



Certain Determinants for New Subclasses of μ – *Fold bi-Univalent Functions*

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Abstract

This paper introduces and investigate a new subclasses of the class Σ_μ of analytic functions that both g and g^{-1} are μ -fold symmetric *bi*-univalent functions in the open unit disk Ω and get the estimates of the initial coefficients $|\alpha_{t+1}|$, $|\alpha_{2t+1}|$ and $|\alpha_{3t+1}|$ for functions in each of these new subclasses. After this, the work will be discussing the Hankel determinant and a Fekete-Szegő functional.

Keywords: Analytic function; univalent function; *bi*-univalent functions; μ – *fold* symmetric *bi*-univalent functions; Hankel determinant; Fekete-Szegő functional.

1. Introduction and Preliminaries

The motivation for this paper has come from two directions: firstly, from a wish to introduce two new subclasses of the function class Σ_μ of *bi*-univalent μ -fold analytic functions in Ω and get the estimates of the initial coefficients $|\alpha_{\mu+1}|$, $|\alpha_{2\mu+1}|$ and $|\alpha_{3\mu+1}|$ for functions in each of these new subclasses. Secondly to discuss the Hankel determinant and a Fekete-Szegő functional for functions belonging in these new subclasses.

Note: In this paper we will represent by $\Omega = \{\gamma \in \mathbb{C} : |\gamma| < 1\}$ to the open unit disk.

Now, we recall some background information on analytic functions g in Ω which satisfy normalization condition $g(0) = g'(0) - 1 = 0$, where:

$$g(\gamma) = \gamma + \sum_{n=2}^{\infty} \alpha_n \gamma^n, \quad (1)$$

In addition to investigate some background information on the class Σ of all *bi*-univalent in Ω given by (1), and some of their properties.

Definition 1.1. Let $\mathcal{H}$ denote the family of functions g satisfying (1) which analytic in Ω such that $g(0) = g'(0) - 1 = 0$. Let $\mathcal{h}$ be the subclass of $\mathcal{H}$ consisting of functions of the form (1) which are also univalent in Ω . It is well known that every $g \in \mathcal{h}$ has an inverse g^{-1} , defined by $g^{-1}(g(\gamma)) = \gamma$, ($\gamma \in \Omega$), and $g^{-1}(g(w)) = w$, ($|w| < k_0(g)$, $k_0(g) \geq \frac{1}{4}$), where

$$g^{-1}(w) = w - \alpha_2 w^2 + (2\alpha_2^2 - \alpha_3) w^3 - (5\alpha_2^3 - 5\alpha_2 \alpha_3 + \alpha_4) w^4 + \dots \quad (2)$$

Then g , is *bi*-univalent in Ω , when both g and g^{-1} are univalent in Ω .

Denoted by Σ , the class of all *bi-univalent* in Ω , given by equation 1. To indeed further details, and examples of functions in the class Σ , can be found in [24], [3], [11], [12] and [17].

Lemma 1.2. For each $g \in \mathcal{H}$, satisfy that:

$$\left| \frac{\gamma^2 g'(\gamma)}{(g(\gamma))^2} - 1 \right| < 1, (\gamma \in \Omega)$$

is univalent in Ω , and hence $g \in \mathcal{h}$. The proof can be found in [12].

Definition 1.3. Denoted by $\mathcal{T}(\tau)$, the class containing the functions $g \in \mathcal{H}$, such that

$$\left| \frac{\gamma^2 g'(\gamma)}{(g(\gamma))^2} - 1 \right| < \tau, \gamma \in \Omega, 0 < \tau \leq 1$$

and $\mathcal{T}(1) = \mathcal{T}$, so we have $\mathcal{T}(\tau) \subset \mathcal{T} \subset \mathcal{h}$ from (lemma1.2.). And

$$\operatorname{Re} \left(\frac{\gamma^2 g'(\gamma)}{(g(\gamma))^2} - 1 \right) > 1 - \tau. \text{ Kuroki et. al. [10].}$$

A function is said to be μ -fold symmetric see ([15] and [9]), if it has the following normalized form:

$$h(\gamma) = \gamma + \sum_{k=1}^{\infty} \alpha_{\mu k+1} \gamma^{\mu k+1} \quad (\gamma \in \Omega; \mu \in \mathbb{N}). \quad (3)$$

Lemma 1.4. For each function $h \in \mathcal{h}$, the function

$$g(\gamma) = \sqrt[\mu]{h(\gamma^\mu)} \quad (\gamma \in \Omega; \mu \in \mathbb{N}), \quad (4)$$

is univalent and maps the unit disk Ω into a region with μ -fold symmetric.

