cdot f(m)) \\
&= \bigvee_{f(m) \in M'} \mu_B(r \cdot f(m)) \\
&= \bigvee_{m' \in M'} \mu_B(r \cdot m') \\
&= \mu_{\overline{B}}(r),
\end{aligned}$$

$$\begin{aligned}
\overline{f^{-1}(\gamma_B)}(r) &= (f^{-1}(\gamma_B) : f^{-1}(M'))(r) \\
&= \bigvee_{m \in f^{-1}(M')} f^{-1}(\gamma_B)(r \cdot m') \\
&= \bigvee_{m \in f^{-1}(M')} \gamma_B(f(r \cdot m)) \\
&= \bigvee_{m \in f^{-1}(M')} \gamma_B(r \cdot f(m)) \\
&= \bigvee_{f(m) \in M'} \gamma_B(r \cdot f(m)) \\
&= \bigvee_{m' \in M'} \gamma_B(r \cdot m') \\
&= \gamma_{\overline{B}}(r),
\end{aligned}$$

and

$$\begin{array}{rcl}\overline{f^{-1}(\psi_B)}(r) & = & (f^{-1}(\psi_B) : f^{-1}(M'))(r) \\ & = & \bigwedge\limits_{m\in f^{-1}(M')} f^{-1}(\psi_B)(r\cdot m') \\ & = & \bigwedge\limits_{m\in f^{-1}(M')} \psi_B(f(r\cdot m)) \\ & = & \bigwedge\limits_{m\in f^{-1}(M')} \psi_B(r\cdot f(m)) \\ & = & \bigwedge\limits_{f(m)\in M'} \psi_B(r\cdot f(m)) \\ & = & \bigwedge\limits_{m'\in M'} \psi_B(r\cdot m') \\ & = & \psi_{\overline{B}}(r).\end{array}$$

Hence $\overline{B} \subseteq \overline{f^{-1}(B)}$. Furthermore, if f is an epimorphism, then $\overline{B} = \overline{f^{-1}(B)}$. □

Proposition 2.15. *Let $A, B \in \mathcal{N}M(M)$. Then we have the following*

1. $\overline{\overline{A}} = A$,
2. $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$,
3. $\overline{A\cap B} = \overline{A} \cap \overline{B}$,
4. $\overline{A+B} \supseteq \overline{A} + \overline{B}$.

Proof. (1). Let $r \in R$. Since $\overline{A}$ is R-module, $\mu_{\overline{A}}(r) = \bigwedge\limits_{r'\in R} \mu_{\overline{A}}(r\cdot r') = \overline{A}$, $\gamma_{\overline{A}}(r) = \bigwedge\limits_{r'\in R} \gamma_{\overline{A}}(r\cdot r') = \overline{A}$ and $\psi_{\overline{A}}(r) = \bigvee\limits_{r'\in R} \psi_{\overline{A}}(r\cdot r') = \overline{A}$. Hence $\overline{\overline{A}} = (\mu_{\overline{A}}, \gamma_{\overline{A}}, \psi_{\overline{A}}) = \overline{A}$.

(2). Since $A \subseteq B$, $\mu_A \subseteq \mu_B$, $\gamma_A \subseteq \gamma_B$ and $\psi_A \supseteq \psi_B$,

$$\mu_{\overline{A}}(r) = \bigwedge_{m\in M} \mu_A(r\cdot m) \leq \bigwedge_{m\in M} \mu_B(r\cdot m) = \mu_{\overline{B}}(r),$$

$$\gamma_{\overline{A}}(r) = \bigwedge_{m\in M} \gamma_A(r\cdot m) \leq \bigwedge_{m\in M} \gamma_B(r\cdot m) = \gamma_{\overline{B}}(r),$$

$$\psi_{\overline{A}}(r) = \bigvee_{m\in M} \psi_A(r\cdot m) \geq \bigvee_{m\in M} \psi_B(r\cdot m) = \psi_{\overline{B}}(r).$$

Hence $\overline{A} \subseteq \overline{B}$.

