



On Orthogonal Reverse α -Derivations of Semi-prime Γ -Ring

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Abstract

In this paper, the concept of reverse α -derivation on Γ -ring M , and some results concerning two reverse α -derivations on M that are related to classical result of E.Ponser are presented. These results shows that if M is α 2-torsion free semiprime Γ -ring, d and g are non-zero reverse derivations of M , then dg are not a non-zero derivation. The new main results present and study notion of orthogonally reverse α -derivations d, g of M .

Keywords: Prime, semiprime; Γ -ring, derivation; α -derivati;n, reverse α -derivati;n, orthogonal reverse α -derivation.

1. Introduction

Notion of Γ -ring M is a more general concept than the ring which was defined by Nobusawa [8], but this definition of Γ -ring has been weakened by Barnes [2]. Following [2] means a Γ -ring is an order pair (M, Γ) where M and Γ are additive group such that there exists a function

$M \times \Gamma \times M \longrightarrow M$ satisfying for all $a, b \in M$ and $\gamma, \beta \in \Gamma$:

- (i) $(a + b)\gamma c = a\gamma c + b\gamma c, a(\gamma + \beta)b = a\gamma b + a\beta b, a\gamma(b + c) = a\gamma b + a\gamma c$
- (ii) $(a\gamma b)\beta c = a\gamma(b\beta c)$

M is 2-torsion free if $2a = 0$ implies $a = 0$, and it is prime if $a\Gamma M\Gamma b = 0$ implies that $a = 0$ or $b = 0$.

Recall that M is semiprime if $a\Gamma M\Gamma a = 0$ implies $a = 0$. It is known that every ring is a Γ -ring, but generally the converse is not true. F. J.Jing in [6] defined a derivation on M by an additive mapping

$d: M \longrightarrow M$ is called derivation if for all $a, b \in M$ and $\gamma \in \Gamma$, then

$$d(a\gamma b) = d(a)\gamma b + a\gamma d(b).$$

J.C. Change defined α -derivation on ring R [5], as follows :

Let α be a non zero mapping of R , an additive mapping $d: R \longrightarrow R$ is called α -derivation, if for $a, b \in R$, then $d(ab) = d(a)\alpha(b) + ad(b)$.

S.M.Salih and H.J.Thahab in [10] expanded definition on Γ -ring M in [5] as follows:

$d(a\gamma b) = d(a)\gamma\alpha(b) + a\gamma d(b)$ for all $a, b \in M$ and $\gamma \in \Gamma$.

It is obvious that every α -derivation is derivation (where $\alpha = \text{id}_M$), but generally the converse is not true. Bresar and Vukman [4] introduced the notion of reverse derivation on R as an additive mapping $d: R \longrightarrow R$ satisfying $d(ab) = d(b)a + bd(a)$, for all $a, b \in R$.

In this paper, the notion of reverse derivation on Γ -ring M is presented.

Bresar and Vukman [3] introduced notion of orthogonality for two derivations d, g on a semiprime ring as follows:

Two derivations d and g on a ring R are called orthogonal, if for all $a, b \in R$, then

$$d(x)Rg(y) = (0) = g(y)Rd(x).$$

Hence, A.H.Majeed introduced the notion of orthogonality for two reverse derivations d, g on semiprime ring [7] as follows:

Two reverse derivations d and g of R are called orthogonal if $d(x)Rg(y) = (0) = g(y)Rd(x)$,

for all $x, y \in R$.

It is obvious that a non-zero reverse derivation can not be orthogonal on itself.

The main aim for this paper is to introduce notion of orthogonal for two reverse α -derivations d and g on a semiprime Γ -ring, and present necessary and sufficient conditions for d and g to be an orthogonal such that these results equivalent to those earlier obtained by M.Ashraf and M.R.Jamal in [1]. First let us suppose that M is a 2-torsion free simeprime

Γ -ring.

2. Reverse Derivation and α -Derivation of Γ -Ring M

In this section, notion of reverse derivation and reverse α -derivation are defined, furthermore concepts of Jordan and triple reverse α -derivations on M .

