A NEW SPECTRAL PROPERTIES FOR LINEAR OPERATORS IN BANACH SPACE

AHFOUDA BELHADI, FARES YAZID, ABDELAZIZ AZEB AHMED, AND KHALED ZENNIR

Abstract. The main results of these studies consisted in finding new spectral properties of as bounded linear operators T and S defined on a Banach space, such as $TST = T^2\sqrt{T}$ and $STS = S^2\sqrt{S}$. The novelty of this work is to extend the study of Christoph Schmoeger [10] where the spectral properties of two operators T and S given as $TST = T^2$ and $STS = S^2$ were addressed and [5] in which the operators are taken $T^\tau S^k T^\tau = T^{\tau+1}$ and $S^\tau T^\tau S^\tau = S^{\tau+1}$, τ is a positive integer . These two works will be a special case of our results.

1. Introduction and preliminary

In the mathematical literature there are various classifications of points and subsets of the spectrum. This is due to the need for spectral theory to be used to solve the corresponding problems. All these classifications either in one sense or another reflect various properties of the operator; not having a bounded inverse throughout the entire space, or are associated with certain stability properties of subsets of the spectrum, for example, under certain perturbations, analytical mappings, see [6, 7].

The spectral theory of linear operators constitutes an essential section of the general spectral theory of operators and occupies a prominent place in mathematical studies of last centuries and applications of mathematics to physical theories and much attention in further research is paid to common spectral properties of linear operators. In almost every physical problem that can be formulated using linear operators, what is the object of primary physical interest? spectrum of the operator under consideration. Sufficient confirmation of this is the use of the term spectrum in both the physical and mathematical sense, see [1].

Throughout this paper H denote a complex Banach space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. For $T \in \mathcal{B}(H)$, let $N(T)$ denote the null space of T, and $R(T)$ denote the range of T. It is well-known and useful when T and S in $\mathcal{B}(H)$, then $\varpi(TS)\backslash\{0\} = \varpi(ST)\backslash\{0\}$, see [2, Prop. 6, p. 16], [11, Lemma 1.4.17]). We use $\varpi(T), \varpi_p(T), \varpi_{ap}(T), \varpi_c(T), \varpi_r(T)$ and $\rho(T)$ to denote spectrum, the point spectrum,the approximute spectrum, the continues spectrum, the residucal spectrum and the resolvent set of T respectively.
As in [11], the operator $T \in \mathcal{B}(H)$ is semi-Fredholm if $R(T)$ is closed, and either $\alpha(T) = \dim N(T)$ or $\beta(T)$= codim $R(T)$ is finite. $T \in \mathcal{B}(H)$ is Fredholm if T is semi-Fredholm, $\alpha(T) < \infty$ and $\beta(T) < \infty$.
The next result will be used later.

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Theorem 1.1. [11]*Let H be a Hilbert space and T and S be bounded linear operators on H. Then their is an equivalence between the following:*

(a) *There is a unique bounded linear operator A on H such that $A^2 = A$ and $T = AA^*$ and $S = A^*A$*
(b) *T and S are self adjoint and satisfy the relations $TST = T^2$ and $STS = S^2$.*

This Theorem was proved by Vidav with two manners, the first one is in geometrical way and the second one is algebric.

Theorem 1.2. [9]

(a) *If $A, B \in \mathbb{S}(H)$, $A^2 = A$, $B^2 = B$, $T = AB$ and $S = BA$, then*
$$TST = T^2,$$
and
$$STS = S^2.$$
(b) *Suppose that $T, S \in \mathbb{S}(H)$ are Drazin invertible, $ind(T) = ind(S) = 1$, $TST = T^2$ and $STS = S^2$. Then there are $A, B \in \mathcal{S}(H)$ such that $A^2 = A$, $B^2 = B$, $T = AB$ and $S = BA$.*

Let $A, B \in \mathcal{B}(H)$ such that $A^2 = A$, $B^2 = B$. If $T = AB$ and $S = BA$. Then have the next properties