(3). We have $A \cap B = (\mu_{A\cap B}, \gamma_{A\cap B}, \psi_{A\cup B})$. Then

$$\begin{array}{rcl}\mu_{\overline{A\cap B}}(r) & = & \bigwedge\limits_{m\in M} \mu_{A\cap B}(r\cdot m) \\ & = & \bigwedge\limits_{m\in M} (\mu_A(r\cdot m) \wedge \mu_B(r\cdot m)) \\ & = & (\bigwedge\limits_{m\in M} (\mu_A(r\cdot m)) \wedge (\bigwedge\limits_{m\in M} \mu_B(r\cdot m)) \\ & = & \mu_A(r) \wedge \mu_B(r) \\ & = & (\mu_A \cap \mu_B)(r),\end{array}$$

$$\begin{array}{rcl}\gamma_{\overline{A\cap B}}(r) & = & \bigwedge\limits_{m\in M} \gamma_{A\cap B}(r\cdot m) \\ & = & \bigwedge\limits_{m\in M} (\gamma_A(r\cdot m) \wedge \gamma_B(r\cdot m)) \\ & = & (\bigwedge\limits_{m\in M} (\gamma_A(r\cdot m)) \wedge (\bigwedge\limits_{m\in M} \gamma_B(r\cdot m)) \\ & = & \gamma_A(r) \wedge \gamma_B(r) \\ & = & (\gamma_A \cap \gamma_B)(r),\end{array}$$

$$\begin{aligned}
\psi_{\overline{A\cup B}}(r) &= \bigvee_{m\in M} \psi_{A\cup B}(r\cdot m)\\
&= \bigvee_{m\in M} (\psi_A(r\cdot m)\vee\psi_B(r\cdot m))\\
&= (\bigvee_{m\in M} (\psi_A(r\cdot m))\vee(\bigvee_{m\in M} \psi_B(r\cdot m))\\
&= \psi_A(r)\vee\psi_B(r)\\
&= (\psi_A\cup\psi_B)(r).
\end{aligned}$$

Hence $\overline{A\cap B}=\overline{A}\cap\overline{B}$.
(4). It is clear that $\mu_A\subseteq\mu_A+\mu_B$, $\mu_B\subseteq\mu_A+\mu_B$, $\gamma_A\subseteq\gamma_A+\gamma_B$, $\gamma_B\subseteq\gamma_A+\gamma_B$, $\psi_A\supseteq\psi_A+\psi_B$, $\psi_B\supseteq\psi_A+\psi_B$. Then $\mu_{\overline{A}}\subseteq\mu_{\overline{A+B}}$, $\gamma_{\overline{B}}\subseteq\gamma_{\overline{A+B}}$ and $\psi_{\overline{A}}\supseteq\psi_{\overline{A+B}}$. Hence $\mu_{\overline{A}}+\mu_{\overline{B}}\subseteq\mu_{\overline{A+B}}$, $\gamma_{\overline{A}}+\gamma_{\overline{B}}\subseteq\gamma_{\overline{A+B}}$ and $\psi_{\overline{A}}+\psi_{\overline{B}}\supseteq\psi_{\overline{A+B}}$. Therefore, $\overline{A+B}\supseteq\overline{A}+\overline{B}$. □

In this part of the paper, we define the neutrosophic primary submodule.

Definition 2.16. Let A be a neutrosophic submodule of R-module M, then A is called neutrosophic primary submodule of M if for $r\in R$, $m\in M$ $\mu_A(r\cdot m)=\mu_A(m)$, $\gamma_A(r\cdot m)=\gamma_A(m)$, $\psi_A(r\cdot m)=\psi_A(m)$ or $\mu_A(r\cdot m)\le\sqrt{\mu_A}(r)$, $\gamma_A(r\cdot m)\le\sqrt{\gamma_A}(r)$, $\psi_A(r\cdot m)\ge\sqrt{\psi_A}(r)$.

Proposition 2.17. *Let M be an R-module and A be a neutrosophic primary submodule of M with* sup *property. Then, $M_A^{(\alpha,\beta,\omega)}$ is a primary submodule of M.*