Let $\mathcal{h}_\mu$ be a class of μ -fold symmetric univalent functions on Ω , which are normalized depending on the series expansion (4). That is, the functions h are 1-fold symmetric. A concept μ -fold symmetric *bi-univalent* functions is introduced, in this paper, similarly the concept of μ -fold symmetric univalent functions. Then for each integer $\mu \in \mathbb{N}$, the *bi-univalent* function g of Σ generates an μ -fold symmetric *bi-univalent* function. The normalized form of the function g is given as shown above (4). The next equation represents the series expansion for its inverse g^{-1} which has been proved by Srivastava et al. [23].

$$\begin{aligned} k(w) &= w - \alpha_{\mu+1} w^{\mu+1} + [(\mu+1)\alpha_{\mu+1}^2 - \alpha_{2\mu+1}] w^{2\mu+1} \\ &\quad - \left[\frac{1}{2}(\mu+1)(3\mu+2)\alpha_{\mu+1}^3 - (3\mu+2)\alpha_{\mu+1}\alpha_{2\mu+1} + \alpha_{3\mu+1} \right] w^{3\mu+1} + \dots, \end{aligned} \quad (5)$$

where $h^{-1} = k$. Denoted by Σ_μ , the class of μ -fold symmetric univalent functions in Ω , we note that for $t = 1$, formula (5) coincide with formula (2).

Noonan and Thomas [13], introduced and studied the j^{th} Hankel determinant for $j \geq 1, n \in \mathbb{N}_0$,

$$H_j(n) = \begin{vmatrix} \alpha_n & \alpha_{n+1} & \cdots & \alpha_{n+q-1} \\ \alpha_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n+q-1} & \cdots & \cdots & \alpha_{n+2(q-1)} \end{vmatrix}$$

where $\alpha_1 = 1$. Many researchers have studied $H_j(n)$, Specifically, the Hankel determinant $H_2(2)$, is equivalent to $|\alpha_2\alpha_4 - \alpha_3^2|$, were obtained in ([8], [7], and [20]), for many classes. A Fekete-Szegő functional $|\alpha_3 - \alpha_2^2|$, is equivalent to the Hankel determinant $H_2(1)$. Also, the Hankel determinant $H_3(1)$, studied and investigated by Babalola [4].

Some other authors have studied various classes of μ -fold symmetric *bi-univalent* functions, such as, see Refs. ([1], [2], [25], [26], [21], [22], [6], [5], [18] and [19]). Such works encouraged us to create the two new subclasses of the function class Σ_μ of analytic functions which both h and h^{-1} are μ -fold symmetric

bi-univalent functions and we obtain estimates on the initial coefficients $|\alpha_{\mu+1}|, |\alpha_{2\mu+1}|, |\alpha_{3\mu+1}|$, Hankel determinant and Fekete-Szegő functional for functions in each of these new subclasses.

for explaining main results, we need the Remark below [26].

Remark 1.5. Let $\phi = \{g: g \text{ analytic in } \Omega \text{ satisfy that } \operatorname{Re}\{g(\gamma)\} > 0\}$, so, $\forall \ell$, and $g \in \phi$ we have $|t_\ell| \leq 2$, and

$$g(\gamma) = 1 + t_1\gamma + t_2\gamma^2 + t_3\gamma^3 + \dots, \quad \gamma \in \Omega.$$

2. Estimates on the Initial Coefficients

Definition 2.1. A function $\zeta \in \Sigma_\mu$ given by (3) is said to be in the class $\mathfrak{T}_{\Sigma_\mu}^\partial$, if the following conditions are satisfied

$$\left| \arg \left(\frac{\gamma^2 \zeta'(\gamma)}{(\zeta(\gamma))^2} \right) \right| < \frac{\partial \pi}{2}, \quad (0 < \partial \leq 1, \gamma \in \Omega) \quad (6)$$

and

$$\left| \arg \left(\frac{w^2 k'(w)}{(k(w))^2} \right) \right| < \frac{\partial \pi}{2}, \quad (0 < \partial \leq 1, w \in \Omega) \quad (7)$$

where the function k is the invers of ζ , given by Equation 5. Denoted by $\mathfrak{T}_{\Sigma_\mu}^\partial$, the class of all μ -fold symmetric bi-univalent functions of order ∂ .

