Generalized fractional Burgers model front travelling wave solution, double soliton, and its interplay

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Abstract:

The fractional Hirota bilinear technique is employed in this publication to calculate the analytical solution for the hyperbolic generalized space-time fractional Burgers model. For the intended fractional differential model under consideration, we develop a double soliton wave. To verify the results, these computations are carried out using symbolic computing tools like Maple. Richer structures can be constructed thanks to the fractional orders' random selection. More applications in the applied sciences may result from soliton alterations based on fractional order adjustments.

Keywords: Burgers equation, conformable fractional derivative, and nonlinear fractional models.

1. Introduction

Numerous scientific issues can be described by nonlinear fractional dynamical systems. To comprehend the characteristics of these issues in practical implementations, numerous scholars have studied these systems extensively [1-4]. In applied research, fractional models—both linear and nonlinear—play crucial functions. Researchers and scientists have been interested in finding a numerical and analytical solution to fractional differential equations (FDEs) for the past 20 years. He [5, Laskin [6, Hassan [14], Draganescu [15], Momani and Odibat [9], Momani and Odibat [10], Xu and Cang [11], Eid et al. [12], Abdel-Salam and Yousif [13], Abdel-Salam and Hassan [14], and Abdel-Salam et al. [15] are a few examples. Nouh and Abdel-Salam [18], Abdel-Salam and Nouh [19], Yousif et al. [16], Abdel-Salam and Mourad [17], Abo-Dahab et al. [20], Nouh et al. [21], Abdel-Salam et al. [22].

It's noteworthy to note that waves with a stable shape are naturally produced by soliton waves. Because a soliton is a localised wave, as a solution, it either approaches a constant at infinity, as in the case of sine-Gordon model solitons, or it decomposes exponentially to zero, as in the case of KdV model solitons. This paper will discuss the double soliton and solitary wave, as well as how these waves

The calculus of the twenty-first century will be fractional and non-integer calculus, which is a generalization of earlier mathematical findings. Fractional calculus advancements and applications are now a popular and unique area of study. Fractional differential models provide an exact and precise representation of several real-life scenarios. Fractional derivatives have been defined and established in a number of ways. Examples of such defensibilities are the modified Riemann-Liouville, Chen's fractal defensibilities, Kolwankar-Gangal, Cresson's, and Caputo [1-3, 20- 29]. The conformable fractional derivative (CFD) was recently introduced by Khalil et al. in their paper [30] based on the constraints as

$$D^{\alpha}M(s)=\lim_{\varepsilon\to 0}\frac{M(s+\varepsilon s^{1-\alpha})-M(s)}{\varepsilon}\quad \forall s>0,\quad \alpha\in(0,1], \tag{1}$$

$$M^{(\alpha)}(0)=\lim_{s\to 0^{+}}M^{(\alpha)}(s). \tag{2}$$

The non-integer derivative reduces to the well-known integer derivative when the final equations are included. The CFD met the requirements listed below:

$$D^{\alpha}s^{n}=n\,s^{n-\alpha},\ \ n\in R,\qquad D^{\alpha}a=0,\quad \forall M(s)=a\ , \tag{3}$$

$$D^{\alpha}(a\,M+b\,N)=a\,D^{\alpha}M+b\,D^{\alpha}N,\qquad \forall\ a,b\in R\ , \tag{4}$$

$$D^{\alpha}(M\,N)=M\,D^{\alpha}N+N\,D^{\alpha}M\ , \tag{5}$$

$$D^{\alpha}\left(\frac{M}{N}\right)=\frac{M\,D^{\alpha}N-N\,D^{\alpha}M}{N^{2}}\ , \tag{6}$$

$$D^{\alpha}\,M(N)=\frac{dM}{dN}D^{\alpha}N,\qquad D^{\alpha}\,M(s)=s^{1-\alpha}\frac{dM}{ds}\ , \tag{7}$$