Proof. Since $A\in\mathcal{N}M(M)$, $M_A^{(\alpha,\beta,\omega)}$ is a submodule of M. Now let $r\in R$, $m\in M$ and $rm\in M_A^{(\alpha,\beta,\omega)}$. Then $\mu_A(rm)\ge\alpha$, $\gamma_A(rm)\ge\beta$, $\psi_A(rm)\le\omega$. Since A is a neutrosophic primary submodule of R-module M, $\alpha\le\mu_A(rm)=\mu_A(m)$, $\beta\le\gamma_A(rm)=\gamma_A(m)$, $\omega\ge\psi_A(rm)=\psi_A(m)$ or $\alpha\le\mu_A(r\cdot m)\le\sqrt{\mu_A}(r)$, $\beta\le\gamma_A(r\cdot m)\le\sqrt{\gamma_A}(r)$, $\omega\ge\psi_A(r\cdot m)\ge\sqrt{\psi_A}(r)$. Hence $\mu_A(m)\ge\alpha$, $\gamma_A(m)\ge\beta$, $\psi_A(m)\le\omega$ or $\sqrt{\mu_A}(r)\ge\alpha$, $\sqrt{\gamma_A}(r)\ge\beta$, $\sqrt{\psi_A}(r)\le\omega$. This means that $m\in M_A^{(\alpha,\beta,\omega)}$ or $r\in(\sqrt{M_A})^{(\alpha,\beta,\omega)}$. But A has sup property and by Proposition 4 we have, $(\sqrt{M_A})^{(\alpha,\beta,\omega)}=\sqrt{M_A^{(\alpha,\beta,\omega)}}$. Hence $m\in M_A^{(\alpha,\beta,\omega)}$ or $r\in\sqrt{M_A^{(\alpha,\beta,\omega)}}$. Therefore, $M_A^{(\alpha,\beta,\omega)}$ is a primary submodule of M. □

Proposition 2.18. *Let A be a neutrosophic submodule of M such that every level cut of A is a primary submodule of M, then A is a neutrosophic primary submodule of M.*

Proof. Let $r\in R$, $m\in M$ and let $\mu_A(r\cdot m)=\alpha$, $\gamma_A(r\cdot m)=\beta$, $\psi_A(r\cdot m)=\omega$. Then $\mu_A(r_1r_2)=\bigwedge_{m\in M}\mu_A(r_1r_2)=\bigwedge_{m\in M}\mu_A(r_1(r_2\cdot m))$, $\gamma_A(r_1r_2)=\bigwedge_{m\in M}\gamma_A(r_1r_2)=\bigwedge_{m\in M}\gamma_A(r_1(r_2\cdot m))$, $\psi_A(r_1r_2)=\bigvee_{m\in M}\psi_A(r_1r_2)=\bigvee_{m\in M}\psi_A(r_1(r_2\cdot m))$. Now since A is a neutrosophic primary submodule, then

$$\begin{aligned}
\Rightarrow\ & \mu_A(m)\ge\alpha,\gamma_A(m)\ge\beta,\psi_A(m)\le\omega \textit{ or}\\
& \mu_A(r\cdot m)\ge\alpha,\gamma_A(r\cdot m)\ge\beta,\psi_A(r\cdot m)\le\omega\ \forall m\in M\\
\Rightarrow\ & \alpha=\mu_A(r\cdot m)\ge\mu_A(m)\ge\alpha,\beta=\gamma_A(r\cdot m)\ge\gamma_A(m)\ge\beta,\\
& \omega=\psi_A(r\cdot m)\le\psi_A(m)\le\omega \textit{ or}\\
& \bigwedge_{m\in M}\mu_A(r\cdot m)\ge\alpha,\ \bigwedge_{m\in M}\gamma_A(r\cdot m)\ge\beta,\ \bigvee_{m\in M}\psi_A(r\cdot m)\le\omega\\
\Rightarrow\ & \mu_A(r\cdot m)=\mu_A(m),\gamma_A(r\cdot m)=\gamma_A(m),\psi_A(r\cdot m)=\psi_A(m),\ \textit{or}\\
& \bigvee_{n\in N}\bigwedge_{m\in M}\mu_A(r\cdot m)\ge\alpha,\ \bigvee_{n\in N}\bigwedge_{m\in M}\gamma_A(r\cdot m)\ge\beta,\ \bigwedge_{n\in N}\bigvee_{m\in M}\psi_A(r\cdot m)\le\omega\\
\Rightarrow\ & \mu_A(r\cdot m)=\mu_A(m),\gamma_A(r\cdot m)=\gamma_A(m),\psi_A(r\cdot m)=\psi_A(m),\ \textit{or}\\
& \sqrt{\mu_A}(r)\ge\alpha,\sqrt{\gamma_A}(r)\ge\beta,\sqrt{\psi_A}(r)\le\omega\\
\Rightarrow\ & \mu_A(r\cdot m)=\mu_A(m),\gamma_A(r\cdot m)=\gamma_A(m),\psi_A(r\cdot m)=\psi_A(m),\ \textit{or}\\
& \mu_A(r\cdot m)\le\sqrt{\mu_A}(r),\gamma_A(r\cdot m)\le\sqrt{\gamma_A}(r),\psi_A(r\cdot m)\ge\sqrt{\psi_A}(r).
\end{aligned}$$

Hence A is a neutrosophic primary submodule of M. □

Definition 2.19. Let A be a neutrosophic submodule of the R-module M. The support of A is defined as $A^*=\{x\in M:\mu_A(x)>0,\gamma_A(x)>0,\psi_A(x)<1\}$.

