



Some properties of neutrosophic nil radicals of neutrosophic ideals in rings

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Abstract

In this paper, we consider the notion of neutrosophic nil radicals of neutrosophic ideals in commutative rings and some properties of such nil radicals. Finally, we study the properties of semiprime neutrosophic ideals of rings.

Keywords: ring; neutrosophic ideal; neutrosophic nil radical; f -invariant; semiprime neutrosophic ideal.

1 Introduction and preliminaries

The concept of fuzzy sets was proposed by Zadeh.¹⁹ The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches, such as soft sets and rough sets, has been discussed in.^{1,3,9} The idea of intuitionistic fuzzy sets suggested by Atanassov² is one of the extensions of fuzzy sets with better applicability. Applications of intuitionistic fuzzy sets appear in various fields, including medical diagnosis, optimization problems, and multicriteria decision-making.¹²⁻¹⁴ The notion of neutrosophic sets was introduced by Smarandache¹⁷ in 1999, which is a more general platform that extends the notions of classic sets (intuitionistic) fuzzy sets and interval-valued (intuitionistic) fuzzy sets. Neutrosophic set theory is applied to various parts, which is referred to on the site <http://fs.unm.edu/neutrosophy.htm>. Smarandache¹⁸ has compiled the essentials of neutrosophic sets in this article, including the distinctions between neutrosophic sets and intuitionistic fuzzy sets. There is a wide variety of studies and research on neutrosophic sets in many domains. In 2018, Cetkin and Aygun¹⁰ introduced the two types of neutrosophic ideals of rings. In 2019, Cetkin and Aygun¹¹ introduced the notion of neutrosophic subrings.

Broumi et al. have explored the idea of an interval-valued Fermatean neutrosophic sets and applied it to graphs, as seen in.⁴⁻⁸

In this paper, we consider the notion of neutrosophic nil radicals of neutrosophic ideals in commutative rings and some properties of such nil radicals. Finally, we study the properties of semiprime neutrosophic ideals of rings.

Let R be a non-empty set. The neutrosophic set¹⁷ A on R is defined to be a structure

$$A := \{ \langle x, \mu(x), \gamma(x), \psi(x) \rangle : x \in R \}, \tag{1}$$

where $\mu : R \rightarrow [0, 1]$ is a truth membership function, $\gamma : R \rightarrow [0, 1]$ is an indeterminate membership function, and $\psi : R \rightarrow [0, 1]$ is a false membership function. The neutrosophic set in (1) is simply denoted by $A = (\mu_A, \gamma_A, \psi_A)$.

Definition 1.1.¹⁵ A neutrosophic set $A = (\mu_A, \gamma_A, \psi_A)$ on a ring R is said to be a neutrosophic subring of R if

$$(\forall x, y \in R) \left(\begin{array}{l} \mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\} \\ \gamma_A(x - y) \geq \min\{\gamma_A(x), \gamma_A(y)\} \\ \psi_A(x - y) \leq \max\{\psi_A(x), \psi_A(y)\} \end{array} \right), \tag{2}$$

$$(\forall x, y \in R) \left(\begin{array}{l} \mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\} \\ \gamma_A(xy) \geq \min\{\gamma_A(x), \gamma_A(y)\} \\ \psi_A(xy) \leq \max\{\psi_A(x), \psi_A(y)\} \end{array} \right). \tag{3}$$

Definition 1.2.¹⁶ A neutrosophic set $A = (\mu_A, \gamma_A, \psi_A)$ on a ring R is said to be a neutrosophic ideal of R if it satisfies (2) and

$$(\forall x, y \in R) \left(\begin{array}{l} \mu_A(xy) \geq \max\{\mu_A(x), \mu_A(y)\} \\ \gamma_A(xy) \geq \max\{\gamma_A(x), \gamma_A(y)\} \\ \psi_A(xy) \leq \min\{\psi_A(x), \psi_A(y)\} \end{array} \right). \tag{4}$$

Sometimes the notation $\min\{a, b\}$ is denoted by $a \wedge b$, and $\max\{a, b\}$ is denoted by $a \vee b$.

Definition 1.3.¹⁵ Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic ideal of a ring R . Then the neutrosophic nil radical of A is defined to be a neutrosophic set $\sqrt{A} = (\mu_{\sqrt{A}}, \gamma_{\sqrt{A}}, \psi_{\sqrt{A}})$ by $\mu_{\sqrt{A}}(x) = \bigvee_{n \geq 1} \mu_A(x^n)$, $\gamma_{\sqrt{A}}(x) = \bigvee_{n \geq 1} \gamma_A(x^n)$, and $\psi_{\sqrt{A}}(x) = \bigwedge_{n \geq 1} \psi_A(x^n)$ for all $x \in R$ and for $n \in \mathbb{N}$.

Theorem 1.4.¹⁵ If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic ideal of a ring R , then \sqrt{A} is a neutrosophic ideal of R .

Proposition 1.5.¹⁵ If $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ are neutrosophic ideals of a ring R , then

- (1) $A \subseteq \sqrt{A}$,
- (2) if $A \subseteq B$, then $\sqrt{A} \subseteq \sqrt{B}$,
- (3) $\sqrt{\sqrt{A}} = \sqrt{A}$,
- (4) $\sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$,
- (5) $\sqrt{A \cup B} = \sqrt{A} \cup \sqrt{B}$.

2 Some properties of neutrosophic ideals

Definition 2.1. Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic set on a set X . The neutrosophic sets $\oplus A$, $\otimes A$, and $\odot A$ are defined as follows: $\oplus A = \{ \langle x, \mu_A(x), \gamma_A(x), \overline{\mu_A}(x) \rangle : x \in X \}$, $\otimes A = \{ \langle x, \mu_A(x), \gamma_A(x), \overline{\gamma_A}(x) \rangle : x \in X \}$, and $\odot A = \{ \langle x, \overline{\psi_A}(x), \overline{\psi_A}(x), \psi_A(x) \rangle : x \in X \}$.

Theorem 2.2. If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic ideal of a ring R , then $\oplus \sqrt{A} = (\mu_{\sqrt{A}}, \gamma_{\sqrt{A}}, \overline{\mu_{\sqrt{A}}})$, $\otimes \sqrt{A} = (\mu_{\sqrt{A}}, \gamma_{\sqrt{A}}, \overline{\gamma_{\sqrt{A}}})$, and $\odot \sqrt{A} = (\overline{\psi_{\sqrt{A}}}, \overline{\psi_{\sqrt{A}}}, \psi_{\sqrt{A}})$ are neutrosophic ideals of R .

Proof. Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic ideal of a ring R . Let $x, y \in R$. By Theorem 1.4, we have \sqrt{A} is a neutrosophic ideal of R . Thus

$$(\forall x, y \in R) \left(\begin{array}{l} \mu_{\sqrt{A}}(x - y) \geq \min\{\mu_{\sqrt{A}}(x), \mu_{\sqrt{A}}(y)\} \\ \gamma_{\sqrt{A}}(x - y) \geq \min\{\gamma_{\sqrt{A}}(x), \gamma_{\sqrt{A}}(y)\} \\ \psi_{\sqrt{A}}(x - y) \leq \max\{\psi_{\sqrt{A}}(x), \psi_{\sqrt{A}}(y)\} \end{array} \right),$$

$$(\forall x, y \in R) \left(\begin{array}{l} \mu_{\sqrt{A}}(xy) \geq \max\{\mu_{\sqrt{A}}(x), \mu_{\sqrt{A}}(y)\} \\ \gamma_{\sqrt{A}}(xy) \geq \max\{\gamma_{\sqrt{A}}(x), \gamma_{\sqrt{A}}(y)\} \\ \psi_{\sqrt{A}}(xy) \leq \min\{\psi_{\sqrt{A}}(x), \psi_{\sqrt{A}}(y)\} \end{array} \right).$$

It is sufficient to show that $\overline{\mu_{\sqrt{A}}}$ satisfies the conditions

$$(\forall x, y \in R) \left(\begin{array}{l} \overline{\mu_{\sqrt{A}}}(x - y) \leq \max\{\overline{\mu_{\sqrt{A}}}(x), \overline{\mu_{\sqrt{A}}}(y)\} \\ \overline{\mu_{\sqrt{A}}}(xy) \leq \min\{\overline{\mu_{\sqrt{A}}}(x), \overline{\mu_{\sqrt{A}}}(y)\} \end{array} \right).$$

For $x, y \in R$, we have

$$\begin{array}{l} \mu_{\sqrt{A}}(x - y) \geq \min\{\mu_{\sqrt{A}}(x), \mu_{\sqrt{A}}(y)\}, \\ 1 - \mu_{\sqrt{A}}(x - y) \leq 1 - \min\{\mu_{\sqrt{A}}(x), \mu_{\sqrt{A}}(y)\}, \\ \overline{\mu_{\sqrt{A}}}(x - y) \leq \max\{1 - \mu_{\sqrt{A}}(x), 1 - \mu_{\sqrt{A}}(y)\}, \\ \overline{\mu_{\sqrt{A}}}(x - y) \leq \max\{\overline{\mu_{\sqrt{A}}}(x), \overline{\mu_{\sqrt{A}}}(y)\}. \end{array}$$