(1) $\varpi(T)\backslash\{0\} = \varpi(TS)\backslash\{0\} = \varpi(ST)\backslash\{0\} = \varpi(S)\backslash\{0\}$,
(2) $\varpi_p(T)\backslash\{0\} = \varpi_p(TS)\backslash 0 = \varpi_p(ST)\backslash\{0\} = \varpi_p(S)\backslash\{0\}$,
(3) $\varpi_{ap}(T)\backslash 0 = \varpi_{ap}(TS)\backslash\{0\} = \varpi_{ap}(ST)\backslash\{0\} = \varpi_{ap}(S)\backslash\{0\}$,
(4) $\varpi_c(T)\backslash\{0\} = \varpi_c(TS)\backslash\{0\} = \varpi_c(ST)\backslash\{0\} = \varpi_c(S)\backslash\{0\}$,
(5) $\varpi_r(T)\backslash 0 = \varpi_r(TS)\backslash\{0\} = \varpi_r(ST)\backslash\{0\} = \varpi_r(S)\backslash\{0\}$.

If T and S are operators in $\mathcal{S}(H)$. Then
$$\varpi_p(T)\backslash\{0\} = \varpi_p(TS)\backslash\{0\} = \varpi_p(ST)\backslash\{0\} = \varpi_p(S)\backslash\{0\}.$$
Let $\lambda \neq 0$. Then, we have the next results ([3, 4])
$$\begin{cases} N(T - \lambda I) = N(TS - \lambda I) = TN(S - \lambda I) \\ N(S - \lambda I) = N(TS - \lambda I) = SN(T - \lambda I). \end{cases} \tag{1.1}$$
$$\varpi_{ap}(T)\backslash\{0\} = \varpi_{ap}(TS)\backslash\{0\} = \varpi_{ap}(ST)\backslash\{0\} = \varpi_{ap}(S)\backslash\{0\}. \tag{1.2}$$
$$\varpi_r(T)\backslash\{0\} = \varpi_r(TS)\backslash\{0\} = \varpi_r(ST)\backslash\{0\} = \varpi_r(S)\backslash\{0\}. \tag{1.3}$$
$$\varpi_r(T)\backslash\{0\} = \varpi_r(S)\backslash\{0\}. \tag{1.4}$$
$$\varpi_c(T)\backslash\{0\} = \varpi_c(TS)\backslash\{0\} = \varpi_c(ST)\backslash\{0\} = \varpi_c(S)\backslash\{0\}. \tag{1.5}$$
$$\varpi(T)\backslash\{0\} = \varpi(TS)\backslash\{0\} = \varpi(ST)\backslash\{0\} = \varpi(S)\backslash\{0\}. \tag{1.6}$$

2. Main results

By weakening the conditions for invertibility of an operator, we arrive at the concepts of normally solvable operators which can be applied to solving specific problems in the theory of differential equations.

Proposition 2.1. *Let $A, B \in \mathcal{B}(H)$ such that $A^2 = A,\ B^2 = B$. If $T = (AB)^2$ and $S = (BA)^2$. Then*

$$TS^2T = T^3\sqrt{T},$$

and

$$ST^2S = S^3\sqrt{S}.$$

Proof. We have

$$\begin{aligned} TS^2A &= ABABBABABABAABAB \\ &= ABABABABABABAB \\ &= (AB)^2(AB)^2(AB)^2AB \\ &= T^3\sqrt{T}. \end{aligned}$$

Similarly we get

$$ST^2S = S^3\sqrt{S}.$$

□

Proposition 2.2. *Let $A, B \in \mathcal{B}(H)$ such that $A^2 = A, B^2 = B$. If $T = (AB)^2$ and $S = (BA)^2$, then $TB = T$ and $SA = S$.*

Proof.

$$\begin{aligned} TB &= ABABB \\ &= ABAB \\ &= (AB)^2 \\ &= T, \end{aligned}$$

and

$$\begin{aligned} SA &= BABAA \\ &= BABA \\ &= (BA)^2 \\ &= S. \end{aligned}$$

□

Proposition 2.3. *Let $T, S \in \mathcal{B}(H)$ Drazin-invertibles. If $ind(T) = ind(S) = k$ and T^D, S^D the Drazn inverses of T, S respectivly. Then*

$$\begin{aligned} (T^D)^kT^{k+1} &= T, (S^D)^kS^{k+1} \\ &= S, \end{aligned}$$

and

$$\begin{aligned}(T^D)^{k+1}T^k &= T^D, (S^D)^{k+1}S^k \\ &= S^D.\end{aligned}$$

Proof.

