Some Results About αc– Continuous Functions

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Abstract
The main aim of this paper is to study new class of continuous function is called αc – continuous function. for this aim, the nation of α- open and pre – open sets and α - compact space are introduced. and we shall study the relationship between αc – continuous and pc– continuous functions.

Keywords: pc continuous function; topology; open set

1. Introduction and preliminaries

Let (X, τ) be a topological space and $A \subseteq X$, we denoted to the complement of A in X byA^c, the closure and the interior of A respectively by $\overline{A}$ and A° we recall the following function :-

Definition 1-1 :- Let (X, τ) be a topological space, A subset $A \subseteq X$ is called

1) **a semi open set [5]** if $A \subseteq \overline{(A^\circ)}$ and semi - closed if $(\overline{A})^\circ \subseteq A$.
2) **a pre – open set [10]** if $A \subseteq (\overline{A})^\circ$ and pre - closed set if $\overline{(A^\circ)} \subseteq A$.
3) **an α - open set [13]** if $A \subseteq (\overline{(A^\circ)})^\circ$ and a -close set if $\overline{((\overline{A})^\circ)} \subseteq A$
4) **a regular open set** if $A = (\overline{A})^\circ$ and regular closed set if $A = \overline{(A^\circ)}$.

The complement of an α - open (resp. semi – open , pre – open , regular open) set is called α - closed (resp. semi – closed , pre – closed , regular closed) set. The smallest α - closed (resp. semi – closed , pre – closed , regular closed) set containing $A \subseteq X$ is called α - closure (resp. semi – closure , pre – closure , regular – closure) of A and shall be denoted by $\overline{A}^\alpha$(resp. $\overline{A}^s$, $\overline{A}^p$, $\overline{A}^R$) .we denote the family of α - open (resp. semi – open, pre – open , regular open) sets by T^α,(resp. T^s , T^p , T^R) sets ,it is shown in [ ] that $T \subset T^\alpha \subset T^s$ and T^α is α topology for X ,It is also shown that $T^\alpha = T^s \cap T^p$. an α - open set is precisely semi – open and pre – open [14, Lemma 1]

In 1970 , Gentry and Hoyle [4] introduce the concept of **C –continuous function**, A function $f: X \to Y$is called **C-continuous function** if and only if for each $x \in X$ and each open set V containing f(x) such that $V^c$is compact ,there exists an open set U containing x such that $f(U) \subset V$,in 1987, F. cammroto and T.noiri [3] define new concept of function is called **WC-continuous function** by replacing **compact** in the definition of C continuous function with **weakly – compact** relative to Y. and A function $f: X \to Y$ is called **PC – continuous function [5]** if and only if for each $x \in X$, and each open set V containing f(x), such that V^c is **P – compact relative to Y** , there exists an open set U containing x such that $f(U) \subset V$.

1- α - compact space.

Definition 2.1 :- A space $(X, \tau)$is said to be α **- compact space** [9] if every α - open cover of X has a finite sub cover .

It is shown in [9] that every α - compact space is compact but not conversely , the following is obvious *from **definition 2.1*** . the subspace $(A, \tau/A)$ is α - compact , where τ/A denotes the induced topology on A.

Proposition 2.2 :- A space (X, τ) is α **- compact** if and only if $(\mathbf{X}, \boldsymbol{\tau}^\alpha)$ **is compact** .

Definition 2.3 :- A subset A of space (X, τ) is said to be α **– compact relative to**$(\mathbf{X}, \boldsymbol{\tau})$, if every cover of A by α **- open sets** of (X, τ) has a finite sub cover.

Theorem 2.4 :- A subset A of a space (X, τ) is α **- compact relative to** (X, τ) if and only if is **compact in** (X, τ^α)

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Proof :-

Necessity : Let $A = \cup \{W_\lambda | \lambda \in \Omega\}$ and $W_\lambda \in \tau^\alpha / A$ for each $\lambda \in \Omega$.

For each $\lambda \in \Omega$, there exists $V_\lambda \in \tau^\alpha$, such that $W_\lambda = V_\lambda \cap A$. since $\{V_\lambda | \lambda \in \Omega\}$ is a cover of A , there exists a finite subset $\Omega_\circ$ of Ω such that $A \subset \cup \{V_\lambda | \lambda \in \Omega_\circ\}$. Therefore , we obtain $A = \cup \{W_\lambda | \lambda \in \Omega_\circ\}$. Hence A is compact set of (X, τ^α).

Sufficiency : Let $A \subset \cup \{W_\lambda | \lambda \in \Omega\}$ and $W_\lambda \in \tau^\alpha$ for each $\lambda \in \Omega$.

Then, we have $A = \cup \{W_\lambda \cap A | \lambda \in \Omega\}$ and $W_\lambda \cap A \in \tau^\alpha / A$ for each $\lambda \in \Omega$.