In the following, we investigated related between neutrosophic submodule A and support of A.

Proposition 2.20. *Let A be a neutrosophic primary submodule of the R-module M. Then A^* is a primary submodule of M.*

Proof. Suppose that $rm \in A^*$ for $r \in R$, $m \in M$. Then, $\mu_A(x) > 0$, $\gamma_A(x) > 0$, $\psi_A(x) < 1$. Since A is a neutrosophic primary submodule, then $\mu_A(rm) = \mu_A(m)$ or $\mu_A(r \cdot m) \leq \sqrt{\mu_A}(r)$, $\gamma_A(rm) = \gamma_A(m)$ or $\gamma_A(r \cdot m) \leq \sqrt{\gamma_A}(r)$, $\psi_A(rm) = \psi_A(m)$ or $\psi_A(r \cdot m) \geq \sqrt{\psi_A}(r)$. So $\mu_A(m) > 0$ or $\sqrt{\mu_A}(r) > 0$, $\gamma_A(m) > 0$ or $\sqrt{\gamma_A}(r) > 0$, $\psi_A(m) < 1$ or $\sqrt{\psi_A}(r) < 1$. Hence $m \in A^*$ or $r \in \sqrt{A^*}$. Therefore A^* is a primary submodule of M. □

Definition 2.21. An neutrosophic ideal A is said to be a neutrosophic weakly primary ideal if for any $x, y \in R$ $\mu_A(x{\cdot}y) = \mu_A(x), \gamma_A(x{\cdot}y) = \gamma_A(x), \psi_A(x{\cdot}y) = \psi_A(x)$ or $\mu_A(x{\cdot}y) \leq \mu_A(y^n), \gamma_A(x{\cdot}y) \leq \gamma_A(y^n), \psi_A(x \cdot y) \geq \psi_A(y^n)$ for some $n \in N$.

Proposition 2.22. *Let A be a neutrosophic primary submodule of M with* sup *property, then $\sqrt{A}$ is a neutrosophic weakly primary submodule of R and $\sqrt{A}$ is the neutrosophic prime ideal of R.*

Proof. Let $r_1, r_2 \in R$. Then $\mu_{\overline{A}}(r_1r_2) = \bigwedge\limits_{m\in M} \mu_A(r_1r_2{\cdot}m) = \bigwedge\limits_{m\in M} \mu_A(r_1(r_2{\cdot}m)), \gamma_{\overline{A}}(r_1r_2) = \bigwedge\limits_{m\in M} \gamma_A(r_1r_2 \cdot m) = \bigwedge\limits_{m\in M} \gamma_A(r_1(r_2 \cdot m)), \psi_{\overline{A}}(r_1r_2) = \bigvee\limits_{m\in M} \psi_A(r_1r_2 \cdot m) = \bigvee\limits_{m\in M} \psi_A(r_1(r_2 \cdot m))$. Now since A is a neutrosophic primary submodule, then