Also, for $x, y \in R$, we have

$$\begin{array}{l} \mu_{\sqrt{A}}(xy) \geq \max\{\mu_{\sqrt{A}}(x), \mu_{\sqrt{A}}(y)\}, \\ 1 - \mu_{\sqrt{A}}(xy) \leq 1 - \max\{\mu_{\sqrt{A}}(x), \mu_{\sqrt{A}}(y)\}, \\ \overline{\mu_{\sqrt{A}}}(xy) \leq \min\{1 - \mu_{\sqrt{A}}(x), 1 - \mu_{\sqrt{A}}(y)\}, \\ \overline{\mu_{\sqrt{A}}}(xy) \leq \min\{\overline{\mu_{\sqrt{A}}}(x), \overline{\mu_{\sqrt{A}}}(y)\}. \end{array}$$

Hence, $\oplus\sqrt{A}$ is a neutrosophic ideal of R . The proof for $\otimes\sqrt{A}$ and $\odot\sqrt{A}$ are similar. □

Theorem 2.3. If $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ are neutrosophic ideals of a ring R , then $\sqrt{A} \cap B$ is a neutrosophic ideal of R .

Proof. Let $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ be neutrosophic ideals of a ring R . By Theorem 1.4, we have \sqrt{A} and \sqrt{B} are neutrosophic ideals of R . Thus

$$(\forall x, y \in R) \left(\begin{array}{l} \mu_{\sqrt{A}}(x - y) \geq \min\{\mu_{\sqrt{A}}(x), \mu_{\sqrt{A}}(y)\} \\ \gamma_{\sqrt{A}}(x - y) \geq \min\{\gamma_{\sqrt{A}}(x), \gamma_{\sqrt{A}}(y)\} \\ \psi_{\sqrt{A}}(x - y) \leq \max\{\psi_{\sqrt{A}}(x), \psi_A(y)\} \end{array} \right),$$

$$(\forall x, y \in R) \left(\begin{array}{l} \mu_{\sqrt{A}}(xy) \geq \max\{\mu_{\sqrt{A}}(x), \mu_{\sqrt{A}}(y)\} \\ \gamma_{\sqrt{A}}(xy) \geq \max\{\gamma_{\sqrt{A}}(x), \gamma_{\sqrt{A}}(y)\} \\ \psi_{\sqrt{A}}(xy) \leq \min\{\psi_{\sqrt{A}}(x), \psi_{\sqrt{A}}(y)\} \end{array} \right),$$

$$(\forall x, y \in R) \left(\begin{array}{l} \mu_{\sqrt{B}}(x - y) \geq \min\{\mu_{\sqrt{B}}(x), \mu_{\sqrt{B}}(y)\} \\ \gamma_{\sqrt{B}}(x - y) \geq \min\{\gamma_{\sqrt{B}}(x), \gamma_{\sqrt{B}}(y)\} \\ \psi_{\sqrt{B}}(x - y) \leq \max\{\psi_{\sqrt{B}}(x), \psi_B(y)\} \end{array} \right),$$

$$(\forall x, y \in R) \left(\begin{array}{l} \mu_{\sqrt{B}}(xy) \geq \max\{\mu_{\sqrt{B}}(x), \mu_{\sqrt{B}}(y)\} \\ \gamma_{\sqrt{B}}(xy) \geq \max\{\gamma_{\sqrt{B}}(x), \gamma_{\sqrt{B}}(y)\} \\ \psi_{\sqrt{B}}(xy) \leq \min\{\psi_{\sqrt{B}}(x), \psi_{\sqrt{B}}(y)\} \end{array} \right).$$

Then

$$\begin{aligned} \min\{\mu_{\sqrt{A} \cap B}(x), \mu_{\sqrt{A} \cap B}(y)\} &= \min\{\min\{\mu_{\sqrt{A}}(x), \mu_{\sqrt{A}}(x)\}, \min\{\mu_{\sqrt{B}}(y), \mu_{\sqrt{B}}(y)\}\} \\ &\leq \min\{\mu_{\sqrt{A}}(x - y), \mu_{\sqrt{B}}(x - y)\} \\ &= \mu_{\sqrt{A} \cap B}(x - y), \\ \min\{\gamma_{\sqrt{A} \cap B}(x), \gamma_{\sqrt{A} \cap B}(y)\} &= \min\{\min\{\gamma_{\sqrt{A}}(x), \gamma_{\sqrt{A}}(x)\}, \min\{\gamma_{\sqrt{B}}(y), \gamma_{\sqrt{B}}(y)\}\} \\ &\leq \min\{\gamma_{\sqrt{A}}(x - y), \gamma_{\sqrt{B}}(x - y)\} \\ &= \gamma_{\sqrt{A} \cap B}(x - y), \end{aligned}$$

$$\begin{aligned} \max\{\psi_{\sqrt{A \cap B}}(x), \psi_{\sqrt{A \cap B}}(y)\} &= \max\{\min\{\psi_{\sqrt{A}}(x), \psi_{\sqrt{A}}(x)\}, \min\{\psi_{\sqrt{B}}(y), \psi_{\sqrt{B}}(y)\}\} \\ &\geq \max\{\psi_{\sqrt{A}}(x - y), \psi_{\sqrt{B}}(x - y)\} \\ &= \psi_{\sqrt{A \cap B}}(x - y). \end{aligned}$$

Also, we have

$$\begin{aligned} \max\{\mu_{\sqrt{A \cap B}}(x), \mu_{\sqrt{A \cap B}}(y)\} &= \max\{\min\{\mu_{\sqrt{A}}(x), \mu_{\sqrt{B}}(x)\}, \min\{\mu_{\sqrt{A}}(y), \mu_{\sqrt{B}}(y)\}\} \\ &= \min\{\max\{\mu_{\sqrt{A}}(x), \mu_{\sqrt{A}}(y)\}, \max\{\mu_{\sqrt{B}}(x), \mu_{\sqrt{B}}(y)\}\} \\ &\leq \min\{\mu_{\sqrt{A}}(xy), \mu_{\sqrt{B}}(xy)\} \\ &= \mu_{\sqrt{A \cap B}}(xy), \\ \max\{\gamma_{\sqrt{A \cap B}}(x), \gamma_{\sqrt{A \cap B}}(y)\} &= \max\{\min\{\gamma_{\sqrt{A}}(x), \gamma_{\sqrt{B}}(x)\}, \min\{\gamma_{\sqrt{A}}(y), \gamma_{\sqrt{B}}(y)\}\} \\ &= \min\{\max\{\gamma_{\sqrt{A}}(x), \gamma_{\sqrt{A}}(y)\}, \max\{\gamma_{\sqrt{B}}(x), \gamma_{\sqrt{B}}(y)\}\} \\ &\leq \min\{\gamma_{\sqrt{A}}(xy), \gamma_{\sqrt{B}}(xy)\} \\ &= \gamma_{\sqrt{A \cap B}}(xy), \\ \min\{\psi_{\sqrt{A \cap B}}(x), \psi_{\sqrt{A \cap B}}(y)\} &= \min\{\min\{\psi_{\sqrt{A}}(x), \psi_{\sqrt{B}}(x)\}, \min\{\psi_{\sqrt{A}}(y), \psi_{\sqrt{B}}(y)\}\} \\ &= \min\{\min\{\psi_{\sqrt{A}}(x), \psi_{\sqrt{A}}(y)\}, \min\{\psi_{\sqrt{B}}(x), \psi_{\sqrt{B}}(y)\}\} \\ &\geq \min\{\psi_{\sqrt{A}}(xy), \psi_{\sqrt{B}}(xy)\} \\ &= \psi_{\sqrt{A \cap B}}(xy). \end{aligned}$$

Hence, $\sqrt{A \cap B}$ is a neutrosophic ideal of R . □

Example 2.4. Let $A = 2\mathbb{Z}$ and $R = \mathbb{Z}$. Define a neutrosophic set A on \mathbb{Z} as follows:

$$\mu_A(x) = \begin{cases} 0.3 & \text{if } x \in 2\mathbb{Z} \\ 0.1 & \text{otherwise,} \end{cases} \quad \gamma_A(x) = \begin{cases} 0.4 & \text{if } x \in 2\mathbb{Z} \\ 0.2 & \text{otherwise,} \end{cases} \quad \psi_A(x) = \begin{cases} 0.2 & \text{if } x \in 2\mathbb{Z} \\ 0.4 & \text{otherwise.} \end{cases}$$

Then $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic ideal of R , also by Theorem 1.4, we have \sqrt{A} is a neutrosophic ideal of R .

Definition 2.5. For any $t \in [0, 1]$ and a fuzzy set μ in a set X , the set $U(\mu, t) = \{x \in X : \mu(x) \geq t\}$ (respectively, $L(\mu, t) = \{x \in X : \mu(x) \leq t\}$) is called an upper (respectively, lower) t -level cut of μ .