Therefore , for some finite subset $\Omega_\circ$ of Ω

$A = \cup \{W_\lambda \cap A | \lambda \in \Omega_\circ\} \subset \cup \{W_\lambda | \lambda \in \Omega_\circ\}$

This is shows that A is α - compact relative to (X, τ^α).

Proposition 2.5 :- Let A be a subset of space (X, τ) if A is α **- compact relative to** (X, τ), then is α **- compact** .

Proof :- Let $A = \cup \{W_\lambda | \lambda \in \Omega\}$ and $W_\lambda \in (\tau/A)^\alpha$ for each $\lambda \in \Omega$. it follows from proposition 2.1 of [8] that $(\tau/A)^\alpha \subset \tau^\alpha / A$.

For each $\lambda \in \Omega$, there exists , $V_\lambda \in \tau^\alpha$ such that $W_\lambda = V_\lambda \cap A$.

Since $\{V_\lambda | \lambda \in \Omega\}$ s a cover of A , there exists $\Omega_\circ$ of Ω , such that $A \subset \cup \{W_\lambda | \lambda \in \Omega_\circ\}$

Therefore, we obtain $A = \cup \{W_\lambda | \lambda \in \Omega\}$.

So A is α - compact of (X, τ).

Lemma 2.6 :- [8] , A space (X, τ) is α**- compact** if and only if every **proper** α **- closed set** of (X, τ) is α**- compact relative to** (X, τ).

Corollary 2.7 :- if a space (X, τ) is α **- compact** and A is an α **- closed set** of (X, τ), then A is α **– compact**.

Proof :-

This is an immediate consequence of lemma 2.6 and proposition 2.5 By corollary 2.7 , we abserve that the assumption " open " of statement (a) in the following corollary is super fluous.

Corollary 2.8 :- [9] Let (X, τ) be an a - compact space and A α subset of (X, τ), then we have

a) if A is open and α - closed set in (X, τ), then it is α - compact.

b) if A is α - closed in (X, τ), then it is compact.

2- αc – continuous function

Definition 3.1 :- A function $f: X \to Y$ is called α**c – continuous function** if and only if for each $x \in X$, and each open set V containing f(x), such that V^c is α **-compact relative to Y**, there exists an open set U containing x such that $f(U) \subset V$.

Theorem 3.2 :- for a function $f: (X, \tau) \to (Y, \tau')$ the following are equivalent :-

1) f is α**c – continuous** function .
2) if V is **open** in Y and V^c is α**- compact relative to Y** then $f^{-1}(V)$ is **open** set in X.
3) if F is **closed** in Y and α **- compact relative to Y** then $f^{-1}(F)$ is **closed** set in X

proof :- of $1 \to 2$

Let V an open set of Y such that V^c is α -compact relative to Y .

Let $x \in f^{-1}(V)$, then $f(x) \in V$, and there exists an open set U of X , such that $f(U) \subset V$.

So, we have $x \in U \subset f^{-1}(V)$.

Then $f^{-1}(V)$ is open set in (X, τ).

of $2 \to 3$

let F is closed set and α-compact relative to Y.

then F^c is open , and since $F = (F^c)^c$ is α-compact relative to Y.

then $f^{-1}(F^c)$ is open set

so $f^{-1}(F)$ is closed.

of $3 \to 1$

let $x \in X$, and V an open containing f (x) , such that V^c is α- compact relative to Y.

so $f^{-1}(V^c)$ is closed set in X .

and hence $U = f^{-1}(V)$is an open set containing x , such that $f(U) \subset V$ so f is αc – continuous function.

Proposition 3.3 :- if $A \subseteq X$ is **p –compact relative to X** then it is α **- compact relative to X**.

proof :-

let $A \subset \cup \{W_\lambda | \lambda \in \Omega\}$, such that W_λ is α - open set .

then W_λ is pre – open set .

since A is P – compact relative to X .

so there exists a finite subset $\Omega_\circ$ of Ω , such that $A \subset \cup \{W_\lambda | \lambda \in \Omega_\circ\}$. There for A is α -compact relative to X.

Proposition 3.4 :- if $A_1$and A_2 are α**- compact relative to X** then $A_1 \cup A_2$ is α**- compact relative to X**

proof:- let $A_1 \cup A_2 \subset \cup \{W_\lambda | \lambda \in \Omega\}$, such that $W_\lambda$is α- open set.

then $A_1 \subset \cup \{W_\lambda | \lambda \in \Omega\}$ and $A_2 \subset \cup \{W_\lambda | \lambda \in \Omega\}$.