Where M, N are two $\alpha-$differentiable functions of a dependent variable s and a is a constant chosen at random. In reference [30], equations (5) through (7) are demonstrated. The **CFD** of some functions

$$\begin{aligned}&D_{s}^{\alpha}e^{ct}=c\,s^{1-\alpha}e^{cs},\quad D_{s}^{\alpha}\sin(cs)=c\,s^{1-\alpha}\cos(cs),\quad D_{s}^{\alpha}\cos(cs)=-c\,s^{1-\alpha}\sin(cs),\\&D_{s}^{\alpha}e^{cs^{\alpha}}=c\,\alpha e^{cs^{\alpha}},\quad D_{s}^{\alpha}\sin(cs^{\alpha})=c\,\alpha\cos(cs^{\alpha}),\quad D_{s}^{\alpha}\cos(cs^{\alpha})=-c\alpha\sin(cs^{\alpha}).\end{aligned} \tag{8}$$

The extension of integer differentiation and integration, or naturally calculus, to non-integer order is currently known as the calculus of fractional order. In recent times, nonlinear fractional models have gained significant attention from a wide range of scholars, including physicists, mathematicians, astronomers, and engineers. Numerous scientific disciplines have utilized fractional models, such as plasma, condensed matter, physics, biomathematics, chemistry, biology, communication, and astronomy [20–30].

Numerous essential applications in engineering and science, such as fractal wave propagations, particle physics, electrical systems, wave mechanics, etc., require an understanding of fractional order calculus.

The fractional Burgers equation is the basic form of the fractional model that explains the interaction between dissipative effects and nonlinear propagation. Fractional Burgers models are used in many fields, including fluid dynamics, acoustic transmission, hydrodynamics, traffic, magneto hydrodynamics, shock waves, supersonic travel, waves affected by diffusion, liquid dynamics, and information sciences [31, 32].

We present the precise characterization of front-wave interaction for the hyperbolic generalized fractional Burgers model (GFBM) in this work. This mathematical model can be thought of as a hyperbolic modification to a fractional Navier-Stokes system. The generalized fractional Burgers model (GBM) has received a lot of attention in research articles over the last few years, the majority of which have used numerical techniques to study it. We want to obtain an explicit characterization of the double soliton solution for the intended model in this work.

This research paper follows the following format. In Section 2, we present a basic version of the hyperbolic generalized fractional burgers model and apply Hirota's approach to analyses its solutions. In Section 3, we examine the double soliton solution using both the conventional and modified Hirota techniques. We conclude the paper with some reflections and suggestions for the future.

2. Model description and solution methodology

We present the hyperbolic GFBM that follows in the format

$$\tau D_t^{\alpha\alpha}u + D_t^{\alpha}u + u\, D_x^{\alpha}u - \kappa D_x^{\alpha\alpha}u = \sum_{n=0}^{3} a_n u^n, \qquad (9)$$

along with $u = u(x,t)$ $u = u(x,t)$ real function of space x, time t, since τ, κ are arbitrary positive constants , a_n are random constants and $D_t^{\alpha\alpha} = D_t^{\alpha}(D_t^{\alpha})$ is the twice fractional derivative with respect to t. When $\alpha = 1$ equation (9) become the well-known hyperbolic GBM studied as

$$\tau u_{tt} + u_t + u\, u_x - \kappa u_{xx} = \sum_{n=0}^{3} a_n u^n.$$

We can rewrite the polynomial on the right-hand side of equation (9) in this equivalent form because it has three real roots.

$$\tau D_t^{\alpha\alpha}u + D_t^{\alpha}u + u\, D_x^{\alpha}u + B\, D_x^{\alpha}u - \kappa D_x^{\alpha\alpha}u = \lambda u(u-s)(u-q), \qquad (10)$$

Although B, λ, s, q are unknown constants. Equation (10) is used in the following to denote hyperbolic **GFBM.**