$$\begin{aligned}(T^D)^k T^{k+1} &= (T^D)^{k-1}(T^D T^2)T^{k-1} \\ &= (T^D)^{k-1}T^k \\ &= (T^D)^{k-2}T^{k-1} \\ &= ... \\ &= T^D T^2 \\ &= T,\end{aligned}$$

and

$$\begin{aligned}(S^D)^k S^{k+1} &= (S^D)^{k-1}(S^D S^2)S^{k-1} = (S^D)^{k-1}S^k \\ &= (S^D)^{k-2}S^{k-1} \\ &= ... \\ &= S^D S^2 \\ &= S,\end{aligned}$$

then

$$\begin{aligned}(T^D)^{k+1}T^k &= (T^D)^{k-1}(T(T^D)^2)T^{k-1} \\ &= (T^D)^k T^{k-1} \\ &= (T^D)^{k-1}T^{k-2} \\ &= ... \\ &= T(T^D)^2 \\ &= T^D,\end{aligned}$$

and

$$\begin{aligned}(S^D)^{k+1}S^k &= (S^D)^{k-1}(S(S^D)^2)S^{k-1} \\ &= (S^D)^k S^{k-1} \\ &= (S^D)^{k-1}S^{k-2} \\ &= ... \\ &= S(S^D)^2 \\ &= S^D.\end{aligned}$$

□

Theorem 2.4. *Let $A, B \in \mathcal{B}(H)$ such that $A^2 = A, B^2 = B$, we have*

(*a*) *If $T = (AB)^2$ and $S = (BA)^2$, then*

$$TST = T^2\sqrt{T},$$

and

$$STS = S^2\sqrt{S}.$$

(b) *suppose that* $T, S \in \mathcal{B}(H)$ *are Drazin invertible. If* $ind(T) = ind(S) = k$, $TST = T^2\sqrt{T}$ *and* $STS = S^2\sqrt{S}$. *Then,*

$$A, B \in \mathcal{B}(H),$$

such that

$$A^2 = A, B^2 = B,$$

and

$$T = (AB)^2 S = (BA)^2.$$

Proof. (a) For the first point, we have

$$\begin{aligned} TST &= ABABBABAABAB \\ &= ABAB^2ABA^2BAB \\ &= ABABABABAB \\ &= (AB)^2(AB)^2AB \\ &= T^2\sqrt{T}. \end{aligned}$$

Similarly we get

$$\begin{aligned} STS &= BABAABABBABA \\ &= S^2\sqrt{S}. \end{aligned}$$

(b) Since $ind(T) = ind(S) = k$. Suppose that

$$A = (T^D)^2 T\sqrt{T}S,$$

and

$$B = ST\sqrt{T}(T^D)^2,$$

then,

$$\begin{aligned} A^2 &= (T^D)^2 T\sqrt{T}S(T^D)^2 T\sqrt{T}S \\ &= (T^D)^2\sqrt{T}TST(T^D)^2\sqrt{T}S \\ &= (T^D)^2\sqrt{T}T^2\sqrt{T}(T^D)^2\sqrt{T}S \\ &= (T^D)^2T^3(T^D)^2T\sqrt{T}S \\ &= (T^D)^2\sqrt{T}TS = A, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} B^2 &= ST\sqrt{T}(T^D)^2ST\sqrt{T}(T^D)^2 \\ &= S\sqrt{T}(T^D)^2TST\sqrt{T}(T^D)^2 \\ &= S\sqrt{T}(T^D)^2T^2\sqrt{T}\sqrt{T}(T^D)^2 \\ &= S\sqrt{T}(T^D)^2T^3(T^D)^2 \\ &= ST\sqrt{T}(T^D)^2 = B, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} (AB)^2 = ABAB &= (T^D)^2T\sqrt{T}SST\sqrt{T}(T^D)^2(T^D)^2T\sqrt{T}SST\sqrt{T}(T^D)^2 \\ &= (T^D)^2\sqrt{T}TS^2T\sqrt{T}(T^D)^2(T^D)^2\sqrt{T}TS^2T\sqrt{T}(T^D)^2 \\ &= (T^D)^2\sqrt{T}T^3\sqrt{T}\sqrt{T}(T^D)^2(T^D)^2\sqrt{T}T^3\sqrt{T}\sqrt{T}(T^D)^2 \\ &= (T^D)^8T^9 \\ &= T, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} (BA)^2 = BABA &= ST\sqrt{T}(T^D)^2(T^D)^2\sqrt{T}TS^2T\sqrt{T}(T^D)^2(T^D)^2T\sqrt{T}S \\ &= ST\sqrt{T}(T^D)^2(T^D)^2\sqrt{T}T^3\sqrt{T}\sqrt{T}(T^D)^2(T^D)^2A\sqrt{T}S \\ &= ST^7(T^D)^8\sqrt{T}S \\ &= STT^6(T^D)^7T^D\sqrt{T}S \\ &= ST(T^D)^2\sqrt{T}S \\ &= SA = S. \end{aligned} \tag{2.4}$$