In our first theorem, we finding the estimates on the initial coefficients $|\alpha_{\mu+1}|, |\alpha_{2\mu+1}|$ and $|\alpha_{3\mu+1}|$ for function in the class $\mathfrak{T}_{\Sigma_\mu}^\partial$.

Theorem 2.2. Let ζ be a function satisfy Eq. (4) in class $\mathfrak{T}_{\Sigma_\mu}^\partial$, $0 < \partial \leq 1$. Then

$$|\alpha_{\mu+1}| \leq \frac{2(2\mu-1)(2\partial-1)}{\partial(3\mu-1)},$$

$$|\alpha_{2\mu+1}| \leq \frac{2\partial}{(2\mu-1)},$$

$$|\alpha_{3\mu+1}| \leq \frac{6\partial}{(3\mu-1)}.$$

Proof. it follows from (6) and (7) that

$$\frac{\gamma^2 \zeta'(\gamma)}{(\zeta(\gamma))^2} = [r(\gamma)]^\partial, \quad (8)$$

and

$$\frac{w^2 k'(w)}{(k(w))^2} = [u(w)]^\partial, \quad (9)$$

where $r(\gamma)$ and $u(w)$ in $\mathcal{P}$ and have the forms

$$r(\gamma) = 1 + r_\mu \gamma^\mu + r_{2\mu} \gamma^{2\mu} + r_{3\mu} \gamma^{3\mu} + \dots, \quad (10)$$

and

$$u(w) = 1 + u_\mu w^\mu + u_{2\mu} w^{2\mu} + u_{3\mu} w^{3\mu} + \dots. \quad (11)$$

Clearly, we have

$$\begin{aligned} [r(\gamma)]^\partial &= 1 + \partial r_\mu \gamma^\mu + \left[\partial r_{2\mu} + \frac{1}{2} \partial(\partial-1) r_\mu^2 \right] \gamma^{2\mu} + \\ &\quad \left[\partial r_{3\mu} + \partial(\partial-1) r_\mu r_{2\mu} + \frac{1}{6} \partial(\partial-1)(\partial-2) r_\mu^3 \right] \gamma^{3\mu} + \dots, \end{aligned}$$

$$[u(w)]^\partial = 1 + \partial u_\mu w^\mu + \left[\partial u_{2\mu} + \frac{1}{2} \partial(\partial - 1) u_\mu^2 \right] w^{2\mu} + \\ \left[\partial u_{2\mu+1} + \partial(\partial - 1) u_\mu u_{2\mu} + \frac{1}{6} \partial(\partial - 1)(\partial - 2) u_\mu^3 \right] w^{3\mu} + \dots$$

And,

$$\frac{\gamma^2 f'(\gamma)}{(f(\gamma))^2} = 1 + (\mu - 1) \alpha_{\mu+1} \gamma^\mu + [(2\mu - 1) \alpha_{2\mu+1} - (2\mu - 1) \alpha_{\mu+1}^2] \gamma^{2\mu} \\ + [(3\mu - 1) \alpha_{2\mu+1} + 2(3\mu - 1) \alpha_{2\mu+1} \alpha_{\mu+1} + (3\mu - 1) \alpha_{\mu+1}^3] \gamma^{3\mu} + \dots,$$

and

$$\frac{w^2 g'(w)}{(g(w))^2} = 1 - (\mu - 1) \alpha_{\mu+1} w^\mu - [(2\mu - 1) \alpha_{2\mu+1} - (2\mu - 1) \alpha_{\mu+1}^2] w^{2\mu} \\ - (3\mu - 1) [\alpha_{2\mu+1} - 3\alpha_{2\mu+1} \alpha_{\mu+1} + 2\alpha_{\mu+1}^3] w^{3\mu} + \dots,$$

Notice that, equating the coefficient in (8) and (9), we get:

$$(\mu - 1) \alpha_{\mu+1} = \partial r_\mu, \quad (12)$$

$$(2\mu - 1)(\alpha_{2\mu+1} - \alpha_{\mu+1}^2) = \partial r_{2\mu} + \frac{1}{2} \partial(\partial - 1) r_\mu^2, \quad (13)$$

$$(3\mu - 1)(\alpha_{\mu+1}^3 + \alpha_{3\mu+1} - 2\alpha_{\mu+1} \alpha_{2\mu+1}) \\ = \partial r_{2\mu} + \partial(\partial - 1) r_\mu r_{2\mu} + \frac{1}{6} \partial(\partial - 1)(\partial - 2) r_\mu^3, \quad (14)$$