As we discussed at the beginning, the primary goal of this manuscript is to use Hirota's technique to gain an explicit description of the double soliton. By applying this strategy, we will be left with a highly complicated and challenging system of nonlinear algebraic equations that, in the absence of any further model parameter information, cannot be solved directly. It becomes simpler to solve these algebraic equation systems that arise while applying Hirota's process if we just search for solutions that have certain known characteristics. Consequently, when the double soliton tends to the equivalent travelling wave, we will evaluate x tends to $\pm\infty$. We shall begin with the analytical characterization of travelling wave solutions, which serve as the primary asymptotic regimes for this purpose. For it, we placed

$$u=\frac{D_x^{\alpha}f}{f}, \tag{11}$$

we suppose

$$f=1+\varepsilon\,\varphi(t^{\alpha},x^{\alpha}), \tag{12}$$

In addition $\varphi(t^{\alpha},x^{\alpha}),$ Whenever is function determined, ε is the parameter. Therefore, increasing the outcomes by after adding equation (11) into (10) ε^{3}. Our three order series consists of ε. Comparing each coefficient of ε by zero lets us to obtain the ordinary fractional differential system in $\varphi(t^{\alpha},x^{\alpha}),$ for instance, based on the coefficient of ε we own

$$\tau D_t^{\alpha\alpha}(D_x^{\alpha}\varphi)+D_t^{\alpha}(D_x^{\alpha}\varphi)+BD_x^{\alpha\alpha}\varphi-\kappa D_x^{\alpha\alpha\alpha}\varphi+qs\lambda D_x^{\alpha}\varphi=0, \tag{13}$$

Assuming that fractional differential equation (13) has been solved in the form

$$\varphi=e^{(ax^{\alpha}-vt^{\alpha}+c)/\alpha}, \tag{14}$$

in which the constants a,v,c will be calculated. Equation (14) was used to convert the fractional differential equation system into an algebraic system.

$$aB-v-a^{2}\kappa+qs\lambda+v^{2}\tau=0 \tag{15}$$

$$-v+a^{2}(1+\kappa)+2qs\lambda+a[B+(s-q)\lambda]-v^{2}\tau=0, \tag{16}$$

$$\lambda(a+q)(a-s)=0. \tag{17}$$

Equation (17) indicates that the value a must equal either s or $-q$, since $\lambda\neq 0,$.

As the first instance, if $a=s,$ we have

$$\varphi_1=e^{(sx^{\alpha}-v_1t^{\alpha}+c_1)/\alpha}, \tag{18}$$

$$v_1=\frac{s[2B+s(\lambda+1)+2\lambda q]}{2}. \tag{19}$$

Equations (19), (18) in (11) and (12) allow us to provide rise kink-like solution with the following constraints:

$$\kappa=\kappa_1=\frac{\tau[2B+s+\lambda(2q+s)]^{2}-2(1+\lambda)}{4}, \tag{20}$$

is fulfilled

Second, we have the situation when $a=-q,$.

$$\varphi_2=e^{(-qx^{\alpha}-v_2t^{\alpha}+c_2)/\alpha}, \tag{21}$$

$$v_2 = \frac{-q[2B - q(\lambda+1) - 2\lambda s]}{2}. \quad (22)$$

Equations (19), (18) in (11) and (12) allow us to provide rise kink-like solution with the following

constraints: $$\kappa = \kappa_2 = \frac{\tau[\lambda(2q+s)+q-2B]^2 - 2(1+\lambda)}{4}, \quad (23)$$

is fulfilled

We need $\kappa_1 = \kappa_2$ in order to study two distinct solutions of hyperbolic GFBM in equations (18) and (21). This is simple to acquire.

$$\kappa = \kappa_1 = \kappa_2 = \frac{\tau(1+3\lambda)(q+s)^2 - 8(1+\lambda)}{16}, \quad (24)$$

$$B = \frac{(s-q)(\lambda-1)}{4}. \quad (25)$$

Hence , we obtain two distinct kink like wave solutions to the hyperbolic **GFBM** where κ & B are supplied by equations (24) & (25).

3 The model's double soliton

We take the form in our search for a solution.

$$u(t^\alpha, x^\alpha) = f(\xi_1, \xi_2), \quad \xi_1 = (s x^\alpha - v_1 t^\alpha + c_1)/\alpha, \; \xi_2 = (-q x^\alpha - v_2 t^\alpha + c_2)/\alpha, \quad (26)$$

When there is no proportional relationship between the two travelling waves, ξ_1 and ξ_2, equation (26) represents the interactions of multiple travelling waves.