□

Theorem 2.5. *Let $A, B \in \mathcal{B}(H)$ such that $A^2 = A, B^2 = B$. If*

$$T = (AB)^2,$$

and

$$S = (BA)^2.$$

Then

(1) $\varpi_p(T\sqrt{T})\backslash\{0\} = \varpi_p(TS)\backslash\{0\} = \varpi_p(S\sqrt{S})\backslash\{0\}$.

(2) $\varpi_{ap}(T\sqrt{T})\backslash\{0\} = \varpi_{ap}(TS)\backslash\{0\} = \varpi_{ap}(S\sqrt{S})\backslash\{0\}$.
(3) $\varpi_{r}(T\sqrt{T})\backslash\{0\} = \varpi_{r}(TS)\backslash\{0\} = \varpi_{r}(S\sqrt{S})\backslash\{0\}$.
(4) $\varpi(T\sqrt{T})\backslash\{0\} = \varpi(TS)\backslash\{0\} = \varpi(S\sqrt{S})\backslash\{0\}$.
(5) $\varpi_{c}(T\sqrt{T})\backslash\{0\} = \varpi_{c}(TS)\backslash\{0\} = \varpi_{c}(S\sqrt{S})\backslash\{0\}$.

The proof of Theorem 2.5 is based on proofs of the next Lemmas.

Lemma 2.6. *Let*

$$T, S \in \mathcal{B}(H),$$

such that

$$TST = T^2\sqrt{T},$$

and

$$STS = S^2\sqrt{S}.$$

Then we have

$$\begin{aligned} \varpi_p(T\sqrt{T})\backslash\{0\} &= \varpi_p(TS)\backslash\{0\} \\ &= \varpi_p(S\sqrt{S})\backslash\{0\}. \end{aligned} \tag{2.5}$$

Proof. We have

$$\begin{aligned} \varpi_p(T\sqrt{T})\backslash\{0\} &\subset \varpi_p(TS)\backslash\{0\} \\ &\subset \varpi_p(S\sqrt{S})\backslash\{0\}. \\ \varpi_p(T\sqrt{T})\backslash\{0\} &\subset \varpi_p(TS)\backslash\{0\} \\ &\subset \varpi_p(S\sqrt{S})\backslash\{0\}. \end{aligned}$$

Now, let $\lambda \in \varpi_p(T\sqrt{T})\backslash\{0\}$ which implies that $T\sqrt{T}x = \lambda x$. Then

$$ST\sqrt{T}x = \lambda Dx,$$

and

$$TST\sqrt{T}x = \lambda TSx,$$

so

$$T^2\sqrt{T}\sqrt{T}x = \lambda TSx$$

$$T\sqrt{T}(\lambda x) = \lambda TSx,$$

$$\lambda(\lambda x) = \lambda TSx,$$

$$\lambda x = TSx$$

$$\lambda \in \varpi_p(TS)\backslash 0.$$

Thus

$$\varpi_p(T\sqrt{T})\backslash\{0\} \subset \varpi_p(TS)\backslash\{0\}, \tag{2.6}$$

then

$$\lambda \in \varpi_p(TS)\backslash\{0\},$$

this implies that

$$\lambda x = TSx.$$

Other hand, we have

$$\begin{aligned} S(S\sqrt{S}x) &= S^2\sqrt{S}x \subset STSx \\ &= S(TSx) \\ &= S(\lambda x), \end{aligned} \tag{2.7}$$