Definition (2.1):

An additive function $d: M \longrightarrow M$ is called reverse derivation on M

if for all $a, b \in M$ and $\gamma \in \Gamma$,

$$d(a\gamma b) = d(b)\gamma a + b\gamma d(a)$$

Definition (2.2):

Let α be endomorphism mapping of Γ -ring M . An additive mapping $d: M \longrightarrow M$ is called a reverse α -derivation of M ,

if for all $a, b \in M$ and $\gamma \in \Gamma$ satisfying:

$$d(a\gamma b) = d(b)\gamma\alpha(a) + b\gamma\alpha(a) \quad \dots(1)$$

d is called Jordan reverse α -derivation of M if

$$d(a\gamma a) = d(a)\gamma\alpha(a) + a\gamma d(a) \quad \dots(2)$$

Also, d is called Jordan triple reverse α -derivation of M ,

if for all $a, b \in M$ and $\gamma, \beta \in \Gamma$ then

$$d(a\gamma b\beta a) = d(a)\beta\alpha(a)\gamma\alpha(b) + a\beta d(b)\gamma\alpha(a) + a\beta b\gamma d(a) \quad \dots(3)$$

The following example is about reverse α -derivation of Γ -ring M .

Example (2.3):

Let d be reverse α -derivation of Γ -ring M . We put $M' = M \oplus M$ and $\Gamma' = \Gamma \oplus \Gamma$, where $\oplus$ direct sum, α' be endomorphism mapping on M' defined by $\alpha'((x,y)) = (\alpha(x), \alpha(y))$, for all $(x,y) \in M'$. Also we define d' on M' by $d'((x,y)) = (d(x), 0)$, for all $(x,y) \in M'$. Then d' is reverse α' -derivation on M' . It is obvious that every reverse α -derivation is reverse derivation, but the converse is not true in general.

Now, we present some properties of reverse α -derivation on M .

Lemma (2.4):

Let α be endomorphism of M , and d be reverse α -derivation on M , then for all $a, b, c \in M$ and $\gamma, \beta \in \Gamma$ that following statements holds:

- (i) $d(\alpha\gamma b + b\gamma a) = d(b)\gamma\alpha(a) + b\gamma d(a) + d(a)\gamma\alpha(b) + \alpha\gamma d(b)$
- (ii) $d(\alpha\gamma b\beta a + a\beta b\gamma a) = d(a)\beta\alpha(a)\gamma\alpha(b) + a\beta d(b)\gamma\alpha(a) + a\beta b\gamma d(a) + d(a)\gamma\alpha(a)\beta\alpha(b) + \alpha\gamma d(b)\beta\alpha(a) + \alpha\gamma b\beta d(a)$ where In particular α is onto mapping.
- (iii) In particular if M be 2-torsion free then
 $d(\alpha\gamma b\gamma a) = d(a)\gamma\alpha(b)\gamma\alpha(a) + \alpha\gamma d(b)\gamma\alpha(a) + \alpha\gamma b\gamma d(a)$
- (iv) $d(\alpha\gamma b\beta c + c\gamma b\beta a) = d(c)\beta\alpha(a)\gamma\alpha(b) + c\beta d(b)\gamma\alpha(a) + c\beta b\gamma d(a) + d(a)\beta\alpha(c)\gamma\alpha(b) + a\beta d(b)\gamma\alpha(c) + a\beta b\gamma d(c)$
- (v) $d(\alpha\gamma b\gamma c + c\gamma b\gamma a) = d(c)\gamma\alpha(a)\gamma\alpha(b) + c\gamma d(b)\gamma\alpha(a) + c\gamma b\gamma d(a) + d(a)\gamma\alpha(c)\gamma\alpha(b) + \alpha\gamma d(b)\gamma\alpha(c) + \alpha\gamma b\gamma d(c)$

Proof :

(i) By calculating we obtain $d((a+b)\gamma(b+a))$ and (ii) is also obtained by replacing $a\beta b + b\beta a$ for b in (i), moreover (iii) can be obtained by replacing β with γ in (ii). If we replace $a+c$ for a in definition (2.2)(3) we can get (iv). While if replacing β by γ in (iv) we obtained (v).

3. The Main Results

In this section, definition of orthogonal reverse α -derivation on Γ -ring M and some properties are introduced.