In an effort to study the double soliton wave using Hirota's method, we must solve incredibly challenging systems of algebraic nonlinear equations. We have to impose some constraints on the chosen parameters in order to make the system solvable. Consequently, we choose the parameters taking into account that the solution we seek is represented asymptotically by fronts of one or two travelling waves, depending on the type of interaction. v_1 and v_2 based on equations (19) and (22). In order to analyses a double soliton wave in a straightforward manner, it is helpful to consider the function f as a superposition of functions corresponding to the travelling wave in the form

$$f = 1 + \varepsilon(e^{\xi_1} + e^{\xi_2}) + R\varepsilon^2 e^{\xi_1+\xi_2}, \quad (27)$$

When we replace equation (11) in (27) with the hyperbolic GFBM (10), we get six order series in the parameter ε. Six nonlinear algebraic equations result from equating each coefficient of ε by zero; these are marked by $E_1, E_2, \ldots, E_6$ in the determined constants and the items from $e^{m\xi_1}.e^{n\xi_1}$ .We utilize the acronyms $X^m = e^{m\xi_1}$ and $Y^n = e^{n\xi_1}$. in the following. Consequently, the set of nonlinear algebraic equations E_k represented by the acronyms X^m and Y^n in the following form

$$\sum_{n=0}^{k} a_n^k X^n Y^{k-n} = 0, \qquad k = 1, 2, ..., 6, \qquad (28)$$

with the calculated constants $\tau, B, \kappa, \lambda, s, q$ and R included in a_n^k.

We treat variables X and Y as independent and unrelated variables since we won't consider the scenario in which waves ξ_1 and ξ_2 are proportionate. Therefore, iff $a_n^k = 0$, system (20) has been met. Drawing from the symbolic programming languages "Maple" and "Mathematica",

The algebraic system $a_n^k = 0$, $0 \le n \le k \le 6$ is solved. Please take note that the answers provided by $q = 0, s = 0$, $q = -s$ and $\lambda = -1/3$ are not taken into consideration.

We now start to describe the process that allows us to derive the double soliton based ansatz (11). First of all, we have since $a_0^1 = a_1^1 = 0$.

$$\begin{aligned} &2\lambda[2qs\tau - 1 + s^2\tau + 2B\tau(2q+s)] - 2 - 4\kappa \\ &+4B^2\tau + 4Bs\tau + (2q+s)^2\lambda^2\tau = 0, \end{aligned} \qquad (29)$$

$$\begin{aligned} &\lambda[2q^2\tau - 2 + 4qs\tau - 4B\tau(2s+q)] - 2 - 4\kappa \\ &+4B^2\tau - 4Bq\tau + q^2\tau + (q+2s)^2\lambda^2\tau = 0, \end{aligned} \qquad (30)$$

After subtracting (30) from (29), we have

$$\tau(q+s)(1+3\lambda)[4B + (q-s)(\lambda-1)] = 0. \qquad (31)$$

As per our earlier agreement, we eliminate the scenarios in which $q = -s$ and $\lambda = -1/3$; so, equation (31) is met only in the case of $\tau = 0$ or the null value of the expression included in brackets. We make the first decision right away since it is evident that in some cases it cannot be ignored. This selection in expression (29), together with this formula,

$$\kappa = -\frac{1+\lambda}{2}. \qquad (32)$$

Second, we have a look at $a_3^6 = 0$, so

$$q\, s\, R^3 (q-s) = 0, \qquad (33)$$

This was fulfilled when $R = 0$ or $q = s$. Since the calculations provide $R = 0$ when we utilize the second pick, it is evident that we are compelled to use the first one. It is confirmed that the system of algebraic equations $E_1, E_2, ..., E_6$ takes the zero value in cases $R = 0$, $\tau = 0$, and $\kappa = -(1+\lambda)/2$

The direct analysis looks into the algebraic system $a_n^k = 0$ and can be thought of as a rather lengthy analysis of some other options. demonstrates that the cases that are introduced are the general ones, and that there are numerous additional non-trivial double soliton solutions that can be obtained by applying various techniques to the algebraic system; these solutions could be the source of all information obtained from this one. On the other hand, non-trivial solutions of this type correspond to the absence of non-zero τ. The introduced cases are the general ones, and there are many other non-trivial double soliton solutions that could be obtained by solving the algebraic system using various techniques. This is despite the fact that non-trivial solutions of this kind correspond to non-zero τ not being present. The direct analysis, which can be thought of as a somewhat lengthy analysis of some other possibilities investigate the algebraic system $a_n^k = 0$.