which implies that

$$S(S\sqrt{S}x) = S(\lambda x), \tag{2.8}$$

and then

$$S\sqrt{S}x = \lambda x, \lambda \in \varpi_p(S\sqrt{S})\backslash\{0\}.$$

Then

$$\varpi_p(TS)\backslash\{0\} = \varpi_p(S\sqrt{S})\backslash\{0\}. \tag{2.9}$$

Conversely

$$\varpi_p(TS)\backslash\{0\} \subset \varpi_p(T\sqrt{T})\backslash\{0\}, \lambda \in \varpi_p(TS)\backslash\{0\},$$

and

$$TSx = \lambda x, TSTx = \lambda Tx,$$

which implies

$$T^2\sqrt{T}x = \lambda Tx,$$

$$T(T\sqrt{T}x) = \lambda Tx,$$

$$(T\sqrt{T}x) = \lambda x,$$

$$\lambda \in \varpi_p(T\sqrt{T})\backslash\{0\}.$$

Then

$$\varpi_p(TS)\backslash\{0\} \subset \varpi_p(T\sqrt{T})\backslash\{0\} \tag{2.10}$$

From (2.6) and (2.10), we have

$$\varpi_p(TS)\backslash\{0\} = \varpi_p(T\sqrt{T})\backslash\{0\},$$

and

$$\varpi_p(S\sqrt{S})\backslash\{0\} \subset \varpi_p(TS)\backslash\{0\}, \lambda \in \varpi_p(S\sqrt{S})\backslash\{0\},$$

and

$$(S\sqrt{S}x) = \lambda x,$$

then

$$S(S\sqrt{S}x) = \lambda Sx,$$

$$S^2\sqrt{S}x = \lambda Sx,$$

$$STSx = \lambda Sx,$$

$$TSx = \lambda x, \lambda \in \varpi_p(TS)\backslash\{0\}.$$

Then

$$\varpi_p(S\sqrt{S})\backslash\{0\} \subset \varpi_p(TS)\backslash\{0\}. \tag{2.11}$$

From (2.9) and (2.11), we have

$$\varpi_p(S\sqrt{S})\backslash\{0\} = \varpi_p(TS)\backslash\{0\}.$$

□

Lemma 2.7. *Let* $T, S \in \mathcal{B}(H)$ *such that*

$$TST = T^2\sqrt{T},$$

$$STS = S^2\sqrt{S},$$

and if $\lambda \neq 0$. *Then*

$$N(T\sqrt{T} - \lambda I) = N(TS - \lambda I) = TN(S\sqrt{S} - \lambda I),$$

$$N(S\sqrt{S} - \lambda I) = N(ST - \lambda I) = SN(T\sqrt{T} - \lambda I).$$

Proof. (1) The proof of this corollary shows that

$$N(T\sqrt{T} - \lambda I) \subset N(TS - \lambda I).$$

Let $x \in N(TS - \lambda I)$ and $TSx = \lambda x$, then

$$T^2\sqrt{T}Sx = \lambda T\sqrt{T}x$$

$$TSTSx = \lambda T\sqrt{T}x$$

$$TT(TSx) = \lambda T\sqrt{T}x$$

$$TS(\lambda x) = \lambda T\sqrt{T}x$$

$$\lambda^2 x = \lambda T\sqrt{T}x$$

$$\lambda x = T\sqrt{T}x$$

$$x \in N(T\sqrt{T} - \lambda I),$$

hence, we have

$$N(S\sqrt{S} - \lambda I) = N(TS - \lambda I).$$

Let

$$x \in N(TS - \lambda I),$$

then

$$TSx = \lambda x,$$

and

$$TSS\sqrt{S}x = \lambda S\sqrt{S}x,$$

$$TS^2\sqrt{S}x = \lambda S\sqrt{S}x,$$

$$TSTSx = \lambda S\sqrt{S}x,$$

$$TS(TSx) = \lambda S\sqrt{S}x,$$

$$TS(\lambda x) = \lambda S\sqrt{S}x,$$

$$\lambda^2 x = \lambda S\sqrt{S}x$$

$$\lambda x = S\sqrt{S}x$$

$$x \in N(S\sqrt{S} - \lambda I).$$

Hence, we have

$$N(S\sqrt{S} - \lambda I) = N(TS - \lambda I)$$

For all $x \in N(ST - \lambda I)$, thus $STx = \lambda x$, hence

$$\lambda Sx = ST(Sx) = STSx = S^2\sqrt{S}x$$

$$\lambda S(\sqrt{S}x) = S^2\sqrt{S}(\sqrt{S}x)$$

$$\lambda S\sqrt{S}x = (S\sqrt{S})^2 x$$

$$\lambda Sx = S\sqrt{S}x,$$

this gives $x \in N(S\sqrt{S} - \lambda I)$ and we deduce that

$$N(ST - \lambda I) \subset N(S\sqrt{S} - \lambda I).$$

On the other hand, let $x \in N(S\sqrt{S} - \lambda I)$ thus $S\sqrt{S}x = \lambda x$, hence