Definition 2.6. A fuzzy set μ in a set X is said to have sup property (respectively, inf property) if for every non-empty subset S of X , there exists $x_0 \in S$ (respectively, $x_1 \in S$) such that $\mu(x_0) = \sup_{x \in S} \mu(x)$ (respectively, $\mu(x_1) = \inf_{x \in S} \mu(x)$).

Theorem 2.7. Let $f : R \rightarrow S$ be a ring epimorphism. Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic ideal of R such that $\mu_{\sqrt{A}}, \gamma_{\sqrt{A}}, \psi_{\sqrt{A}}, \mu_A$, and γ_A have sup property and ψ_A has inf property, then for every $t \in [0, 1]$,

- (1) $U(f(\psi_{\sqrt{A}}), t) \subseteq f(U(\psi_A, t))$,
- (2) $L(f(\mu_{\sqrt{A}}), t) \subseteq f(L(\mu_A, t))$,
- (3) $L(f(\gamma_{\sqrt{A}}), t) \subseteq f(L(\gamma_A, t))$,
- (4) $U(f(\mu_{\sqrt{A}}), t) \supseteq f(U(\mu_A, t))$,
- (5) $U(f(\gamma_{\sqrt{A}}), t) \supseteq f(U(\gamma_A, t))$,
- (6) $L(f(\psi_{\sqrt{A}}), t) \supseteq f(L(\psi_A, t))$.

Proof. (1) Now,

$$\begin{aligned} y \in U(f(\psi_{\sqrt{A}}), t) &\Rightarrow f(\psi_{\sqrt{A}})(y) \geq t \\ &\Rightarrow \sup_{x \in f^{-1}(y)} \psi_{\sqrt{A}}(x) \geq t \\ &\Rightarrow \exists x_0 \in f^{-1}(y), \psi_{\sqrt{A}}(x_0) \geq t \\ &\Rightarrow f(x_0) = y, \inf_{n \geq 1} \psi_A(x_0^n) \geq t \\ &\Rightarrow f(x_0) = y, \exists n_0 \in \mathbb{N}, \psi_A(x_0^{n_0}) \geq t \\ &\Rightarrow f(x_0) = y, \psi_A(x_0) \geq t \\ &\Rightarrow f(x_0) = y, x_0 \in U(\psi_A, t) \\ &\Rightarrow y \in f(U(\psi_A, t)). \end{aligned}$$

Hence, $U(f(\psi_{\sqrt{A}}), t) \subseteq f(U(\psi_A, t))$.

(2) Now,

$$\begin{aligned} y \in L(f(\mu_{\sqrt{A}}), t) &\Rightarrow f(\mu_{\sqrt{A}})(y) \leq t \\ &\Rightarrow \sup_{x \in f^{-1}(y)} \mu_{\sqrt{A}}(x) \leq t \\ &\Rightarrow \forall x \in f^{-1}(y), \mu_{\sqrt{A}}(x) \leq t \\ &\Rightarrow \forall x \in f^{-1}(y), \sup_{n \geq 1} \mu_A(x^n) \leq t \\ &\Rightarrow \forall x \in f^{-1}(y), \forall n \in \mathbb{N}, \mu_A(x^n) \leq t \\ &\Rightarrow \forall x \in f^{-1}(y), \mu_A(x) \leq t \\ &\Rightarrow \forall x \in f^{-1}(y), x \in L(\mu_A, t) \\ &\Rightarrow y \in f(L(\mu_A, t)). \end{aligned}$$

Hence, $L(f(\mu_{\sqrt{A}}), t) \subseteq f(L(\mu_A, t))$.

(3) Now,

$$\begin{aligned} y \in L(f(\gamma_{\sqrt{A}}), t) &\Rightarrow f(\gamma_{\sqrt{A}})(y) \leq t \\ &\Rightarrow \sup_{x \in f^{-1}(y)} \gamma_{\sqrt{A}}(x) \leq t \\ &\Rightarrow \forall x \in f^{-1}(y), \gamma_{\sqrt{A}}(x) \leq t \\ &\Rightarrow \forall x \in f^{-1}(y), \sup_{n \geq 1} \gamma_A(x^n) \leq t \\ &\Rightarrow \forall x \in f^{-1}(y), \forall n \in \mathbb{N}, \gamma_A(x^n) \leq t \\ &\Rightarrow \forall x \in f^{-1}(y), \gamma_A(x) \leq t \\ &\Rightarrow \forall x \in f^{-1}(y), x \in L(\gamma_A, t) \\ &\Rightarrow y \in f(L(\gamma_A, t)). \end{aligned}$$

Hence, $L(f(\gamma_{\sqrt{A}}), t) \subseteq f(L(\gamma_A, t))$.

(4) Now,

$$\begin{aligned} y \in f(U(\mu_A, t)) &\Rightarrow f(x_0) = y, x_0 \in U(\mu_A, t) \\ &\Rightarrow f(x_0) = y, \mu_A(x_0) \geq t \\ &\Rightarrow f(x_0) = y, \exists n_0 \in \mathbb{N}, \mu_A(x_0^{n_0}) \geq t \\ &\Rightarrow f(x_0) = y, \inf_{n \geq 1} \mu_A(x_0^n) \geq t \\ &\Rightarrow \exists x_0 \in f^{-1}(y), \mu_{\sqrt{A}}(x_0) \geq t \\ &\Rightarrow \sup_{x \in f^{-1}(y)} (\mu_{\sqrt{A}})(y) \geq t \\ &\Rightarrow f(\mu_{\sqrt{A}})(y) \geq t \\ &\Rightarrow y \in U(f(\mu_{\sqrt{A}}), t). \end{aligned}$$

Hence, $U(f(\mu_{\sqrt{A}}), t) \supseteq f(U(\mu_A, t))$.

(5) Now,

$$\begin{aligned} y \in f(U(\gamma_A, t)) &\Rightarrow f(x_0) = y, x_0 \in U(\gamma_A, t) \\ &\Rightarrow f(x_0) = y, \gamma_A(x_0) \geq t \\ &\Rightarrow f(x_0) = y, \exists n_0 \in \mathbb{N}, \gamma_A(x_0^{n_0}) \geq t \\ &\Rightarrow f(x_0) = y, \inf_{n \geq 1} \gamma_A(x_0^n) \geq t \\ &\Rightarrow \exists x_0 \in f^{-1}(y), \gamma_{\sqrt{A}}(x_0) \geq t \\ &\Rightarrow \sup_{x \in f^{-1}(y)} (\gamma_{\sqrt{A}})(y) \geq t \\ &\Rightarrow f(\gamma_{\sqrt{A}})(y) \geq t \\ &\Rightarrow y \in U(f(\gamma_{\sqrt{A}}), t). \end{aligned}$$

Hence, $U(f(\gamma_{\sqrt{A}}), t) \supseteq f(U(\gamma_A, t))$.

(6) Now,

$$\begin{aligned} y \in f(L(\psi_A, t)) &\Rightarrow f(x_0) = y, x_0 \in L(\psi_A, t) \\ &\Rightarrow f(x_0) = y, \psi_A(x_0) \leq t \\ &\Rightarrow f(x_0) = y, \exists n_0 \in \mathbb{N}, \psi_A(x_0^{n_0}) \leq t \\ &\Rightarrow f(x_0) = y, \inf_{n \geq 1} \psi_A(x_0^n) \leq t \\ &\Rightarrow \exists x_0 \in f^{-1}(y), \psi_{\sqrt{A}}(x_0) \leq t \\ &\Rightarrow \sup_{x \in f^{-1}(y)} (\psi_{\sqrt{A}})(y) \leq t \\ &\Rightarrow f(\psi_{\sqrt{A}})(y) \leq t \\ &\Rightarrow y \in L(f(\psi_{\sqrt{A}}), t). \end{aligned}$$

Hence, $L(f(\psi_{\sqrt{A}}), t) \supseteq f(L(\psi_A, t))$. □

Theorem 2.8. Let $f : R \rightarrow S$ be a ring homomorphism. Let $B = (\mu_B, \gamma_B, \psi_B)$ be a neutrosophic ideal of S such that $\mu_{\sqrt{A}}, \gamma_{\sqrt{A}}, \psi_{\sqrt{A}}, \mu_A$, and γ_A have sup property and ψ_A has inf property, then for every $t \in [0, 1]$,

- (1) $U(f^{-1}(\psi_{\sqrt{B}}), t) \subseteq f^{-1}(U(\psi_B, t))$,
- (2) $L(f^{-1}(\mu_{\sqrt{B}}), t) \subseteq f^{-1}(L(\mu_B, t))$,
- (3) $L(f^{-1}(\gamma_{\sqrt{B}}), t) \subseteq f^{-1}(L(\gamma_B, t))$,
- (4) $U(f^{-1}(\mu_{\sqrt{B}}), t) \supseteq f^{-1}(U(\mu_B, t))$,
- (5) $U(f^{-1}(\gamma_{\sqrt{B}}), t) \supseteq f^{-1}(U(\gamma_B, t))$,
- (6) $L(f^{-1}(\psi_{\sqrt{B}}), t) \supseteq f^{-1}(L(\psi_B, t))$.