$$\begin{aligned}
&\Rightarrow \quad \mu_A(r_1(r_2 \cdot m)) = \mu_A(r_1 \cdot m) \text{ or } \mu_A(r_1(r_2 \cdot m)) = \sqrt{\mu_A}(r_2)\\
&\Rightarrow \quad \mu_A(r_1r_2) = \bigwedge_{m\in M} \mu_A(r_1 \cdot m) \text{ or } \mu_A(r_1r_2) = \bigwedge_{m\in M} \sqrt{\mu_A}(r_2) = \sqrt{\mu_A}(r_2)\\
&\Rightarrow \quad \mu_A(r_1r_2) = \mu_A(r_1) \text{ or } \mu_A(r_1r_2) \leq \sqrt{\mu_A}(r_2) = \bigvee_{n\in N} \bigwedge_{m\in M} \mu_A(r_2^n \cdot m)\\
&\Rightarrow \quad \mu_A(r_1r_2) = \mu_A(r_1) \text{ or } \mu_A(r_1r_2) \leq \bigwedge_{m\in M} \mu_A(r_2^n \cdot m) \; for \; some \; n \in N\\
&\Rightarrow \quad \mu_A(r_1r_2) = \mu_A(r_1) \text{ or } \mu_A(r_1r_2) \leq \bigwedge_{m\in M} \mu_A(r_2^{n_2} \cdot m) \leq \bigwedge_{m\in M} \mu_A(r_1^{n_2}r_2^{n_2} \cdot m), n_1, n_2 \in N,
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \quad \gamma_A(r_1(r_2 \cdot m)) = \gamma_A(r_1 \cdot m) \text{ or } \gamma_A(r_1(r_2 \cdot m)) = \sqrt{\gamma_A}(r_2)\\
&\Rightarrow \quad \gamma_A(r_1r_2) = \bigwedge_{m\in M} \gamma_A(r_1 \cdot m) \text{ or } \gamma_A(r_1r_2) = \bigwedge_{m\in M} \sqrt{\gamma_A}(r_2) = \sqrt{\gamma_A}(r_2)\\
&\Rightarrow \quad \gamma_A(r_1r_2) = \gamma_A(r_1) \text{ or } \gamma_A(r_1r_2) \leq \sqrt{\gamma_A}(r_2) = \bigvee_{n\in N} \bigwedge_{m\in M} \gamma_A(r_2^n \cdot m)\\
&\Rightarrow \quad \gamma_A(r_1r_2) = \gamma_A(r_1) \text{ or } \gamma_A(r_1r_2) \leq \bigwedge_{m\in M} \gamma_A(r_2^n \cdot m) \; for \; some \; n \in N\\
&\Rightarrow \quad \gamma_A(r_1r_2) = \gamma_A(r_1) \text{ or } \gamma_A(r_1r_2) \leq \bigwedge_{m\in M} \gamma_A(r_2^{n_2} \cdot m) \leq \bigwedge_{m\in M} \gamma_A(r_1^{n_2}r_2^{n_2} \cdot m), n_1, n_2 \in N,
\end{aligned}$$

and

$$\begin{aligned}
&\Rightarrow \quad \psi_A(r_1(r_2 \cdot m)) = \psi_A(r_1 \cdot m) \text{ or } \psi_A(r_1(r_2 \cdot m)) = \sqrt{\psi_A}(r_2)\\
&\Rightarrow \quad \psi_A(r_1r_2) = \bigvee_{m\in M} \psi_A(r_1 \cdot m) \text{ or } \psi_A(r_1r_2) = \bigvee_{m\in M} \sqrt{\psi_A}(r_2) = \sqrt{\psi_A}(r_2)\\
&\Rightarrow \quad \psi_A(r_1r_2) = \psi_A(r_1) \text{ or } \psi_A(r_1r_2) \geq \sqrt{\psi_A}(r_2) = \bigwedge_{n\in N} \bigvee_{m\in M} \psi_A(r_2^n \cdot m)\\
&\Rightarrow \quad \psi_A(r_1r_2) = \psi_A(r_1) \text{ or } \psi_A(r_1r_2) \geq \bigvee_{m\in M} \psi_A(r_2^n \cdot m) \; for \; some \; n \in N\\
&\Rightarrow \quad \psi_A(r_1r_2) = \psi_A(r_1) \text{ or } \psi_A(r_1r_2) \geq \bigvee_{m\in M} \psi_A(r_2^{n_2} \cdot m) \geq \bigvee_{m\in M} \psi_A(r_1^{n_2}r_2^{n_2} \cdot m), n_1, n_2 \in N.
\end{aligned}$$

If we consider $n' = n_1 + n_2$, then we have $\mu_A(r_1r_2) = \mu_A(r_1)$, $\gamma_A(r_1r_2) = \gamma_A(r_1)$, $\psi_A(r_1r_2) = \psi_A(r_1)$ or $\mu_A(r_1r_2) \leq \mu_A(r_1^n)$, $\gamma_A(r_1r_2) \leq \gamma_A(r_1^n)$, $\psi_A(r_1r_2) \geq \psi_A(r_1^n)$. This proves that A is a neutrosophic weakly primary ideal of R. Next, by Remark 1 we have $\sqrt{\overline{A}} = \sqrt{A}$ and since $\sqrt{\overline{A}}$ is neutrosophic prime ideal then $\sqrt{A}$ is a neutrosophic prime ideal of R. □

The opposite of the above proposition is true if A is an ideal.