$$\begin{aligned} \lambda Sx &= S\sqrt{S} \\ &= STSx \end{aligned}$$

Then

$$\begin{aligned} \lambda STx &= STSTx \\ &= S^2\sqrt{S}x \\ &= STSx. \end{aligned}$$

Thus $\lambda x = STx$ and we deduce that $x \in N(ST - \lambda I)$ Which implies that

$$N(S\sqrt{S} - \lambda I) \subset N(ST - \lambda I).$$

(2) From [1, Proposition 2], we see that

$$N(TS - \lambda I) = TN(ST - \lambda I).$$

Thus

$$N(TS - \lambda I) = TN(T\sqrt{T} - \lambda I).$$

On the other hand

$$N(ST - \lambda I) = SN(AB - \lambda I).$$

Thus

$$N(ST - \lambda I) = SN(T\sqrt{T} - \lambda I).$$

This complets the proof. □

Lemma 2.8. *Let $T, S \in \mathcal{B}(H)$ such that*

$$TST = T^2\sqrt{T},$$

$$STS = S^2\sqrt{S},$$

and if $\lambda \neq 0$. Then we have

$$\begin{aligned} \varpi_{ap}(T\sqrt{T})\backslash\{0\} &= \varpi_{ap}(TS)\backslash\{0\} \\ &= \varpi_{ap}(S\sqrt{S})\backslash\{0\}. \end{aligned}$$

Proof. It suffices to show that

$$\begin{aligned} \varpi_{ap}(T\sqrt{T})\backslash\{0\} &\subset \varpi_{ap}(TS)\backslash\{0\} \\ &\subset \varpi_{ap}(S\sqrt{S})\backslash\{0\}. \end{aligned}$$

To this end let $\lambda \in \varpi_{ap}(T\sqrt{T})\backslash\{0\}$, then there exist a sequence (x_n) in H, with $\|x_n\| = 1$ for all $n \in \mathbb{N}$, $(\lambda I - T\sqrt{T})x_n \to 0$ when $n \to \infty$.
Let $z_n = (\lambda I - T\sqrt{T})x_n$, hence $T\sqrt{T}x_n = \lambda x_n - z_n$ and $z_n \to 0$ when $n \to \infty$, which give us

$$T\sqrt{T}^2 x_n = \lambda T\sqrt{T}x_n - T\sqrt{T}z_n,$$

and there

$$\begin{aligned} (T\sqrt{T})^2 x_n &= \lambda(\lambda x_n - z_n) - T\sqrt{T}z_n \\ &= \lambda^2 x_n - \lambda z_n - T\sqrt{T}z_n, \end{aligned}$$

and

$$T\sqrt{T}x_n = \lambda x_n - Z_n$$

$$ST\sqrt{T}x_n = S\lambda x_n - SZ_n$$

$$TST\sqrt{T}x_n = TS\lambda x_n - TSZ_n$$

$$(T\sqrt{T})^2 x_n = TS\lambda x_n - TSZ_n$$

$$(T\sqrt{T})^2 x_n - TS\lambda x_n = -TSZ_n$$

$$\lambda^2 x_n - \lambda z_n - T\sqrt{T}z_n - TS\lambda x_n = -TSz_n$$

$$\lambda^2 x_n - TS\lambda x_n = \lambda z_n + T\sqrt{T}z_n - TSz_n$$

$$= (\lambda I + T\sqrt{T} - TS)z_n \to 0 \ as \ n \to \infty.$$

This gives $(\lambda I - TS)x_n \to 0 \ as \ n \to \infty)$.
Hence

$$\lambda \in \varpi_{ap}(TS)\backslash\{0\},$$

and

$$\varpi_{ap}(TS)\backslash\{0\} \subset \varpi_{ap}(S\sqrt{S})\backslash\{0\}.$$