Proof. (1) Now,

$$\begin{aligned} x_0 \in U(f^{-1}(\psi_{\sqrt{B}}), t) &\Rightarrow f^{-1}(\psi_{\sqrt{B}})(x_0) \geq t \\ &\Rightarrow \psi_{\sqrt{B}}(f(x_0)) \geq t \\ &\Rightarrow \inf_{n \geq 1} \psi_B(f^n(x_0)) \geq t \\ &\Rightarrow \psi_B(f^n(x_0)) \geq t, \forall n \in \mathbb{N} \\ &\Rightarrow \psi_B(f(x_0)) \geq t \\ &\Rightarrow f(x_0) \in U(\psi_B, t) \\ &\Rightarrow x_0 \in f^{-1}(U(\psi_B, t)). \end{aligned}$$

Hence, $U(f^{-1}(\psi_{\sqrt{B}}), t) \subseteq f^{-1}(U(\psi_B, t))$.

(2) Now,

$$\begin{aligned} x_0 \in L(f^{-1}(\mu_{\sqrt{B}}), t) &\Rightarrow f^{-1}(\mu_{\sqrt{B}})(x_0) \leq t \\ &\Rightarrow \mu_{\sqrt{B}}(f(x_0)) \leq t \\ &\Rightarrow \sup_{n \geq 1} \mu_B(f^n(x_0)) \leq t \\ &\Rightarrow \mu_B(f^n(x_0)) \leq t, \forall n \in \mathbb{N} \\ &\Rightarrow \mu_B(f(x_0)) \leq t \\ &\Rightarrow f(x_0) \in L(\mu_B, t) \\ &\Rightarrow x_0 \in f^{-1}(L(\mu_B, t)). \end{aligned}$$

Hence, $L(f^{-1}(\mu_{\sqrt{B}}), t) \subseteq f^{-1}(L(\mu_B, t))$.

(3) Now,

$$\begin{aligned} x_0 \in L(f^{-1}(\gamma_{\sqrt{B}}), t) &\Rightarrow f^{-1}(\gamma_{\sqrt{B}})(x_0) \leq t \\ &\Rightarrow \gamma_{\sqrt{B}}(f(x_0)) \leq t \\ &\Rightarrow \sup_{n \geq 1} \gamma_B(f^n(x_0)) \leq t \\ &\Rightarrow \gamma_B(f^n(x_0)) \leq t, \forall n \in \mathbb{N} \\ &\Rightarrow \gamma_B(f(x_0)) \leq t \\ &\Rightarrow f(x_0) \in L(\gamma_B, t) \\ &\Rightarrow x_0 \in f^{-1}(L(\gamma_B, t)). \end{aligned}$$

Hence, $L(f^{-1}(\gamma_{\sqrt{B}}), t) \subseteq f^{-1}(L(\gamma_B, t))$. The proofs of (4), (5), and (6) are similar to those of (1), (2), and (3). □

3 Neutrosophic nil radicals of neutrosophic ideals

Definition 3.1. ¹⁵ Let $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ be neutrosophic sets on a ring R (not necessarily commutative). The neutrosophic product of $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ is defined to be the neutrosophic set $AB = (\mu_{AB}, \gamma_{AB}, \psi_{AB})$ on R given by

$$\mu_{AB}(x) = \bigvee \left\{ \bigwedge_{1 \leq i \leq k} \mu_A(a_i) \wedge \mu_B(b_i) : \sum_{i=1}^n a_i b_i = x, k \in \mathbb{N} \right\},$$

$$\gamma_{AB}(x) = \bigvee \left\{ \bigwedge_{1 \leq i \leq k} \gamma_A(a_i) \wedge \gamma_B(b_i) : \sum_{i=1}^n a_i b_i = x, k \in \mathbb{N} \right\},$$

$$\psi_{AB}(x) = \bigwedge \left\{ \bigvee_{1 \leq i \leq k} \psi_A(a_i) \vee \psi_B(b_i) : \sum_{i=1}^n a_i b_i = x, k \in \mathbb{N} \right\},$$

if we can express $x = \sum_{i=1}^n a_i b_i$ for some $a_i, b_i \in \mathbb{R}$, where each $a_i b_i \neq 0$ and $k \in \mathbb{N}$. Otherwise, we define $AB = 0$, that is, $\mu_{AB}(x) = 0$, $\gamma_{AB}(x) = 0$, and $\psi_{AB}(x) = 1$.

Obviously, the product AB is commutative if R is a commutative ring.

Theorem 3.2. *If $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ are neutrosophic ideals of a ring R , then AB is a neutrosophic ideal of R .*

Proof. Let $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ be neutrosophic ideals of a ring R . For any $x, y \in R$ and $k, m, n \in \mathbb{N}$, we have

$$\begin{aligned} \mu_{AB}(x - y) &= \bigvee_{x-y=\sum_{i=1}^k a_i b_i} \left\{ \bigwedge_{1 \leq i \leq k} \mu_A(a_i) \wedge \mu_B(b_i) \right\} \\ &\geq \bigvee_{x=\sum_{i=1}^n a_i b_i, -y=\sum_{i=1}^n -c_i d_i} \left\{ \left(\bigwedge_{1 \leq i \leq m} \mu_A(a_i) \wedge \mu_B(b_i) \right) \wedge \left(\bigwedge_{1 \leq i \leq n} \mu_A(-c_i) \wedge \mu_B(d_i) \right) \right\} \\ &= \bigvee_{x=\sum_{i=1}^n a_i b_i, y=\sum_{i=1}^n c_i d_i} \left\{ \left(\bigwedge_{1 \leq i \leq m} \mu_A(a_i) \wedge \mu_B(b_i) \right) \wedge \left(\bigwedge_{1 \leq i \leq n} \mu_A(c_i) \wedge \mu_B(d_i) \right) \right\} \\ &= \bigvee_{x=\sum_{i=1}^n a_i b_i} \left(\bigwedge_{1 \leq i \leq m} \mu_A(a_i) \wedge \mu_B(b_i) \right) \wedge \bigvee_{y=\sum_{i=1}^n c_i d_i} \left(\bigwedge_{1 \leq i \leq n} \mu_A(c_i) \wedge \mu_B(d_i) \right) \\ &= \mu_{AB}(x) \wedge \mu_{AB}(y), \\ \gamma_{AB}(x - y) &= \bigvee_{x-y=\sum_{i=1}^k a_i b_i} \left\{ \bigwedge_{1 \leq i \leq k} \gamma_A(a_i) \wedge \gamma_B(b_i) \right\} \\ &\geq \bigvee_{x=\sum_{i=1}^n a_i b_i, -y=\sum_{i=1}^n -c_i d_i} \left\{ \left(\bigwedge_{1 \leq i \leq m} \gamma_A(a_i) \wedge \gamma_B(b_i) \right) \wedge \left(\bigwedge_{1 \leq i \leq n} \gamma_A(-c_i) \wedge \gamma_B(d_i) \right) \right\} \\ &= \bigvee_{x=\sum_{i=1}^n a_i b_i, y=\sum_{i=1}^n c_i d_i} \left\{ \left(\bigwedge_{1 \leq i \leq m} \gamma_A(a_i) \wedge \gamma_B(b_i) \right) \wedge \left(\bigwedge_{1 \leq i \leq n} \gamma_A(c_i) \wedge \gamma_B(d_i) \right) \right\} \\ &= \bigvee_{x=\sum_{i=1}^n a_i b_i} \left(\bigwedge_{1 \leq i \leq m} \gamma_A(a_i) \wedge \gamma_B(b_i) \right) \wedge \bigvee_{y=\sum_{i=1}^n c_i d_i} \left(\bigwedge_{1 \leq i \leq n} \gamma_A(c_i) \wedge \gamma_B(d_i) \right) \\ &= \gamma_{AB}(x) \wedge \gamma_{AB}(y), \end{aligned}$$

$$\begin{aligned}
 \psi_{AB}(x - y) &= \bigwedge_{x-y=\sum_{i=1}^k a_i b_i} \left\{ \bigvee_{1 \leq i \leq k} \psi_A(a_i) \vee \psi_B(b_i) \right\} \\
 &\leq \bigwedge_{x=\sum_{i=1}^n a_i b_i, -y=\sum_{i=1}^n -c_i d_i} \left\{ \left(\bigvee_{1 \leq i \leq m} \psi_A(a_i) \vee \psi_B(b_i) \right) \vee \left(\bigvee_{1 \leq i \leq n} \psi_A(-c_i) \vee \psi_B(d_i) \right) \right\} \\
 &= \bigwedge_{x=\sum_{i=1}^n a_i b_i, y=\sum_{i=1}^n c_i d_i} \left\{ \left(\bigvee_{1 \leq i \leq m} \psi_A(a_i) \vee \psi_B(b_i) \right) \vee \left(\bigvee_{1 \leq i \leq n} \psi_A(c_i) \vee \psi_B(d_i) \right) \right\} \\
 &= \bigwedge_{x=\sum_{i=1}^n a_i b_i} \left(\bigvee_{1 \leq i \leq m} \psi_A(a_i) \vee \psi_B(b_i) \right) \vee \bigwedge_{y=\sum_{i=1}^n c_i d_i} \left(\bigvee_{1 \leq i \leq n} \psi_A(c_i) \vee \psi_B(d_i) \right) \\
 &= \psi_{AB}(x) \vee \psi_{AB}(y).
 \end{aligned}$$