Now, we begin with the following definition:

Definition (3.1): Let M be Γ -ring and α be an endomorphism mapping of M . Then two reverse α -derivations d and g on M are called orthogonal if for all $x, y \in M$ then:

$$d(x)\Gamma\alpha(M)\Gamma g(y) = (0) = g(y)\Gamma\alpha(M)\Gamma d(x).$$

It is obvious that a non-zero reverse α -derivation can not be orthogonal on itself.

Example (3.2):

Let d' is reverse α' -derivations on M' as in example (2.3) and g' is reverse α' -derivations on M' defined by $g'((x,y)) = (0, g(x))$ for all $(x,y) \in M'$ then:

d' and g' are orthogonal reverse α' -derivations on M' .

The following are useful to prove the main results :-

Lemma (3.3): ([9], lemma 3)

Let M be Γ -ring and a, b the elements of M . Then the following conditions are equivalent. For all $\gamma, \beta \in \Gamma$ then

- (i) $a\gamma M\beta b = (0)$.
- (ii) $b\gamma M\beta a = (0)$.
- (iii) $a\gamma M\beta b + b\gamma M\beta a = (0)$.

If one of these conditions are fulfilled then $a\delta b = b\delta a = 0$, for all $\delta \in \Gamma$.

$$0 = d(x)\delta\alpha(z)\beta g(x) + d(x)\delta\alpha(z)\beta g(y) + d(y)\delta\alpha(z)\beta g(x) + d(y)\delta\alpha(z)\beta g(y)$$

$$0 = d(x)\delta\alpha(z)\beta g(y) + d(y)\delta\alpha(z)\beta g(y)$$

Therefore by our assumption we get

$$d(x)\delta\alpha(z)\beta g(y)\gamma\alpha(t)\lambda d(x)\delta\alpha(z)\beta g(y) = 0$$

$$-d(x)\delta\alpha(z)\beta g(y)\gamma\alpha(t)\lambda d(x)\delta\alpha(z)\beta g(y) = 0$$

Since M is semiprime, then $d(x)\delta\alpha(z)\beta g(y) = 0$.

Hence $d(x)\Gamma\alpha(M)\Gamma g(y) = (0)$, for all $x, y \in M$

Lemma (3.4):

Let α be an endomorphism of semiprime Γ -ring M . Suppose that additive mappings d and g are reverse α -derivations of M satisfy $d(x)\Gamma\alpha(M)\Gamma g(x) = (0)$, for all $x \in M$. Then $d(x)\Gamma\alpha(M)\Gamma g(y) = (0)$, for all $x, y \in M$.

Proof :

Since $d(x)\delta\alpha(z)\beta h(x) = 0$, for all $x, z \in M$ and $\delta, \beta \in \Gamma$.

On linearizing we get;

$$0 = d(x+y)\delta\alpha(z)\beta g(x+y)$$

Lemma (3.5):

Let α be an endomorphism of M , d and g are reverse α -derivation of M . Then d and g are orthogonal if and only if for all $x, y \in M$ and $\delta, \beta \in \Gamma$ then

$$d(\alpha(x))\delta g(y) + g(\alpha(x))\beta d(y) = 0, \text{ where } \alpha^2 = \alpha.$$

Proof :

Suppose that $d(\alpha(x))\delta g(y) + g(\alpha(x))\beta d(y) = 0$ for all $x, y \in M$. Replacing x by $x\gamma y$ we get

$$d(\alpha(x\gamma y))\delta g(y) + g(\alpha(x\gamma y))\beta d(y) = 0, \text{ then } d(\alpha(x)\gamma\alpha(y))\delta g(y) + g(\alpha(x)\gamma\alpha(y))\beta d(y) = 0, \text{ thus}$$

$$(d(\alpha(y)\gamma\alpha^2(x)) + \alpha(y)\gamma d(\alpha(x))\delta g(y) + (g(\alpha(y))\gamma\alpha^2(x) + \alpha(y)\gamma g(\alpha(x))\beta d(y) = 0.$$

$$d(\alpha(y))\gamma\alpha(x)\delta g(y) + \alpha(y)\gamma d(\alpha(x))\gamma g(y) + g(\alpha(y))\gamma\alpha(x)\beta d(y) + \alpha(y)\gamma g(\alpha(x))\beta d(y) = 0, \text{ therefore}$$

$$\alpha(y)\gamma(d(\alpha(x))\delta g(y) + g(\alpha(x))\beta d(y)) + d(\alpha(y))\gamma\alpha(x)\delta g(y) + g(\alpha(y))\gamma\alpha(x)\beta d(y) = 0, \text{ by assumption we}$$

$$\text{reduce this relation to form } d(\alpha(y))\gamma\alpha(x)\delta g(y) + g(\alpha(y))\gamma\alpha(x)\beta d(y) = 0.$$