For $\lambda \in \varpi_{ap}(TS)\backslash\{0\}$, there exist a sequence (x_n) in X,with $\|x_n\| = 1$ for all $n \in \mathbb{N}$ $(\lambda I - TS)x_n \to 0$ when $n \to \infty$.
Let $Z_n = (\lambda I - TS)x_n$, hence $TSx_n = \lambda x_n - Z_n$ and $Z_n \to 0$ when $n \to \infty$, which implies

$$STSx_n = S\lambda x_n - SZ_n,$$

and

$$(S)^2\sqrt{S}x_n = S\lambda x_n - SZ_n,$$

and

$$S\sqrt{S}x_n = \lambda x_n - Z_n,$$

then

$$\lambda \in \varpi_{ap}(S\sqrt{S})\backslash\{0\},$$

hence, we have

$$\varpi_{ap}(TS)\backslash\{0\} = \varpi_{ap}(S\sqrt{S})\backslash\{0\}.$$

□

Lemma 2.9. *Let $T = (AB)^2$, $S = (BA)^2$ and*

$$TST = T^2\sqrt{T},$$

$$TS = S^2\sqrt{S}.$$

Then we have

$$\begin{aligned} \varpi_r(T\sqrt{T})\backslash\{0\} &= \varpi_r(TS)\backslash\{0\} \\ &= \varpi_r(S\sqrt{S})\backslash\{0\}. \end{aligned}$$

Proof. Let

$$\lambda \in \varpi_r(T\sqrt{T})\backslash\{0\},$$

hence

$$\lambda \notin \varpi_p(A\sqrt{T})\backslash\{0\},$$

and

$$(\lambda I - T\sqrt{T})(X) \neq X.$$

Thus

$$N(\lambda I^* - (T\sqrt{T})^*)\backslash\{0\} \neq \{0\}.$$

By (1.1), we have

$$N(\lambda I^* - (TS)^*) = N(\lambda I^* - S^*T^*)\backslash\{0\} \neq \{0\},$$

hence

$$\overline{((\lambda I - TS))}(X) \neq X.$$

Since $\lambda \notin \varpi_p(TS)\backslash\{0\}$ and owing to (1.3), we have

$$\lambda \in \varpi_r(T\sqrt{T})\backslash\{0\}.$$

Now let

$$\lambda \in \varpi_r(TS)\backslash\{0\},$$

hence

$$\lambda \notin \varpi_p(TS)\backslash\{0\},$$

and

$$((\lambda I - TS)(X) \neq X),$$

it follows that

$$N(\lambda I^* - (TS)^*) = N(\lambda I^* - S^*T^*)\backslash\{0\} \neq \{0\}.$$

From (1.4), we get

$$N(\lambda I^* - (T\sqrt{T})^*)\backslash\{0\} \neq \{0\},$$

thus

$$((\lambda I - T\sqrt{T}))(X) \neq X.$$

Since

$$\lambda \notin \varpi_p(T\sqrt{T})\backslash\{0\}.$$

Using (1.1)), we have

$$\lambda \in \varpi_r(T\sqrt{T})\backslash\{0\},$$

and by (1.4), we have

$$\varpi_r(T\sqrt{T})\backslash\{0\} = \varpi_r(S\sqrt{S})\backslash\{0\}.$$

The proof is now completed. □

Lemma 2.10. *Let*

$$T = (AB)^2, S = (BA)^2,$$

and

$$TST = T^2\sqrt{T}, STS = S^2\sqrt{S}.$$

Then

$$\begin{aligned} \varpi(T\sqrt{T})\backslash\{0\} &= \varpi(TS)\backslash\{0\} \\ &= \varpi(S\sqrt{S})\backslash\{0\}. \end{aligned}$$

Proof. Let $\lambda \in \varpi(T)\backslash\{0\}$ and assume to the contrary that $\lambda \in \rho(TS)$ then

$$\alpha(\lambda I - TS) = 0,$$

and

$$\lambda \notin \varpi_{ap}(TS).$$

By Lemma 2.6 and Lemma 2.8, we deduce that

$$\alpha(\lambda I - T\sqrt{T}) = 0,$$

and

$$\lambda \in \varpi_{ap}(T\sqrt{T}).$$

Therefore

$$\chi(\lambda I - T) = \inf\{\|(\lambda I - T\sqrt{T})x\|,\ x \in H,\ \|x\| = 1\} > 0,$$