Also, we have

$$\begin{aligned}
 \mu_{AB}(x) &= \bigvee_{x=\sum_{i=1}^m a_i b_i, m \in \mathbb{N}} \left\{ \bigwedge_{1 \leq i \leq m} \mu_A(a_i) \wedge \mu_B(b_i) \right\} \\
 &\leq \bigvee_{xy=\sum_{i=1}^m a_i(b_i y), m \in \mathbb{N}} \left\{ \bigwedge_{1 \leq i \leq m} \mu_A(a_i) \wedge \mu_B(b_i y) \right\} \\
 &\leq \bigvee_{xy=\sum_{i=1}^k \alpha_i \beta_i, k \in \mathbb{N}} \left\{ \bigwedge_{1 \leq i \leq k} \mu_A(\alpha_i) \wedge \mu_B(\beta_i) \right\} \\
 &= \mu_{AB}(xy), \\
 \gamma_{AB}(x) &= \bigvee_{x=\sum_{i=1}^m a_i b_i, m \in \mathbb{N}} \left\{ \bigwedge_{1 \leq i \leq m} \gamma_A(a_i) \wedge \gamma_B(b_i) \right\} \\
 &\leq \bigvee_{xy=\sum_{i=1}^m a_i(b_i y), m \in \mathbb{N}} \left\{ \bigwedge_{1 \leq i \leq m} \gamma_A(a_i) \wedge \gamma_B(b_i y) \right\} \\
 &\leq \bigvee_{xy=\sum_{i=1}^k \alpha_i \beta_i, k \in \mathbb{N}} \left\{ \bigwedge_{1 \leq i \leq k} \gamma_A(\alpha_i) \wedge \gamma_B(\beta_i) \right\} \\
 &= \gamma_{AB}(xy), \\
 \psi_{AB}(x) &= \bigwedge_{x=\sum_{i=1}^m a_i b_i, m \in \mathbb{N}} \left\{ \bigwedge_{1 \leq i \leq m} \psi_A(a_i) \wedge \psi_B(b_i) \right\} \\
 &\leq \bigwedge_{xy=\sum_{i=1}^m a_i(b_i y), m \in \mathbb{N}} \left\{ \bigwedge_{1 \leq i \leq m} \psi_A(a_i) \wedge \psi_B(b_i y) \right\} \\
 &\leq \bigwedge_{xy=\sum_{i=1}^k \alpha_i \beta_i, k \in \mathbb{N}} \left\{ \bigwedge_{1 \leq i \leq k} \psi_A(\alpha_i) \wedge \psi_B(\beta_i) \right\} \\
 &= \psi_{AB}(xy).
 \end{aligned}$$

Hence, $\mu_{AB}(x) \leq \mu_{AB}(xy)$, $\gamma_{AB}(x) \leq \gamma_{AB}(xy)$, and $\psi_{AB}(x) \geq \psi_{AB}(xy)$. Similarly, we get $\mu_{AB}(y) \leq \mu_{AB}(xy)$, $\gamma_{AB}(y) \leq \gamma_{AB}(xy)$, and $\psi_{AB}(y) \geq \psi_{AB}(xy)$. Therefore, AB is a neutrosophic ideal of R . \square

Theorem 3.3. If $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ are neutrosophic ideals of a ring R , then $\sqrt{AB} = \sqrt{A \cap B}$.

Proof. Let $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ be neutrosophic ideals of a ring R . Let $x \in R$ and

$x = \sum_{i=1}^n a_i b_i$ (\star), where $a_i, b_i \neq 0$ in R . Then for $1 \leq i \leq m$,

$$\begin{aligned} \mu_A(a_i) \wedge \mu_B(b_i) &\leq \mu_A(a_i) \leq \mu_A(a_i b_i), \\ \gamma_A(a_i) \wedge \gamma_B(b_i) &\leq \gamma_A(a_i) \leq \gamma_A(a_i b_i), \\ \psi_A(a_i) \vee \psi_B(b_i) &\geq \psi_A(a_i) \geq \psi_A(a_i b_i). \end{aligned}$$

Hence,

$$\begin{aligned} \bigwedge_{1 \leq i \leq m} \mu_A(a_i) \wedge \mu_B(b_i) &\leq \bigwedge_{1 \leq i \leq m} \mu_A(a_i b_i) \leq \mu_A\left(\sum_{i=1}^n a_i b_i\right) = \mu_A(x), \\ \bigwedge_{1 \leq i \leq m} \mu_A(a_i) \wedge \gamma_B(b_i) &\leq \bigwedge_{1 \leq i \leq m} \gamma_A(a_i b_i) \leq \gamma_A\left(\sum_{i=1}^n a_i b_i\right) = \gamma_A(x), \\ \bigvee_{1 \leq i \leq m} \psi_A(a_i) \vee \psi_B(b_i) &\geq \bigvee_{1 \leq i \leq m} \psi_A(a_i b_i) \geq \psi_A\left(\sum_{i=1}^n a_i b_i\right) = \psi_A(x). \end{aligned}$$

Taking the supremum and infimum, respectively, overall expressions like (\star), we have $\mu_{AB}(x) \leq \mu_A(x)$, $\gamma_{AB}(x) \leq \gamma_A(x)$, and $\psi_{AB}(x) \geq \psi_A(x)$. Similarly, we get $\mu_{AB}(x) \leq \mu_B(x)$, $\gamma_{AB}(x) \leq \gamma_B(x)$, and $\psi_{AB}(x) \geq \psi_B(x)$. Hence, $\mu_{AB}(x) \leq \mu_A(x) \wedge \mu_B(x)$, $\gamma_{AB}(x) \leq \gamma_A(x) \wedge \gamma_B(x)$, and $\psi_{AB}(x) \geq \psi_A(x) \vee \psi_B(x)$ for all $x \in R$, that is, $AB \subseteq A \cap B$. Using Proposition 1.5 (2), we obtain $\sqrt{AB} \subseteq \sqrt{A \cap B}$. Now for any $x \in R$, we have

$$\begin{aligned} \mu_{\sqrt{AB}}(x) &= \bigvee_k \mu_{AB}(x^k) \geq \mu_{AB}(x^{2n}) \geq \mu_A(x^n) \wedge \mu_B(x^n) = \mu_{A \cap B}(x^n), \\ \gamma_{\sqrt{AB}}(x) &= \bigvee_k \gamma_{AB}(x^k) \geq \gamma_{AB}(x^{2n}) \geq \gamma_A(x^n) \wedge \gamma_B(x^n) = \gamma_{A \cap B}(x^n), \\ \psi_{\sqrt{AB}}(x) &= \bigwedge_k \psi_{AB}(x^k) \leq \psi_{AB}(x^{2n}) \leq \psi_A(x^n) \vee \psi_B(x^n) = \psi_{A \cap B}(x^n), \end{aligned}$$

for all $n \geq 1$. Taking the supremum and infimum, respectively, over all $n \geq 1$, we get $\mu_{\sqrt{A \cap B}}(x) \leq \mu_{\sqrt{AB}}(x)$, $\gamma_{\sqrt{A \cap B}}(x) \leq \gamma_{\sqrt{AB}}(x)$, and $\psi_{\sqrt{A \cap B}}(x) \geq \psi_{\sqrt{AB}}(x)$. \square

Definition 3.4. ¹⁵ Let $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ be neutrosophic sets in a ring R (not necessarily commutative). The neutrosophic sum $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ is defined to be the neutrosophic set $A + B = (\mu_{A+B}, \gamma_{A+B}, \psi_{A+B})$ in R given by

$$\begin{aligned} \mu_{A+B}(x) &= \begin{cases} \bigvee_{x=y+z} (\mu_A(y) \wedge \mu_B(z)) & \text{if } x = y + z \\ 0 & \text{otherwise,} \end{cases} \\ \gamma_{A+B}(x) &= \begin{cases} \bigvee_{x=y+z} (\gamma_A(y) \wedge \gamma_B(z)) & \text{if } x = y + z \\ 0 & \text{otherwise,} \end{cases} \\ \psi_{A+B}(x) &= \begin{cases} \bigwedge_{x=y+z} (\psi_A(y) \vee \psi_B(z)) & \text{if } x = y + z \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Theorem 3.5. If $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ are neutrosophic ideals of a ring R , then $A + B$ is a neutrosophic ideal of R .