Hence by lemma (3.3) we get $d(\alpha(y))\gamma\alpha(x)\delta g(y) = 0 = g(\alpha(y))\gamma\alpha(x)\beta d(y)$ for all $x, y \in M$ and $\gamma, \delta, \beta \in \Gamma$.

therefore d, g are orthogonal.

Conversely: suppose that d, g are orthogonal then $d(x)\gamma\alpha(z)\delta g(y) = 0 = g(y)\gamma\alpha(z)\delta d(x)$. Therefore by lemma (3.3) we have $d(x)\gamma\alpha(z)\delta g(y) + g(y)\gamma\alpha(z)\delta d(x) = 0$.

Thus $d(x)\beta g(y) = 0 = g(x)\beta d(y)$, since α onto then we can take $x = \alpha(x)$, therefore

$$d(\alpha(x))\beta g(y) = 0 = g(\alpha(x))\beta d(y).$$

$$\text{Hence } d(\alpha(x))\beta g(y) + g(\alpha(x))\beta d(y) = 0.$$

Now, we prove ([1], theorem 2.1) by using less condition, as follows:

Theorem (3.6):

Let α be an onto endomorphism mapping of Γ -ring M , suppose d and g are reverse α -derivations of M . Then the following conditions are equivalent, where $\alpha = \alpha^2$ and α commutative mapping:

- (i) d and g are orthogonal reverse α -derivation.
- (ii) $dg = 0$ where $d^2 = d$.
- (iii) $gd = 0$ where $g^2 = g$.
- (iv) $dg + gd = 0$.
- (v) dg is a reverse α -derivation of M .
- (vi) gd is a reverse α -derivation of M .
- (vii) There exists a, b and $\beta, \delta \in \Gamma$ such that $dg(x) = a\beta\alpha(x) + \alpha(x)\delta b$, for all $x \in M$.

Proof :

(i) $\leftrightarrow$ (ii) Since d and g are orthogonal, then $d(x)\delta\alpha(z)\beta g(y) = 0$, for all $x, y \in M$ and $\delta, \beta \in \Gamma$.
 $0 = d(d(x)\delta\alpha(z)\beta g(y))$, therefore by definition (2.2)

$$0 = d(g(y)\beta\alpha(d(x))\delta\alpha^2(z)) = g(y)\beta d(\alpha(z))\delta\alpha(d(x)) + g(y)\beta\alpha(z)\delta d(d(x))$$

$$0 = d(g(y))\beta d(\alpha(x))\delta\alpha(z) + g(y)\beta\alpha(d(z))\delta d(\alpha(x)) + g(y)\beta\alpha(z)\delta d^2(x)$$

$$0 = d(g(y))\beta d(\alpha(x))\delta\alpha(z) + g(y)\beta\alpha(d(z))\delta d(\alpha(x)) + g(y)\beta\alpha(z)\delta d(x)$$

The second and third summands are zero since d and g are orthogonal therefore this relation is reduced to $d(g(y)\beta d(\alpha(x))\delta\alpha(z)) = 0$. Hence by concept semiprimeness we have $d(g(y)\beta d(\alpha(x))\delta\alpha(z)\delta d(g(y))\beta d(\alpha(x))) = 0$, thus $d(g(y)\beta d(\alpha(x))) = 0$, since M is semiprime where x, y are arbitrary elements in M , so take $\alpha(x) = g(y)$ in the above relation we get $d(g(y)\beta d(g(y))) = 0$, for all $y \in M$. Since M is semiprime then $(dg)(y) = 0$, for all $y \in M$.