hence $(\lambda I - T\sqrt{T})x = 0$ is closed. Thus we have show that $\lambda I - T$ is semi-Fredholm. Since

$$\lambda \in \rho((TS)^*) = \rho(S^*T^*),$$

it follows from [3, Proposition 5.3] that $\lambda \in \rho(S^*T^*)$, since

$$T^*S^*T^* = (T^2\sqrt{T})^*,$$

and

$$S^*T^*S^* = (S^2\sqrt{S})^*,$$

the same arguments as above show that

$$\alpha(\lambda I^* - (T\sqrt{T})^*) = 0,$$

and that

$$I^* - (T\sqrt{T})^*,$$

is semi-Fredholm.
It follows now that

$$\beta(\lambda I - T\sqrt{T}) = \alpha(\lambda I^* - (T\sqrt{T})^*),$$

thus

$$0 \in \rho(T\sqrt{T}),$$

is a contradiction. Hence

$$\varpi(T\sqrt{T})\backslash\{0\} \subset \varpi(TS)\backslash\{0\}.$$

The converse

$$\varpi(TS)\backslash\{0\} \subset \varpi(T\sqrt{T})\backslash\{0\},$$

goes similarly.
Use arguments similar to those in above, we can prove

$$\varpi(T\sqrt{T})\backslash\{0\} = \varpi(S\sqrt{S})\backslash\{0\}.$$

□

Lemma 2.11. *Let*

$$T = (AB)^2, S = (BA)^2,$$

and

$$TST = T^2\sqrt{T}, STS = S^2\sqrt{S}.$$

Then

$$\begin{aligned} \varpi_c(T\sqrt{T})\backslash\{0\} &= \varpi_c(TS)\backslash\{0\} \\ &= \varpi_c(S\sqrt{S})\backslash\{0\}. \end{aligned}$$

Proof. By Lemma 2.9 and Lemma 2.10, we have

$$\begin{aligned} \varpi_c(T\sqrt{T})\backslash\{0\} &= \varpi(T\sqrt{T}) \setminus \{\varpi_p(T\sqrt{T}) \cup \varpi_r(T\sqrt{T}) \cup \{0\}\} \\ &= \varpi(TS) \setminus \{\varpi_p(TS) \cup \varpi_r(TS) \cup \{0\}\} \\ &= \varpi_c(TS)\backslash\{0\}. \end{aligned}$$

On the other hand using arguments similar to those above we find

$$\varpi_c(TS)\backslash\{0\} = \varpi_c(S\sqrt{S})\backslash\{0\}$$

□

Proof. (Of Theorem 2.5) By Lemma 2.6- Lemma 2.11, one can easily established the proof of Theorem 2.5. □

3. Conclusion

The paper is devoted to the modern mathematical problem of spectral analysis - the theory of essential spectra of bounded linear operators. It studies the structure of the spectrum and essential spectra determined by properties of operators. This problem is of a fundamental nature, since its solution, on the one hand, allows us to obtain qualitative characteristics of the essential spectra of specific classes of linear operators, and on the other hand, indicates algorithms and exact formulas for finding the spectrum and essential spectra of these operators. The paper contains new scientifically substantiated theoretical results in the spectral of various classes of bounded and linear operators in Banach spaces. A variety of techniques are known to achieve the desired result including the algebraic techniques which is used in our work. We have also obtained new results related to the qualitative aspect of the operators. This field of research is very important for researchers who are interested in modern science; engineers and new physical principles. In many practical applications it is possible to formulate extremely important problems, the solution of which requires newly derived methods such as geometrical way (se [8, 12]).

Declaration of Competing Interest. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Ahfouda Belhadi: Department of Mathematics, Exact Sciences Faculty, University of El-Oued, P.O.Box789, El Oued39000, Algeria
E-mail address: belhadi67@gmail.com

Fares Yazid: Laboratory of pure and applied Mathematics, Amar Teledji Laghouat University, Algeria.
E-mail address: f.yazid@lagh-univ.dz

Abdelaziz Azeb Ahmed: Laboratory of Operator Theory and PDE: Foundations and Applications University of El Oued, Algeria.
E-mail address: aziz-azebahmed@univ-eloued.dz

Khaled Zennir: Department of Mathematics, College of Sciences and Arts in Ar-Rass, Qassim University, Saudi Arabia.
E-mail address: k.zennir@qu.edu.sa