Proof. Let $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ be neutrosophic ideals of a ring R . Let $a, b \in R$. Then

$$\begin{aligned} \mu_{A+B}(x) \wedge \mu_{A+B}(y) &= \bigvee_{x=a+b} (\mu_A(a) \wedge \mu_B(b)) \wedge \bigvee_{y=c+d} (\mu_A(c) \wedge \mu_B(d)) \\ &= \bigvee_{\substack{x=a+b \\ y=c+d}} ((\mu_A(a) \wedge \mu_B(b)) \wedge (\mu_A(c) \wedge \mu_B(d))) \\ &= \bigvee_{\substack{x=a+b \\ -y=-c-d}} ((\mu_A(a) \wedge \mu_B(b)) \wedge (\mu_A(-c) \wedge \mu_B(-d))) \\ &= \bigvee_{\substack{x=a+b \\ -y=-c-d}} (((\mu_A(a) \wedge \mu_A(-c)) \wedge (\mu_B(b) \wedge \mu_B(-d)))) \\ &\leq \bigvee_{\substack{x-y= \\ (a-c)+(b-d)}} (\mu_A(a-c) \wedge \mu_B(b-d)) \\ &= \mu_{A+B}(x-y), \end{aligned}$$

$$\begin{aligned}
 \gamma_{A+B}(x) \wedge \gamma_{A+B}(y) &= \bigvee_{x=a+b} (\gamma_A(a) \wedge \gamma_B(b)) \wedge \bigvee_{y=c+d} (\gamma_A(c) \wedge \gamma_B(d)) \\
 &= \bigvee_{\substack{x=a+b \\ y=c+d}} ((\gamma_A(a) \wedge \gamma_B(b)) \wedge (\gamma_A(c) \wedge \gamma_B(d))) \\
 &= \bigvee_{\substack{x=a+b \\ -y=-c-d}} ((\gamma_A(a) \wedge \gamma_B(b)) \wedge (\gamma_A(-c) \wedge \gamma_B(-d))) \\
 &= \bigvee_{\substack{x=a+b \\ -y=-c-d}} (((\gamma_A(a) \wedge \gamma_A(-c)) \wedge (\gamma_B(b) \wedge \gamma_B(-d)))) \\
 &\leq \bigvee_{\substack{x-y= \\ (a-c)+(b-d)}} (\gamma_A(a-c) \wedge \gamma_B(b-d)) \\
 &= \gamma_{A+B}(x-y), \\
 \psi_{A+B}(x) \vee \psi_{A+B}(y) &= \bigwedge_{x=a+b} (\psi_A(a) \vee \psi_B(b)) \vee \bigwedge_{y=c+d} (\psi_A(c) \vee \psi_B(d)) \\
 &= \bigwedge_{\substack{x=a+b \\ y=c+d}} ((\psi_A(a) \vee \psi_B(b)) \vee (\psi_A(c) \vee \psi_B(d))) \\
 &= \bigwedge_{\substack{x=a+b \\ -y=-c-d}} ((\psi_A(a) \vee \psi_B(b)) \vee (\psi_A(-c) \vee \psi_B(-d))) \\
 &= \bigwedge_{\substack{x=a+b \\ -y=-c-d}} (((\psi_A(a) \vee \psi_A(-c)) \vee (\psi_B(b) \vee \psi_B(-d)))) \\
 &\leq \bigwedge_{\substack{x-y= \\ (a-c)+(b-d)}} (\psi_A(a-c) \vee \psi_B(b-d)) \\
 &= \psi_{A+B}(x-y).
 \end{aligned}$$

Also, we have

$$\begin{aligned}
 \mu_{A+B}(x) &= \bigvee_{x=a+b} (\mu_A(a) \wedge \mu_B(b)) \\
 &\leq \bigvee_{xy=ay+by} (\mu_A(ay) \wedge \mu_B(by)) \\
 &\leq \bigvee_{xy=\alpha+\beta} (\mu_A(\alpha) \wedge \mu_B(\beta)) \\
 &= \mu_{A+B}(xy), \\
 \gamma_{A+B}(x) &= \bigvee_{x=a+b} (\gamma_A(a) \wedge \gamma_B(b)) \\
 &\leq \bigvee_{xy=ay+by} (\gamma_A(ay) \wedge \gamma_B(by)) \\
 &\leq \bigvee_{xy=\alpha+\beta} (\gamma_A(\alpha) \wedge \gamma_B(\beta)) \\
 &= \gamma_{A+B}(xy), \\
 \psi_{A+B}(x) &= \bigwedge_{x=a+b} (\psi_A(a) \vee \psi_B(b)) \\
 &\geq \bigwedge_{xy=ay+by} (\psi_A(ay) \vee \psi_B(by)) \\
 &\geq \bigwedge_{xy=\alpha+\beta} (\psi_A(\alpha) \vee \psi_B(\beta)) \\
 &= \psi_{A+B}(xy).
 \end{aligned}$$

Hence, $\mu_{A+B}(x) \leq \mu_{A+B}(xy)$, $\gamma_{A+B}(x) \leq \gamma_{A+B}(xy)$, and $\psi_{A+B}(x) \geq \psi_{A+B}(xy)$. Similarly, we get $\mu_{A+B}(y) \leq \mu_{A+B}(xy)$, $\gamma_{A+B}(y) \leq \gamma_{A+B}(xy)$, and $\psi_{A+B}(y) \geq \psi_{A+B}(xy)$. Therefore, $A + B$ is a neutrosophic ideal of R . □

Theorem 3.6. If $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ are neutrosophic ideals of a ring R , then $\sqrt{A} + \sqrt{B} \subseteq \sqrt{\sqrt{A} + \sqrt{B}} = \sqrt{A + B}$.

Proof. The first inclusion follows from Proposition 1.5 (1). Since $A \subseteq \sqrt{A}$ and $B \subseteq \sqrt{B}$ (by Proposition 1.5

(1)), it follows from Proposition 1.5 (2) that $\sqrt{A+B} \subseteq \sqrt{\sqrt{A}+\sqrt{B}}$. Let $x \in R$. Then

$$\begin{aligned} \mu_{\sqrt{A+B}}(x) &= \bigvee_{x=a+b} \{ \mu_{\sqrt{A}}(a) \wedge \mu_{\sqrt{B}}(b) \} \\ &= \bigvee_{x=a+b} \left\{ \left(\bigvee_{m \geq 1} \mu_A(a^m) \right) \wedge \left(\bigvee_{n \geq 1} \mu_A(b^n) \right) \right\} \\ &= \bigvee_{x=a+b} \left(\bigvee_{m \geq 1} \bigvee_{n \geq 1} \{ \mu_A(a^m) \wedge \mu_A(b^n) \} \right), \\ \gamma_{\sqrt{A+B}}(x) &= \bigvee_{x=a+b} \{ \gamma_{\sqrt{A}}(a) \wedge \gamma_{\sqrt{B}}(b) \} \\ &= \bigvee_{x=a+b} \left\{ \left(\bigvee_{m \geq 1} \gamma_A(a^m) \right) \wedge \left(\bigvee_{n \geq 1} \gamma_A(b^n) \right) \right\} \\ &= \bigvee_{x=a+b} \left(\bigvee_{m \geq 1} \bigvee_{n \geq 1} \{ \gamma_A(a^m) \wedge \gamma_A(b^n) \} \right), \\ \psi_{\sqrt{A+B}}(x) &= \bigwedge_{x=a+b} \{ \psi_{\sqrt{A}}(a) \vee \psi_{\sqrt{B}}(b) \} \\ &= \bigwedge_{x=a+b} \left\{ \left(\bigwedge_{m \geq 1} \psi_A(a^m) \right) \vee \left(\bigwedge_{n \geq 1} \psi_A(b^n) \right) \right\} \\ &= \bigwedge_{x=a+b} \left(\bigwedge_{m \geq 1} \bigwedge_{n \geq 1} \{ \psi_A(a^m) \vee \psi_A(b^n) \} \right). \end{aligned}$$

Now, let $x = a + b$ for $a, b \in R$ and $m, n \in \mathbb{N}$. Since R is commutative, we have $x^{m+n} = ta^m + rb^n$ for some $t, r \in R$. Hence,

$$\begin{aligned} \mu_A(a^m) \wedge \mu_A(b^n) &\leq \mu_A(ta^m) \wedge \mu_A(rb^n) \\ &\leq \mu_{A+B}(ta^m + rb^n) \\ &= \mu_{A+B}(x^{m+n}) \\ &\leq \bigvee_{k \geq 1} \mu_{A+B}(x^k) \\ &= \mu_{\sqrt{A+B}}(x), \\ \gamma_A(a^m) \wedge \gamma_A(b^n) &\leq \gamma_A(ta^m) \wedge \gamma_A(rb^n) \\ &\leq \gamma_{A+B}(ta^m + rb^n) \\ &= \gamma_{A+B}(x^{m+n}) \\ &\leq \bigvee_{k \geq 1} \gamma_{A+B}(x^k) \\ &= \gamma_{\sqrt{A+B}}(x), \\ \psi_A(a^m) \vee \psi_A(b^n) &\geq \psi_A(ta^m) \vee \psi_A(rb^n) \\ &\geq \psi_{A+B}(ta^m + rb^n) \\ &= \psi_{A+B}(x^{m+n}) \\ &\geq \bigwedge_{k \geq 1} \psi_{A+B}(x^k) \\ &= \psi_{\sqrt{A+B}}(x). \end{aligned}$$