Theorem 2.23. *Let $A \in \mathcal{N}I(R)$ with sup property. Then, A is a neutrosophic primary submodule if and only if A is an neutrosophic weakly primary ideal of R.*

Proof. Suppose that A is a neutrosophic primary submodule of R, then A is a neutrosophic primary ideal of R. Now since A is a neutrosophic primary submodule, then for $r_1, r_2 \in R$ $\mu_A(r_1r_2) = \mu_A(r_1), \gamma_A(r_1r_2) = \gamma_A(r_1)$, $\psi_A(r_1r_2) = \psi_A(r_1)$ or $\mu_A(r_1r_2) \leq \sqrt{\mu_A}(r_2)$, $\gamma_A(r_1r_2) \leq \sqrt{\gamma_A}(r_2)$, $\psi_A(r_1r_2) \geq \sqrt{\psi_A}(r_2)$

$$\begin{aligned}
\Rightarrow \quad & \mu_A(r_1r_2) \leq \bigvee_{n\in N} \mu_A(r_2^n), \gamma_A(r_1r_2) \leq \bigvee_{n\in N} \gamma_A(r_2^n), \psi_A(r_1r_2) \geq \bigwedge_{n\in N} \psi_A(r_2^n) \\
\Rightarrow \quad & \mu_A(r_1r_2) \leq \mu_A(r_2^{n_1}), \gamma_A(r_1r_2) \leq \gamma_A(r_2^{n_2}), \psi_A(r_1r_2) \geq \psi_A(r_2^{n_3}), n_1, n_2, n_3 \in N \\
\Rightarrow \quad & \mu_A(r_1r_2) \leq \mu_A(r_2^{n_1}) \leq \mu_A(r_1^{n_1}r_2^{n_1}), \gamma_A(r_1r_2) \leq \gamma_A(r_2^{n_1}) \leq \gamma_A(r_1^{n_1}r_2^{n_1}), \\
& \psi_A(r_1r_2) \geq \psi_A(r_2^{n_1}) \geq \psi_A(r_1^{n_1}r_2^{n_1}) \\
\Rightarrow \quad & \mu_A(r_1r_2) \leq \mu_A(r_2^{n'}), \gamma_A(r_1r_2) \leq \gamma_A(r_2^{n'}), \psi_A(r_1r_2) \geq \psi_A(r_2^{n'}).
\end{aligned}$$

Hence $\mu_A(r_1r_2) = \mu_A(r_1)$, $\gamma_A(r_1r_2) = \gamma_A(r_1)$, $\psi_A(r_1r_2) = \psi_A(r_1)$ or $\mu_A(r_1r_2) \leq \mu_A(r_2^{n'}), \gamma_A(r_1r_2) \leq \gamma_A(r_2^{n'})$, $\psi_A(r_1r_2) \geq \psi_A(r_2^{n'})$. Hence A is a neutrosophic weakly primary ideal of R. Conversely, let A be a neutrosophic weakly primary ideal of R, then $\mu_A(r_1r_2) = \mu_A(r_1)$, $\gamma_A(r_1r_2) = \gamma_A(r_1)$, $\psi_A(r_1r_2) = \psi_A(r_1)$ or $\mu_A(r_1r_2) \leq \mu_A(r_2^n), \gamma_A(r_1r_2) \leq \gamma_A(r_2^n)$, $\psi_A(r_1r_2) \geq \psi_A(r_2^n)$ for some $n \in N$. Hence $\mu_A(r_1r_2) = \mu_A(r_1)$, $\gamma_A(r_1r_2) = \gamma_A(r_1)$, $\psi_A(r_1r_2) = \psi_A(r_1)$ or $\mu_A(r_1r_2) \leq \bigvee_{n\in N} \mu_A(r_2^n), \gamma_A(r_1r_2) \leq \bigvee_{n\in N} \gamma_A(r_2^n)$, $\psi_A(r_1r_2) \geq \bigwedge_{n\in N} \psi_A(r_2^n)$. Therefore, $\mu_A(r_1r_2) = \mu_A(r_1)$, $\gamma_A(r_1r_2) = \gamma_A(r_1)$, $\psi_A(r_1r_2) = \psi_A(r_1)$ or $\mu_A(r_1r_2) \leq \sqrt{\mu_A}(r_2), \gamma_A(r_1r_2) \leq \bigvee_{n\in N} \sqrt{\gamma_A}(r_2)$, $\psi_A(r_1r_2) \geq \bigwedge_{n\in N} \sqrt{\psi_A}(r_2)$ for all $r_1, r_2 \in R$. Hence A is a neutrosophic primary submodule of R. ☐