$$-(\mu - 1) \alpha_{\mu+1} = \partial u_\mu, \quad (15)$$

$$-(2\mu - 1)(\alpha_{2\mu+1} - \alpha_{\mu+1}^2) = \partial u_{2\mu} + \frac{1}{2} \partial(\partial - 1) u_\mu^2, \quad (16)$$

and

$$-(3\mu - 1)(2\alpha_{\mu+1}^3 + \alpha_{3\mu+1} - 3\alpha_{\mu+1} \alpha_{2\mu+1}) = \\ \partial u_{2\mu+1} + \partial(\partial - 1) u_\mu u_{2\mu} + \frac{1}{6} \partial(\partial - 1)(\partial - 2) u_\mu^3. \quad (17)$$

From Equations 12 and 15, we get:

$$r_\mu = -u_\mu \quad (18)$$

From Equation 13, and using Lemma 1.5, we obtain

$$|(2\mu - 1)(\alpha_{2\mu+1} - \alpha_{\mu+1}^2)| = |2\mu - 1| |\alpha_{2\mu+1} - \alpha_{\mu+1}^2| = \left| \partial r_{2\mu} + \frac{1}{2} \partial(\partial - 1) r_\mu^2 \right| \\ \leq \partial |r_{2\mu}| + \frac{1}{2} \partial(\partial - 1) |r_\mu^2| \leq \frac{2\partial^2}{(2\mu - 1)}. \quad (19)$$

By compensation 14, in 17, and use 18, we find that:

$$(3\mu - 1) \alpha_{\mu+1} (\alpha_{2\mu+1} - \alpha_{\mu+1}^2) = \partial(r_{2\mu} + u_{2\mu}) + \partial(\partial - 1)(r_\mu r_{2\mu} + u_\mu u_{2\mu}), \quad (20)$$

which gives,

$$\begin{aligned} |(3\mu - 1)\alpha_{\mu+1}(\alpha_{2\mu+1} - \alpha_{\mu+1}^2)| &= (3\mu - 1)|\alpha_{\mu+1}||\alpha_{2\mu+1} - \alpha_{\mu+1}^2| \\ &\leq \frac{4\partial(2\partial - 1)}{(3\mu - 1)}. \end{aligned} \quad (21)$$

Note that, from Equations 19 and 21, we get that:

$$|\alpha_{\mu+1}| \leq \frac{2(2\mu - 1)(2\partial - 1)}{\partial(3\mu - 1)}.$$

Now, by subtracting Equation 17 from Equation 14, and use Equation 18, we have:

$$\begin{aligned} (3\mu - 1)(3\alpha_{\mu+1}^3 + 2\alpha_{3\mu+1} - 5\alpha_{\mu+1}\alpha_{2\mu+1}) &= \partial(r_{2\mu} - u_{2\mu}) \\ &+ \partial(\partial - 1)(r_{\mu}r_{2\mu} - u_{\mu}u_{2\mu}) + \frac{1}{3}\partial(\partial - 1)(\partial - 2)r_{\mu}^3, \end{aligned} \quad (22)$$

By using Equations 20 and 22, removing $\alpha_{\mu+1}^3$, we have the following:

$$\begin{aligned} 2(3\mu - 1)(\alpha_{3\mu+1} - \alpha_{\mu+1}\alpha_{2\mu+1}) &= \partial(r_{2\mu} - u_{2\mu}) + 3\partial(r_{2\mu} + u_{2\mu}) \\ &= \partial(4r_{2\mu} + 2u_{2\mu}), \end{aligned} \quad (23)$$

which gives, via Lemma 1.5,

$$|\alpha_{3\mu+1} - \alpha_{\mu+1}\alpha_{2\mu+1}| \leq \frac{6\partial}{(3\mu - 1)}. \quad (24)$$

Also, by using Equations 18 and 21, removing $\alpha_{\mu+1}\alpha_{2\mu+1}$, we have the following:

$$\begin{aligned} 2(3\mu - 1)(\alpha_{3\mu+1} - \alpha_{\mu+1}^3) &= \partial(r_{2\mu+1} - u_{2\mu+1}) + 5\partial(r_{2\mu+1} + u_{2\mu+1}) \\ &= \partial(6r_{2\mu+1} + 4u_{2\mu+1}), \end{aligned} \quad (25)$$

which gives, via Lemma 1.5,

$$|\alpha_{3\mu+1} - \alpha_{\mu+1}^3| \leq \frac{10\partial}{(3\mu - 1)}. \quad (26)$$