Conversely:

Let $dg = 0$ where $d^2 = d$, therefore

$$\begin{aligned} dg(x\gamma y) &= d(g(x\gamma y)) = d(g(y)\gamma\alpha(x) + y\gamma g(x)) \\ &= d(\alpha(x))\gamma\alpha(g(y)) + \alpha(x)\gamma d(g(y)) + d(g(x))\gamma\alpha(y) + g(x)\gamma d(y), \end{aligned}$$

then

$$\begin{aligned} dg(x\gamma y) &= d(\alpha(x))\gamma g(\alpha(y)) + \alpha(x)\gamma dg(y) + dg(x)\gamma\alpha(y) + g(x)\gamma d(y) \\ &= d(\alpha(x))\gamma g(\alpha(y)) + g(x)\gamma d(y) \end{aligned}$$

Since α is onto then we can take $\alpha(y) = y$ and $x = \alpha(x)$.

Thus, $d(\alpha(x))\gamma g(y) + g(\alpha(x))\gamma d(y) = 0$, hence by lemma (3.5) we have d, g are orthogonal reverse α -derivation of M .

(i) $\leftrightarrow$ (iii) Similar way used in the part of (ii).

(iv) $\leftrightarrow$ (i) Suppose that g and d are orthogonal, therefore from (ii) and (iii) theorem (3.6) we obtain $dg + gd = 0$.

Conversely: Suppose that $dg + gd = 0$, then

$$0 = (dg + gd)(x\gamma y) = (dg)(x\gamma y) + (gd)(x\gamma y). \text{ Therefore}$$

$$0 = d(\alpha(x))\gamma\alpha(g(y)) + \alpha(x)\gamma dg(y) + dg(x)\gamma\alpha(y) + g(x)\gamma d(y) + g(\alpha(x))\gamma\alpha(d(y)) + \\ \alpha(x)\gamma gd(y) + gd(x)\gamma\alpha(y) + d(x)\gamma g(y)$$

$$0 = (d(\alpha(x))\gamma g(\alpha(y)) + g(\alpha(x))\gamma d(\alpha(y)) + \alpha(x)\gamma(dg(y) + gd(y)) + (dg(x) + gd(x))\gamma\alpha(y) + \\ g(x)\gamma d(y) + d(x)\gamma g(y)$$

$$0 = d(\alpha(x))\gamma g(\alpha(y)) + g(\alpha(x))\gamma d(\alpha(y)) + g(x)\gamma d(y) + d(x)\gamma g(y).$$

Hence since α is onto then we can take $x = \alpha(x)$, $\alpha = y$. $0 = d(\alpha(x))\gamma g(y) + g(\alpha(x))\gamma d(y) + g(x)\gamma d(y) + d(\alpha(x))\gamma g(y)$.

$$0 = 2(d(\alpha(x))\gamma g(y) + g(\alpha(x))\gamma d(y)).$$

Since M be 2-torsion free then $d(\alpha(x))\gamma g(y) + g(\alpha(x))\gamma d(y) = 0$, be lemma (3.5) we get d, g are orthogonal.

(v) $\leftrightarrow$ (i) Suppose that dg is a reverse α -derivation of M .

$$(dg)(x\gamma y) = (dg)(y)\alpha(x) + y\gamma(dg)(x) \quad \dots(1)$$

But,

$$(dg)(x\gamma y) = d(\alpha(x))\gamma\alpha(g(y)) + \alpha(x)\gamma d(g(y)) + d(g(x))\gamma\alpha(y) + g(x)\gamma d(y).$$

Since α is commutative mapping then

$$(dg)(x\gamma y) = d(\alpha(x))\gamma\alpha(g(y)) + dg(y)\gamma\alpha(x) + \alpha(y)\gamma dg(x) + g(x)\gamma d(y) \quad \dots(2)$$

α be onto, then replacing $\alpha(y)$ by y and comparing (1) with (2) we get

$$d(\alpha(x))\gamma\alpha(g(y)) + g(x)\gamma d(y) = 0. \text{ Replace } x \text{ by } \alpha(x) \text{ then}$$

$$d(\alpha(x))\gamma\alpha(g(y)) + g(\alpha(x))\gamma d(y) = 0, \text{ therefore by lemma (3.5) we have } d, g \text{ are orthogonal.}$$

Conversely: Suppose d, g are orthogonal, then $d(x)\gamma\alpha(z)\beta g(y) = 0$.