Proposition 2.24. *Let $f : M \to M'$ be an epimorphism of R-module M to R-module M'. If $A \in \mathcal{N}M(M)$, then $\sqrt{A} \subseteq \sqrt{f(A)}$. Equality holds if A is constant on kernel f.*

Proof. We have $\overline{A} \subseteq \overline{f(A)}$, $\sqrt{\overline{A}} \subseteq \sqrt{\overline{f(A)}}$ and hence $\sqrt{A} \subseteq \sqrt{f(A)}$. Also $\overline{A} = \overline{f(A)}$ if A is constant on kernal f, hence $\sqrt{\overline{A}} = \sqrt{\overline{f(A)}}$ and hence $\sqrt{A} = \sqrt{f(A)}$. ☐

Proposition 2.25. *Let $f : M \to M'$ be a homomorphism of R-modules. If $B \in \mathcal{N}M(M')$, then $\sqrt{B} \subseteq \sqrt{f^{-1}(B)}$. Equality holds if f is an epimorphism.*

Proof. We have $\overline{B} \subseteq \overline{f^{-1}(B)}$, hence $\sqrt{\overline{B}} \subseteq \sqrt{\overline{f^{-1}(B)}}$ and hence $\sqrt{B} \subseteq \sqrt{f^{-1}(B)}$. If f is epimorphism, then $\overline{f^{-1}(B)} = \overline{B}$. Therefore, $\sqrt{B} = \sqrt{f^{-1}(B)}$. ☐

Theorem 2.26. *Let $f : M \to M'$ be an epimorphism of R-module M to R-module M', and let A be a neutrosophic primary submodule of M, which is constant on kernel f, then the image f, $f(A)$ is a neutrosophic primary submodule of M'.*

Proof. We have $f(A)$ is a neutrosophic submodule of M'. Let $r \in R$, $m' \in M'$, then

$$\begin{aligned}
f(\mu_A)(rm') &= \bigvee_{m\in f^{-1}(rm')} \mu_A(rm'), \\
f(\gamma_A)(rm') &= \bigvee_{m\in f^{-1}(rm')} \gamma_A(rm'), \\
f(\psi_A)(rm') &= \bigwedge_{m\in f^{-1}(rm')} \psi_A(rm').
\end{aligned}$$

Since A constant on kernel f,

$$\begin{aligned}
m \in f^{-1}(rm') \quad &\Rightarrow \quad f(m) = rm' = rf(m_1) \\
&\Rightarrow \quad f(m) - f(rm_1) = 0 \\
&\Rightarrow \quad m - rm_1 \in kerf.
\end{aligned}$$

Then $\mu_A(m) = \mu_A(rm_1), \gamma_A(m) = \gamma_A(rm_1), \psi_A(m) = \psi_A(rm_1)$. So $f(\mu_A)(rm') = \bigvee_{m\in f^{-1}(rm')} \mu_A(rm_1)$, $f(\gamma_A)(rm') = \bigvee_{m\in f^{-1}(rm')} \gamma_A(rm_1)$, $f(\psi_A)(rm') = \bigwedge_{m\in f^{-1}(rm')} \psi_A(rm_1)$, but A is a neutrosophic fuzzy primary submodule of M and hence $\mu_A(rm_1) = \mu_A(m_1), \gamma_A(rm_1) = \gamma_A(m_1), \psi_A(rm_1) = \psi_A(m_1)$ or