Now, to be discovered the bound on $|\alpha_{2\mu+1}|$. Using the inequality:

$$||\gamma_1| - |\gamma_2|| \leq |\gamma_1 - \gamma_2|, \quad (27)$$

in Equation 19, we have:

$$|\alpha_{2\mu+1}| - |\alpha_{\mu+1}^2| \leq |\alpha_{2\mu+1} - \alpha_{\mu+1}^2| \leq 2\partial \frac{1}{(2\mu - 1)},$$

for this, the following is evident:

$$|\alpha_{2\mu+1}| \leq \frac{2\partial}{(2\mu - 1)}.$$

Also, to be discovered the bound on $|\alpha_{3\mu+1}|$. By using the inequality (27) in (24) and (26), we obtain:

$$|\alpha_{3\mu+1}| \leq \frac{6\partial}{(3\mu - 1)}.$$

Remark 2.3. In case of 1-fold symmetric *bi*-univalent functions, we obtain from Theorem 1, the following results

$$|\alpha_2| \leq 2 - \frac{1}{\theta}, |\alpha_3| \leq 2\theta \text{ and } |\alpha_4| \leq 3\theta. (0 < \theta \leq 1)$$

Remark 2.4. In case of 1-fold symmetric bi -univalent functions and $\theta = 1$, the coefficients initial $|\alpha_{\mu+1}|$, $|\alpha_{2\mu+1}|$ and $|\alpha_{3\mu+1}|$ for function in the class $\mathfrak{T}_{\Sigma\mu}^\theta$, is given through

$$|\alpha_2| \leq 1, |\alpha_3| \leq 2 \text{ and } |\alpha_4| \leq 3.$$

Corollary 2.5. Let ζ be a function satisfy 4, in class $\mathfrak{T}_{\Sigma\mu}^\theta$. The Hankel determinant $H_2(2)$ and Fekete-Szegő functional are given by $|\alpha_2\alpha_4 - \alpha_3^2| \leq 7$, and $|\alpha_3 - \alpha_2^2| \leq 3$, respectively.

Definition 2.6. A function h given by Equation 4, is said to be μ -fold symmetric bi -univalent function for order τ , ($0 < \tau \leq 1$), if

$$\operatorname{Re} \left\{ \frac{z^2 h'(z)}{(h(z))^2} \right\} > 1 - \tau, \quad (z \in \Omega) \quad (28)$$

and

$$\operatorname{Re} \left\{ \frac{\omega^2 k'(\omega)}{(k(\omega))^2} \right\} > 1 - \tau, \quad (\omega \in \Omega) \quad (29)$$

where the function k is the invers of h , given by Equation 5. Denoted by $\mathfrak{T}_{\Sigma\mu}^\tau$, the class of all μ -fold symmetric bi -univalent functions of order τ .

Theorem 2.7. For $0 < \tau \leq 1$, if a function h given by Equation 3, be in the class $\mathfrak{T}_{\Sigma\mu}^\tau$, then the coefficients $\alpha_{\mu+1}$, $\alpha_{2\mu+1}$ and $\alpha_{3\mu+1}$ satisfy the inequalities

$$|\alpha_{\mu+1}| \leq \frac{2(2\mu-1)}{3\mu-1}, |\alpha_{2\mu+1}| \leq \frac{2\tau}{(2\mu-1)} \text{ and } |\alpha_{3\mu+1}| \leq \frac{6\tau}{(3\mu-1)}.$$

Proof. Using the Equations 26 and 27, we can write:

$$\frac{\gamma^2 h'(\gamma)}{(h(\gamma))^2} = (1 - \tau) + \tau r(\gamma), \quad (30)$$

and

$$\frac{\omega^2 k'(\omega)}{(k(\omega))^2} = (1 - \tau) + \tau u(\omega), \quad (31)$$

where, $r(\gamma)$, and $u(\omega)$ are given by Equations 10 and 11, respectively.