4 The method implemented through the use of Hirota's modified protocol

A minor adjustment to relation (11) will enable us to achieve the double-soliton solutions of fractional model (10) successfully. Considering that

$$u = \frac{g}{f}, \tag{34}$$

that is we have

$$f = 1 + \varepsilon(e^{\xi_1} + e^{\xi_2}) + R\varepsilon^2 e^{\xi_1+\xi_2}, \tag{35}$$

$$g = \varepsilon(r_1 e^{\xi_1} + r_2 e^{\xi_2}) + A\varepsilon^2 e^{\xi_1+\xi_2}, \tag{36}$$

Contrary to what happened in Hirota's method, it is evident from the hypothesis of the two functions f and g that we do not initially propose a link between these two functions.

We derive a series of degree six in ε by direct substitution into hyperbolic GFBM (10), utilizing the relations (34)–(36). Six nonlinear algebraic equations (denoted by $E_1, E_2, ..., E_6$) in the determined constants and the products of $e^{m\xi_1}.e^{n\xi_1}$ result from equating each coefficient of ε by zero. When $a_n^k = 0, 0 \le n \le k \le 6.$, we can get fifteen nonlinear algebraic equations using the same abbreviations as earlier. This system $a_n^k = 0$, can be solved using the symbolic programmer "Maple" or "Mathematica". As previously discussed, we ignore the scenarios in which $q = 0, s = 0,$, $q = -s$, and $\lambda = -1/3$ occur. Furthermore, since the solutions in those cases $r_2 = 0,\ R = 1$ and $A = r_1$, describe single travelling waves, we shall not consider them. Of course, the changes are apparent.

$$r_1 \to r_2, \quad r_2 \to r_1, \quad s \to -q, \qquad q \to -s, \tag{37}$$

be an equivalency relationship in the double soliton group, meaning that each pair has a single investigation.

First, we choose the state in which $r_1 \ne 0,\ r_2 \ne 0,$ using $E_1 = 0$ We own

$$B = \frac{(s-q)(\lambda-1)}{4}. \tag{38}$$

$$\kappa = \frac{\tau(1+3\lambda)^2(q+s)^2 - 8(1+\lambda)}{16}, \tag{39}$$

Applying (38) and (39) to $a_2^2 = a_0^2 = a_3^3 = a_0^3 = 0$,yields

$$\begin{aligned} &r_1(s-r_1)[s+\lambda(s-q)]=0,\\ &r_2(q+r_2)[q+\lambda(q-s)]=0,\\ &r_1\lambda(s-r_1)(q+r_1)=0,\\ &r_2\lambda(s-r_2)(q+r_2)=0. \end{aligned} \tag{40}$$

We discover that the previous set of algebraic equations may be solved if we provide $r_1 = r_2 = s$ and

$$\lambda = \frac{q}{s-q}, \tag{41}$$

Given $a_3^6 = 0$, we have

$$A(A+qR)(A-sR)=0, \tag{42}$$

that, for instance, is obtained when $A = -q\,R$; using this choice in $a_2^5 = 0$, we have

$$R^2(q+s)^3 = 0, \tag{43}$$

As a result, $R = 0$ must be used in this equation. Recall that $a_1^2 = 0$ in the equation, thus we obtain

$$s^2(q+s)^2[2(q-s)+\tau q(2q+s)^2]=0, \tag{44}$$

that provides

$$\tau = \frac{2(s-q)}{q(2q+s)^2}. \tag{45}$$

Therefore, the double soliton in the hyperbolic **GFBM** (10) takes the following form:

$$\begin{aligned} &u(t^\alpha, x^\alpha) = \frac{s\left(e^{(s x^\alpha - v_1 t^\alpha + c_1)/\alpha} + e^{(-q x^\alpha - v_2 t^\alpha + c_2)/\alpha}\right)}{1+e^{(s x^\alpha - v_1 t^\alpha + c_1)/\alpha} + e^{(-q x^\alpha - v_2 t^\alpha + c_2)/\alpha}},\\ &v_1 = \frac{s(2q^2+3qs+s^2)}{4(s-q)}, \quad ,v_2 = \frac{q(2q^2+3qs+s^2)}{4(s-q)}, \end{aligned} \tag{46}$$

Whereas $\tau = \dfrac{2(s-q)}{q(2q+s)^2}, \lambda = \dfrac{q}{s-q}, \mathrm{B} = \dfrac{2q-s}{4}$ and $\kappa = \dfrac{s-q}{8q}$.