Received: April 26, 2023 Revised: July 08, 2023 Accepted: October 04, 2023

$\mu_A(rm_1) \leq \sqrt{\mu_A}(r), \gamma_A(rm_1) \leq \sqrt{\gamma_A}(r), \psi_A(rm_1) \geq \sqrt{\psi_A}(r)$. Now if $\mu_A(rm_1) = \mu_A(m_1), \gamma_A(rm_1) = \gamma_A(m_1), \psi_A(rm_1) = \psi_A(m_1)$, then $f(\mu_A)(rm') = \bigvee\limits_{m \in f^{-1}(rm')} \mu_A(m'), f(\gamma_A)(rm') = \bigvee\limits_{m \in f^{-1}(rm')} \gamma_A(m')$, $f(\psi_A)(rm') = \bigwedge\limits_{m \in f^{-1}(rm')} \psi_A(m')$. And if $\mu_A(rm_1) \leq \sqrt{\mu_A}(r)$, $\gamma_A(rm_1) \leq \sqrt{\gamma_A}(r)$, $\psi_A(rm_1) \geq \sqrt{\psi_A}(r)$, then $\bigvee\limits_{m \in f^{-1}(rm')} \mu_A(rm_1) \leq \bigvee\limits_{m \in f^{-1}(rm')} \mu_A(rm_1)\sqrt{\mu_A}(r_1) \Rightarrow f(\mu_A)(rm') \leq \bigvee\limits_{m \in f^{-1}(m')} \sqrt{\mu_A}(r) = \sqrt{\mu_A}(r) \leq \sqrt{f(\mu_A)}(r)$, and $\bigvee\limits_{m \in f^{-1}(rm')} \gamma_A(rm_1) \leq \bigvee\limits_{m \in f^{-1}(rm')} \gamma_A(rm_1)\sqrt{\gamma_A}(r_1) \Rightarrow f(\gamma_A)(rm') \leq \bigvee\limits_{m \in f^{-1}(m')} \sqrt{\gamma_A}(r) = \sqrt{\gamma_A}(r) \leq \sqrt{f(\gamma_A)}(r)$ and $\bigwedge\limits_{m \in f^{-1}(rm')} \psi_A(rm_1) \geq \bigwedge\limits_{m \in f^{-1}(rm')} \psi_A(rm_1)\sqrt{\psi_A}(r_1)$ $\Rightarrow f(\psi_A)(rm') \geq \bigwedge\limits_{m \in f^{-1}(m')} \sqrt{\psi_A}(r) = \sqrt{\psi_A}(r) \geq \sqrt{f(\psi_A)}(r)$. Then $f(\mu_A)(rm') = f(\mu_A)(m')$, $f(\gamma_A)(rm') = f(\gamma_A)(m')$, $f(\psi_A)(rm') = f(\psi_A)(m')$ or $f(\mu_A)(rm') \leq \sqrt{f(\mu_A)}(r)$, $f(\gamma_A)(rm') \leq \sqrt{f(\gamma_A)}(r)$, $f(\psi_A)(rm') \geq \sqrt{f(\psi_A)}(r)$. Hence $f(A)$ is a neutrosophic primary submodule of M'. □

Theorem 2.27. *Let $f : M \to M'$ be a homomorphism of R-module M to R-module M'. If B is a neutrosophic primary submodule of M', then $f^{-1}(B)$ is a neutrosophic primary submodule of M.*

Proof. Let $r \in R, m \in M$, then $f^{-1}(rm) = \mu_B(f(rm)) = \mu_B(rf(m))$, $f^{-1}(rm) = \gamma_B(f(rm)) = \gamma_B(rf(m))$, $f^{-1}(rm) = \psi_B(f(rm)) = \psi_B(rf(m))$. Then $\mu_B(rf(m)) = \mu_B(f(m)) = \mu_B(rf(m))$, $\gamma_B(rf(m)) = \gamma_B(f(m)) = \gamma_B(rf(m))$, $\psi_B(rf(m)) = \psi_B(f(m)) = \psi_B(rf(m))$ or $\mu_B(rf(m)) \leq \sqrt{\mu_B}(r) \leq \sqrt{f^{-1}(\mu_B)}(r), \gamma_B(rf(m)) \leq \sqrt{\gamma_B}(r) \leq \sqrt{f^{-1}(\gamma_B)}(r), \psi_B(rf(m)) \geq \sqrt{\psi_B}(r) \geq \sqrt{f^{-1}(\psi_B)}(r)$. Hence $f^{-1}(\mu_B)(rm) = f^{-1}(\mu_B)(m)$, $f^{-1}(\gamma_B)(rm) = f^{-1}(\gamma_B)(m)$, $f^{-1}(\psi_B)(rm) = f^{-1}(\psi_B)(m)$ or $f^{-1}(\mu_B)(rm) \leq \sqrt{f^{-1}(\mu_B)}(r)$, $f^{-1}(\gamma_B)(rm) \leq \sqrt{f^{-1}(\gamma_B)}(r)$, $f^{-1}(\psi_B)(rm) \geq \sqrt{f^{-1}(\psi_B)}(r)$. Hence $f^{-1}(B)$ is a neutrosophic primary submodule of M. □

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Received: April 26, 2023 Revised: July 08, 2023 Accepted: October 04, 2023