It follows that $\mu_{\sqrt{A+B}}(x) \geq \mu_{\sqrt{A}+\sqrt{B}}(x)$, $\gamma_{\sqrt{A+B}}(x) \geq \gamma_{\sqrt{A}+\sqrt{B}}(x)$, and $\psi_{\sqrt{A+B}}(x) \leq \psi_{\sqrt{A}+\sqrt{B}}(x)$ for all $x \in R$. Thus $\sqrt{A+B} \supseteq \sqrt{\sqrt{A}+\sqrt{B}}$, it follows from Proposition 1.5 (2) and (3) that $\sqrt{A+B} = \sqrt{\sqrt{A}+\sqrt{B}} \supseteq \sqrt{\sqrt{A}+\sqrt{B}}$. Hence, $\sqrt{A}+\sqrt{B} \subseteq \sqrt{\sqrt{A}+\sqrt{B}} = \sqrt{A+B}$. \square

Theorem 3.7. Let $f : R \rightarrow S$ be a ring homomorphism. If $A = (\mu_A, \gamma_A, \psi_A)$ and $B = (\mu_B, \gamma_B, \psi_B)$ are neutrosophic ideals of rings R and S , respectively, then

- (1) $\sqrt{f^{-1}(B)} = f^{-1}(\sqrt{B})$,
- (2) $\sqrt{f(A)} = f(\sqrt{A})$, provided $A = (\mu_A, \gamma_A, \psi_A)$ is f -invariant, that is, $f(x) = f(y)$ implies $\mu_A(x) = \mu_A(y)$, $\gamma_A(x) = \gamma_A(y)$, and $\psi_A(x) = \psi_A(y)$, and f is onto.

Proof. (1) For any $x \in R$, we have

$$\begin{aligned} \mu_{\sqrt{f^{-1}(B)}}(x) &= \bigvee_n \mu_{f^{-1}(B)}(x^n) \\ &= \bigvee_n \mu_B(f(x^n)) \\ &= \bigvee_n \mu_B((f(x))^n) \\ &= \mu_{\sqrt{B}}(f(x)) \\ &= \mu_{f^{-1}(\sqrt{B})}(x), \end{aligned}$$

$$\begin{aligned} \gamma_{\sqrt{f^{-1}(B)}}(x) &= \bigvee_n \gamma_{f^{-1}(B)}(x^n) \\ &= \bigvee_n \gamma_B(f(x^n)) \\ &= \bigvee_n \gamma_B((f(x))^n) \\ &= \gamma_{\sqrt{B}}(f(x)) \\ &= \gamma_{f^{-1}(\sqrt{B})}(x), \end{aligned}$$

$$\begin{aligned} \psi_{\sqrt{f^{-1}(B)}}(x) &= \bigwedge_n \psi_{f^{-1}(B)}(x^n) \\ &= \bigwedge_n \psi_B(f(x^n)) \\ &= \bigwedge_n \psi_B((f(x))^n) \\ &= \psi_{\sqrt{B}}(f(x)) \\ &= \psi_{f^{-1}(\sqrt{B})}(x). \end{aligned}$$

Hence, $\sqrt{f^{-1}(B)} = f^{-1}(\sqrt{B})$.

(2) Let $y \in S$. Since f is onto, there exists $x \in R$ such that $f(x) = y$. Furthermore, $f^{-1}(f(A)) = A$, that is, $\mu_{f^{-1}(f(A))} = \mu_A$, $\gamma_{f^{-1}(f(A))} = \gamma_A$, and $\psi_{f^{-1}(f(A))} = \psi_A$, as A is f -invariant. Let $(\varepsilon, \theta, \delta) > 0_\sim$, that is, $\varepsilon > 0$, $\theta > 0$, and $\delta < 1$. Then there exists $m \in \mathbb{N}$ such that

$$\begin{aligned} \mu_{\sqrt{f(A)}}(y) - \varepsilon &< \mu_{f(A)}(y^m) \\ &= \mu_{f(A)}(f(x^m)) \\ &= \mu_{f^{-1}(f(A))}(x^m) \\ &= \mu_A(x^m) \\ &\leq \mu_{\sqrt{A}}(x) \\ &\leq \bigvee_{t \in f^{-1}(y)} \mu_{\sqrt{A}}(t) \\ &= \mu_{f(\sqrt{A})}(y), \end{aligned}$$

$$\begin{aligned} \gamma_{\sqrt{f(A)}}(y) - \theta &< \gamma_{f(A)}(y^m) \\ &= \gamma_{f(A)}(f(x^m)) \\ &= \gamma_{f^{-1}(f(A))}(x^m) \\ &= \gamma_A(x^m) \\ &\leq \gamma_{\sqrt{A}}(x) \\ &\leq \bigvee_{t \in f^{-1}(y)} \gamma_{\sqrt{A}}(t) \\ &= \gamma_{f(\sqrt{A})}(y), \end{aligned}$$

$$\begin{aligned} \psi_{\sqrt{f(A)}}(y) + \delta &> \psi_{f(A)}(y^m) \\ &= \psi_{f(A)}(f(x^m)) \\ &= \psi_{f^{-1}(f(A))}(x^m) \\ &= \psi_A(x^m) \\ &\geq \psi_{\sqrt{A}}(x) \\ &\geq \bigwedge_{t \in f^{-1}(y)} \psi_{\sqrt{A}}(t) \\ &= \psi_{f(\sqrt{A})}(y). \end{aligned}$$

Since $(\varepsilon, \theta, \delta)$ is arbitrary, we have $\mu_{\sqrt{f(A)}}(y) \leq \mu_{f(\sqrt{A})}(y)$, $\gamma_{\sqrt{f(A)}}(y) \leq \gamma_{f(\sqrt{A})}(y)$, and $\psi_{\sqrt{f(A)}}(y) \geq$

$\psi_{f(\sqrt{A})}(y)$ for all $y \in S$. On the other hand, there exists $x_0 \in R$ such that $f(x_0) = y$. Thus

$$\begin{aligned} \mu_{\sqrt{f(A)}}(y) - \varepsilon &< \mu_{\sqrt{A}}(x_0) = \bigvee_n \mu_A(x_0^n), \\ \gamma_{\sqrt{f(A)}}(y) - \theta &< \gamma_{\sqrt{A}}(x_0) = \bigvee_n \gamma_A(x_0^n), \\ \psi_{\sqrt{f(A)}}(y) + \delta &> \psi_{\sqrt{A}}(x_0) = \bigwedge_n \psi_A(x_0^n). \end{aligned}$$

Also, there exists $k \in \mathbb{N}$ such that

$$\begin{aligned} \mu_{\sqrt{f(A)}}(y) - \varepsilon &< \mu_A(x_0^k) = \bigvee_{t \in f^{-1}(y^k)} \mu_A(t) = \mu_{f(A)}(y^k) \leq \bigvee_n \mu_{f(A)}(y^n) = \mu_{\sqrt{f(A)}}(y), \\ \gamma_{\sqrt{f(A)}}(y) - \theta &< \gamma_A(x_0^k) = \bigvee_{t \in f^{-1}(y^k)} \gamma_A(t) = \gamma_{f(A)}(y^k) \leq \bigvee_n \gamma_{f(A)}(y^n) = \gamma_{\sqrt{f(A)}}(y), \\ \psi_{\sqrt{f(A)}}(y) + \delta &> \psi_A(x_0^k) = \bigwedge_{t \in f^{-1}(y^k)} \psi_A(t) = \psi_{f(A)}(y^k) \geq \bigwedge_n \psi_{f(A)}(y^n) = \psi_{\sqrt{f(A)}}(y). \end{aligned}$$

Since $(\varepsilon, \theta, \delta)$ is arbitrary, we have $\mu_{f(\sqrt{A})}(y) \leq \mu_{\sqrt{f(A)}}(y)$, $\gamma_{f(\sqrt{A})}(y) \leq \gamma_{\sqrt{f(A)}}(y)$, and $\psi_{f(\sqrt{A})}(y) \geq \psi_{\sqrt{f(A)}}(y)$ for all $y \in S$. Hence, $\sqrt{f(A)} = f(\sqrt{A})$. \square

Definition 3.8. A neutrosophic ideal $A = (\mu_A, \gamma_A, \psi_A)$ of a ring R is said to be semiprime if $\sqrt{A} = A$.