Therefore, we have

$$(1 - \tau) + \tau r(\gamma) = 1 + \tau r_\mu \gamma^\mu + \tau r_{2\mu} \gamma^{2\mu} + \tau r_{3\mu} \gamma^{3\mu} + \dots,$$

$$(1 - \tau) + \tau u(\omega) = 1 + \tau u_\mu \omega^\mu + \tau u_{2\mu} \omega^{2\mu} + \tau u_{3\mu} \omega^{3\mu} + \dots.$$

Using Equations 4 and 5, we obtain:

$$\begin{aligned} \frac{\gamma^2 f'(\gamma)}{(f(\gamma))^2} &= 1 + (\mu - 1)\alpha_{\mu+1}\gamma^\mu + [(2\mu - 1)\alpha_{2\mu+1} - (2\mu - 1)\alpha_{\mu+1}^2]\gamma^{2\mu} \\ &+ [(3\mu - 1)\alpha_{2\mu+1} + 2(3\mu - 1)\alpha_{2\mu+1}\alpha_{\mu+1} + (3\mu - 1)\alpha_{\mu+1}^3]\gamma^{3\mu} + \dots, \end{aligned}$$

and

$$\begin{aligned} \frac{\omega^2 g'(\omega)}{(g(\omega))^2} &= 1 - (\mu - 1)\alpha_{\mu+1}\omega^\mu - [(2\mu - 1)\alpha_{2\mu+1} - (2\mu - 1)\alpha_{\mu+1}^2]\omega^{2\mu} \\ &- (3\mu - 1)[\alpha_{2\mu+1} - 3\alpha_{2\mu+1}\alpha_{\mu+1} + 2\alpha_{\mu+1}^3]\omega^{3\mu} + \dots, \end{aligned}$$

Now, arrangement the coefficients in Equations 30 and 31, we get:

$$(\mu - 1)\alpha_{\mu+1} = \tau r_{\mu}, \quad (32)$$

$$(\alpha_{2\mu+1} - \alpha_{\mu+1}^2) = \tau r_{2\mu}, \quad (33)$$

$$2(\alpha_{\mu+1}^3 + \alpha_{3\mu+1} - 2\alpha_{\mu+1}\alpha_{2\mu+1}) = \tau r_{3\mu}, \quad (34)$$

$$-(\mu - 1)\alpha_{\mu+1} = \tau u_{\mu}, \quad (35)$$

$$-(\alpha_{2\mu+1} - \alpha_{\mu+1}^2) = \tau u_{2\mu}, \quad (36)$$

and

$$-2(2\alpha_{\mu+1}^3 + \alpha_{3\mu+1} - 3\alpha_{\mu+1}\alpha_{2\mu+1}) = \tau u_{3\mu}. \quad (37)$$

We can now complete the further proof by continuing in the same way as with Theorem 2.2 $\square$

Now, from this theorem and when $\mu = 1$ we could get the following results:

Remark 2.8.

- I. $|\alpha_2| \leq 1$, $|\alpha_3| \leq 2\tau$ and $|\alpha_4| \leq 3\tau$.
 II. Also, Theorem 2.7 restricts the findings to the Naik and Patil (2017), that we remember here below as Corollary 2.9.

Corollary 2.9. (Naik and Patil, 2017) Let the function h , Given through Equation 3, be in the class $\mathfrak{T}_{\Sigma\mu}^{\tau}$. Then $|\alpha_2| \leq 1$, $|\alpha_3| \leq 2\tau$ and $|\alpha_4| \leq 3\tau$.

Remark 2.10. Set $m = 1$ and $\tau = 1$, in results of Theorem 2.7. The Hankel determinant $H_2(2)$ and Fekete-Szegő functional are given by $|\alpha_2\alpha_4 - \alpha_3^2| \leq 7$, and $|\alpha_3 - \alpha_2^2| \leq 3$, respectively.

3. Conclusions.

- i- The estimates $|\alpha_{\mu+1}|$, dependent on ∂ and independent on τ , for the subclasses $\mathfrak{T}_{\Sigma\mu}^{\partial}$ and $\mathfrak{T}_{\Sigma\mu}^{\tau}$, respectively.
 ii- In case of 1-fold symmetric *bi*-univalent functions, when you look at the estimates $|\alpha_{2\mu+1}|$ and $|\alpha_{3\mu+1}|$, It's intriguing to see here that, we can expanding it to $|\alpha_{n\mu+1}| \leq nm\partial$, ($n \geq 2$) for the subclass $\mathfrak{T}_{\Sigma\mu}^{\partial}$ and $|\alpha_{n\mu+1}| \leq nm\tau$, ($n \geq 2$) for the subclass $\mathfrak{T}_{\Sigma\mu}^{\tau}$.
 iii- The value of $H_2(2)$ and Fekete-Szegő functional for the subclasses $\mathfrak{T}_{\Sigma\mu}^{\partial}$ and $\mathfrak{T}_{\Sigma\mu}^{\tau}$ are similar.

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