Thus by lemma (3.3)

$$d(x)\delta g(y) = 0 = g(y)\delta d(x) \quad \dots(1)$$

$$\text{Now, } (dg)(x\gamma y) = d(\alpha(x))\gamma g(\alpha(y)) + (dg)(y)\gamma\alpha(x) + \alpha(y)\gamma(dg)(x) + g(x)\gamma d(y).$$

α be onto, then replacing $\alpha(y)$ by y

$$(dg)(x\gamma y) = d(\alpha(x))\gamma g(y) + dg(y)\gamma\alpha(x) + y\gamma dg(x) + g(x)\gamma d(y).$$

By (1) we can reduce above relation to $(dg)(x\gamma y) = dg(y)\gamma\alpha(x) + y\gamma dg(x)$.

Hence dg is a reverse α -derivation.

(vi) $\leftrightarrow$ (i) Similar way use in the proof (v). (i) $\leftrightarrow$ (iii) Let d and g are orthogonal,

then by (ii) we have $dg = 0$, so we can choose $a = b = 0$ and $\beta, \delta \in \Gamma$.

So that $dg(x) = a\beta\alpha(x) + \alpha(x)\delta b$.

Theorem (3.7):

Let M be a Γ -ring, and d, g be reverse α -derivation of M and α be onto endomorphism mapping of M . Suppose $d^2 = g^2$. Then $d + g$ and $d - g$ are orthogonal.

Proof :

Suppose that $d^2 = g^2$, now for all $x \in M$

$$\begin{aligned} & \{(d - g)(d + g) + (d + g)(d - g)\}(x) \\ &= (d - g)(d + g)(x) + (d + g)(d - g)(x) \\ &= d^2(x) + dg(x) - gd(x) - g^2(x) + d^2(x) - dg(x) + gd(x) - g^2(x) = 2(d^2(x) - g^2(x)). \end{aligned}$$

$$= 2(d^2(x) - g^2(x)).$$

Since M be a 2-torsion free, therefore $d^2(x) = 2(d^2(x) - g^2(x))$. $g^2(x) = 0$, thus by using assumption we get

$$(d - g)(d + g) + (d + g)(d - g) = 0.$$

Hence by (iv), (i) of theorem (3.6) we $d + g$ and $d - g$ are orthogonal reverse α -derivation.

References

- [1] M.Ashraf and M.Rashid Jamal, "Orthogonal Derivation in Γ -Rings", Advance in Algebra, Vol.3, No.1, pp.1-6, 2010.
- [2] W.E.Barnes, "On The Γ -Rings of Nobusawa", Pacific J.of Math., Vol.18, No.3, pp.411-422, 1966.
- [3] M.Bresar and J.Vukman, "Orthogonal Derivation and an Extension of a Theorem of Posner", Radovi Mathematics, Vol.5, pp.237-246, 1989.
- [4] Bresar, M. and Vukman, "On Some Additive Mappings in Rings with Involution", Aequations Math., Vol.38, pp.178-185, J.1989.
- [5] J.C.Change, " α -Derivation with Invertible Values", Bulletin of the Institute of Math. Acad.Sinica, Vol.13, 1985.
- [6] F.J.Jing, "On Derivations of Γ -Ring", QUFU Shi Fan Daxue Xuebao Ziran Kexue Ban, Vol.13, No.4, pp.159-161, 1987.
- [7] A.H.Majeed, "On Orthogonal Reverse Derivations of Semiprime Rings", Iraqi Journal of Science, Vol.50, No.1, pp.84-88, 2009.
- [8] N.Nobosawa, "On a Generalization of the Ring Theory", Osaka J. Math., Vol.1, pp.81-89, 1964.
- [9] M.A.Ozturk, M.Sapanci, M.Soyturk and K.H.Kim, "Symmetric bi-Derivation on Prime Gamma-Rings", Sci.Math., Vol.3(2), pp.273-281, 2000.
- [10] H.J.Thahab, "On The Orthogonal α -Derivations of Semiprime Γ -Rings", M.Sc. Thesis, Department of Mathematics, College of Education, Al-Mustansiriyah University, 2013.