One can verify that an arbitrary intersection of semiprime neutrosophic ideals of a ring R is a semiprime neutrosophic ideal of R .

Example 3.9. Let S be a semiprime ideal of a ring R . Define a neutrosophic set $A = (\mu_A, \gamma_A, \psi_A)$ on R by $\mu_A(x) = \begin{cases} \frac{9}{10} & \text{if } x \in S \\ \frac{1}{3} & \text{otherwise,} \end{cases}$ $\gamma_A(x) = \begin{cases} \frac{1}{10} & \text{if } x \in S \\ \frac{1}{3} & \text{otherwise,} \end{cases}$ $\psi_A(x) = \begin{cases} 0 & \text{if } x \in S \\ \frac{2}{3} & \text{otherwise.} \end{cases}$ Then $A = (\mu_A, \gamma_A, \psi_A)$ is a semiprime neutrosophic ideal of R .

Theorem 3.10. If $A = (\mu_A, \gamma_A, \psi_A)$ is a neutrosophic ideal of a ring R , then the following assertions are equivalent:

- (1) A is a semiprime neutrosophic ideal of R ,
- (2) $\mu_A(x^n) = \mu_A(x)$, $\gamma_A(x^n) = \gamma_A(x)$, and $\psi_A(x^n) = \psi_A(x)$ for all $x \in R$ and $n \geq 1$,
- (3) $\mu_A(x^m) = \mu_A(x)$, $\gamma_A(x^m) = \gamma_A(x)$, and $\psi_A(x^m) = \psi_A(x)$ for all $x \in R$ and $m \geq 2$.

Proof. (1) \Rightarrow (2) Let $x \in R$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} \mu_A(x) &\leq \mu_A(x^n) \leq \bigvee_n \mu_A(x^n) = \mu_{\sqrt{A}}(x) = \mu_A(x), \\ \gamma_A(x) &\leq \gamma_A(x^n) \leq \bigvee_n \gamma_A(x^n) = \gamma_{\sqrt{A}}(x) = \gamma_A(x), \\ \psi_A(x) &\geq \psi_A(x^n) \geq \bigwedge_n \psi_A(x^n) = \psi_{\sqrt{A}}(x) = \psi_A(x). \end{aligned}$$

Hence, $\mu_A(x^n) = \mu_A(x)$, $\gamma_A(x^n) = \gamma_A(x)$, and $\psi_A(x^n) = \psi_A(x)$.

(2) \Rightarrow (3) Obvious.

(3) \Rightarrow (1) Let $x \in R$ and $m < n$. Then $m^k > n$ for some $k \in \mathbb{N}$. Hence,

$$\begin{aligned} \mu_A(x) &= \mu_A(x^m) = \mu_A(x^{m^2}) = \dots = \mu_A(x^{m^k}) \geq \mu_A(x^n) = \mu_A(x), \\ \gamma_A(x) &= \gamma_A(x^m) = \gamma_A(x^{m^2}) = \dots = \gamma_A(x^{m^k}) \geq \gamma_A(x^n) = \gamma_A(x), \\ \psi_A(x) &= \psi_A(x^m) = \psi_A(x^{m^2}) = \dots = \psi_A(x^{m^k}) \geq \psi_A(x^n) = \psi_A(x), \end{aligned}$$

and so, $\mu_A(x^m) = \mu_A(x)$, $\gamma_A(x^m) = \gamma_A(x)$, and $\psi_A(x^m) = \psi_A(x)$. It follows that

$$\begin{aligned} \mu_{\sqrt{A}}(x) &= \lim_n \mu_A(x^n) = \mu_A(x), \\ \gamma_{\sqrt{A}}(x) &= \lim_n \gamma_A(x^n) = \gamma_A(x), \\ \psi_{\sqrt{A}}(x) &= \lim_n \psi_A(x^n) = \psi_A(x), \end{aligned}$$

so that $\sqrt{A} = A$. Hence, A is a semiprime neutrosophic ideal of R . \square

Theorem 3.11. Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic set on R such that $\mu_A(0) = \alpha$, $\gamma_A(0) = \beta$, and $\psi_A(0) = \theta$. If A is a semiprime neutrosophic ideal of R , then $\mu_A^{-1}([\delta_\alpha, \alpha])$, $\gamma_A^{-1}([\delta_\beta, \beta])$, and $\psi_A^{-1}([\theta, \delta_\theta])$ are semiprime neutrosophic ideal of R for all $\delta_\alpha \in (0, \alpha]$, $\delta_\beta \in (0, \beta]$, and $\delta_\theta \in [\theta, 1)$.

Proof. Assume that A is a semiprime neutrosophic ideal of R . Let $\delta_\alpha \in (0, \alpha]$, $\delta_\beta \in (0, \beta]$, and $\delta_\theta \in [\theta, 1)$. Let $x, y \in \mu_A^{-1}([\delta_\alpha, \alpha])$. Then $\mu_A(x), \mu_A(y) \in [\delta_\alpha, \alpha]$, and so $\delta_\alpha \leq \mu_A(x) \wedge \mu_A(y) \leq \mu_A(x - y) \leq \mu_A(0) = \alpha$ and $\delta_\alpha \leq \mu_A(x) \leq \mu_A(rx) \leq \mu_A(0) = \alpha$ for all $r \in R$. Hence, $x - y \in \mu_A^{-1}([\delta_\alpha, \alpha])$ and $rx \in \mu_A^{-1}([\delta_\alpha, \alpha])$. Similarly, $xr \in \mu_A^{-1}([\delta_\alpha, \alpha])$. Therefore, $\mu_A^{-1}([\delta_\alpha, \alpha])$ is an ideal of R . Let $x, y \in \gamma_A^{-1}([\delta_\beta, \beta])$. Then $\gamma_A(x), \gamma_A(y) \in [\delta_\beta, \beta]$, and so $\delta_\beta \leq \gamma_A(x) \wedge \gamma_A(y) \leq \gamma_A(x - y) \leq \gamma_A(0) = \beta$ and $\delta_\beta \leq \gamma_A(x) \leq \gamma_A(rx) \leq \mu_A(0) = \beta$ for all $r \in R$. Hence, $x - y \in \gamma_A^{-1}([\delta_\beta, \beta])$ and $rx \in \gamma_A^{-1}([\delta_\beta, \beta])$. Similarly, $xr \in \gamma_A^{-1}([\delta_\beta, \beta])$. Therefore, $\gamma_A^{-1}([\delta_\beta, \beta])$ is an ideal of R . Let $x, y \in \psi_A^{-1}([\theta, \delta_\theta])$ and $r \in R$. Then

$$\begin{aligned}\theta &= \psi_A(0) \leq \psi_A(x - y) \leq \psi_A(x) \vee \psi_A(y) \leq \delta_\theta, \\ \theta &= \psi_A(0) \leq \psi_A(rx) \leq \psi_A(x) \leq \delta_\theta, \\ \theta &= \psi_A(0) \leq \psi_A(xr) \leq \psi_A(x) \leq \delta_\theta.\end{aligned}$$

Then $x - y, xr, rx \in \psi_A^{-1}([\theta, \delta_\theta])$. Therefore, $\psi_A^{-1}([\theta, \delta_\theta])$ is an ideal of R . Now assume that $a^n \in \mu_A^{-1}([\delta_\alpha, \alpha])$ for some $a \in R$. It follows from Theorem 3.10 that $\delta_\alpha \leq \mu_A(a^n) = \mu_A(a) \leq \alpha$, and so $a \in \mu_A^{-1}([\delta_\alpha, \alpha])$. Consequently, $\mu_A^{-1}([\delta_\alpha, \alpha])$, $\gamma_A^{-1}([\delta_\beta, \beta])$, and $\psi_A^{-1}([\theta, \delta_\theta])$ are semiprime neutrosophic ideals of R . \square

Theorem 3.12. If R is a regular ring, then every neutrosophic ideal of R is semiprime.

Proof. Let $A = (\mu_A, \gamma_A, \psi_A)$ be a neutrosophic ideal of a regular ring R . Given $x \in R$, there exists $a \in R$ such that $x = xax = x^2a$. Repeatedly, we get $x = x^n a^{n-1}$ for all $n \geq 2$. Then

$$\begin{aligned}\mu_A(x) &= \mu_A(x^n a^{n-1}) \geq \mu_A(x^n) \geq \mu_A(x), \\ \gamma_A(x) &= \gamma_A(x^n a^{n-1}) \geq \gamma_A(x^n) \geq \gamma_A(x), \\ \psi_A(x) &= \psi_A(x^n a^{n-1}) \geq \psi_A(x^n) \geq \psi_A(x),\end{aligned}$$

and so $\mu_A(x^n) = \mu_A(x)$, $\gamma_A(x^n) = \gamma_A(x)$, and $\psi_A(x^n) = \psi_A(x)$ for all $x \in R$ and $n \geq 1$. It follows from Theorem 3.10 that A is a semiprime neutrosophic ideal of R . \square

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