As of right now, we propose that $r_2 = 0$ and $r_1 \neq 0$,. According to this hypothesis, the expression $a_1^1 = 0$, still yields the continuous constant κ. Furthermore, we may deduce that $r_1 = s$ and $A = Rs$. from the formulae $a_2^2 = a_3^3 = a_3^6 = 0$. Using $a_2^3 = 0$, we obtain

$$\tau s\,q(R-1)(q+s)^2(1-3\lambda)[4B+q(\lambda-1)+2s(\lambda+1)]=0, \tag{47}$$

after $R = 1$.We only have one soliton answer; the double soliton solution is absent. Thus, we have to have

$$B = \frac{q(1-\lambda)-2s(\lambda+1)}{4}, \tag{48}$$

(48) allows us to express $a_1^3 = 0$ as

$$s\,q\,(R-1)[q+s+\lambda(q+2s)]=0, \tag{49}$$

that provides

$$\lambda = -\frac{q+s}{q+2s}, \tag{50}$$

Recall that the two equations with the same factor, $a_1^2 = 0$ and $a_2^4 = 0$

$$2q+4s+\tau(q+s)(2q+s)^2,$$

calculate the parameter τ as a result.

$$\tau = -\frac{2(q+2s)}{(q+s)(2q+s)^2}, \tag{51}$$

Therefore, the double soliton in the hyperbolic **GFBM** (10) takes the following form:

$$\begin{aligned} u &= \frac{s\,e^{(s x^\alpha - v_1 t^\alpha + c_1)/\alpha}\left(1+R\,e^{(-qx^\alpha - v_2 t^\alpha + c_2)/\alpha}\right)}{1+e^{(s x^\alpha - v_1 t^\alpha + c_1)/\alpha} + e^{(-qx^\alpha - v_2 t^\alpha + c_2)/\alpha} R\,e^{(s x^\alpha - v_1 t^\alpha + c_1)/\alpha + (-qx^\alpha - v_2 t^\alpha + c_2)/\alpha}}, \\ v_1 &= \frac{s\,q(2q+s)}{q+s}, \quad v_2 = \frac{q(2q+s)}{4}, \end{aligned} \tag{52}$$

For $B = \frac{2q-s}{4}, \lambda = -\frac{q+s}{q+2s}, \tau = -\frac{2(q+2s)}{(q+s)(2q+s)^2}, \kappa = -\frac{q+2s}{8(q+s)}$, and $s, q, R, c_1\, c_2$ are constants. The following inequality $(q+2s)(q+s)<0$ must be investigated according to the specifications, using the constants' physical meanings τ, κ as well as the parameters s, q in the above. It is important to remember that when $R \geq 0$ and $R < 0$, respectively, the solution (52) denotes both regular and singular solutions for the random selection of the parameter R. All of the outcomes in [31] are recovered when $\alpha = 1$

5 conclusions and discussion

The analytical solution of the hyperbolic generalized space-time fractional equation is calculated in this study. Burgers model using the fractional Hirota bilinear approach. We design a twin soliton wave for the required fractional differential model under discussion. For the treatment of nonlinear differential models of integer and fractional orders, the Hirota bilinear technique provides a straightforward and promising method. Recently, researchers have been using symbolic computation like Maple to perform these calculations. We investigated if the results demonstrate the simplicity, effectiveness, and ease of computation of the approach for a range of engineering and physics models. The flexible and random selection of the fractional orders allows us to build deeper structures. Soliton modifications based on fractional order changes enable further applications in the applied sciences.

For the above procedure to be used successfully, further information is needed. Finally, let us point out that the solutions found in this work only describe the interaction of kinks. As a result, an effective analysis of the interactions

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