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So for each i = 1 , 2 , there exists a finite subset Ω_i of Ω,
such that $A_i \subset \cup \{W_\lambda | \lambda \in \Omega_i\}$
so, we have $A_1 \cup A_2 \subset \cup \{W_\lambda | \lambda \in \Omega_i\}$.
Then $A_1 \cup A_2$ is α- compact relative to X

Let (X, τ) be a topological space , it follows from **proposition 3 . 4** that the family of open sets having the complement α- compact relative to X may be used as a base for a topology $\tau_{\alpha c}$. it has been shown that the family of open sets having the complement P –compact relative to X may be used as base for a topology τ_{PC} and the family of open sets having the complement compact relative to X may be used as base for a topology τ_c and the family of open sets having the complement almost – compact relative to X may be used as base for a topology τ_A and the family of open sets having the complement weakly – compact relative to X may be used as base for a topology τ_{WC}.

Remark 3.5 :- for space X, we have $\tau_{\alpha c} \subset \tau_{PC} \subset \tau_c \subset \tau_A \subset \tau_{WC} \subset \tau$.

Definition 3.6 :- A function $f: (X, \tau) \to (Y, \tau')$is said to be

1) α- **continuous** function [12] if $f^{-1}(V)$ is α- open in X for every V is open in Y.
2) **Weakly** α- **continuous** function if for each x ∈ X, and each open set V in Y, such that f(x) ∈ V, there exists an α- open set U containing x such that $f(U) \subset \overline{V}$.

Theorem 3.7 :- A function $f: (X, \tau) \to (Y, \tau'_{\alpha c})$ is **continuous** if and only if $f: (X, \tau) \to (Y, \tau'_{\alpha c})$ α- **continuous** function .

Proof :- **Necessity** :let V is open in Y, since $f: (X, \tau) \to (Y, \tau'_{\alpha c})$ is continuous.
Then $f^{-1}(V)$ is open in X.
So, $f^{-1}(V)$ is α- open in X.
Therefore $f: (X, \tau) \to (Y, \tau'_{\alpha c})$ is α- continuous function.
Sufficiency : let V is open in Y.
Then V^c is α- compact relative to Y.
From **proposition 3.2** , Then $f^{-1}(V)$ is open in X.
Then $f: (X, \tau) \to (Y, \tau'_{\alpha c})$ is continuous function.

Theorem 3.8 :- A function $f: (X, \tau) \to (Y, \tau'_{\alpha c})$ is α**c – continuous** function if and only if $f: (X, \tau) \to (Y, \tau'_{\alpha c})$is **continuous** function.

Proof :- **Necessity** : let V is open in Y.
Then V^c is α- compact relative to Y.
From **proposition 3.2** , Then $f^{-1}(V)$ is open in X.
Then $f: (X, \tau) \to (Y, \tau'_{\alpha c})$ is continuous function.
Sufficiency : Let x ∈ X, and each open set V containing f(x), such that V^c is α-compact relative to Y.
Then $f^{-1}(V)$ is open set in X containing x.
Let $f^{-1}(V) = U$, then $f(U) \subset V$.
Therefore $f: (X, \tau) \to (Y, \tau'_{\alpha c})$ is αC- continuous function.

Corollary 3.9 :- A function $f: (X, \tau) \to (Y, \tau'_{\alpha c})$ is αc–continuous function if and only if $f: (X, \tau) \to (Y, \tau'_{\alpha c})$ is α-continuous function.

Proof :- this is an immediate from **theorem 3.8**.

Theorem 3.10 :- if $f: (X, \tau) \to (Y, \tau'_{\alpha c})$ is α**c – continuous** function then $f: (X, \tau) \to (Y, \tau'_{\alpha c})$ is **weakly** α **– continuous** function.

Proof :- let x ∈ X, and each an open set V in Y.
Then V^c is α- compact relative to Y.
Since $f: (X, \tau) \to (Y, \tau'_{\alpha c})$ is αc – continuous function , then there exists an open set U containing x, such that $f(U) \subset V$.
So , $f(U) \subset \overline{V}$.
Therefore $f: (X, \tau) \to (Y, \tau'_{\alpha c})$ is weakly α – continuous function.

Theorem 3.11:- Let $(X, \tau), (Y, \tau')$ be tow topological spaces ,if $f: (X, \tau) \to (Y, \tau')$ is α - **continuous function** then $f: (X, \tau) \to (Y, \tau')$ is **PC – continuous function**.

Proof :- let x ∈ X and V is open set containing f (x) ,such that V^c is P -compact space relative to Y
Then V^c is α- compact relative to Y.
Since $f: X \to Y$ is αc– continuous function.
So there exists open set U containing x, such that $f(U) \subset V$.
Therefore $f: X \to Y$ is PC – continuous function.

Theorem 3.12 :- if $f: (X, \tau) \to (Y, \tau')$ is αc **– continuous** function, and A is subset of X, then restriction $f/A : A \to Y$ is αc– continuous function.

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Theorem 3.13 :- if $f: (X, \tau) \rightarrow (Y, \tau')$ is **continuous** function and $g: (Y, \tau') \rightarrow (Z, \sigma)$ is α**c- continuous** function, then the composition g o $f: X \rightarrow Z$ is αc – continuous.

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