SOME NEW RESULTS ON DIFFERENCE PARACOMPACT NOTION IN TOPOLOGICAL SPACES

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Abstract

An interesting area of research in topology is D −paracompact spaces. It is a significant type of topological space that retain compactness while benefiting from paracompactness and is considered as generalization of compact spaces. The concept of D −paracompactness was introduced and its basic characteristics were examined by the author in [17]. In this research, we introduce and improve this concept further by using a special type of covering and the difference sets (called as D −sets), which contains new and impactful properties. As a result, we obtained several new properties and results. We discuss the concept, characteristics, and theorems that related of D −paracompact space. We also studied different characterizations of D −paracompact spaces and discussed how they relate to other topological characteristics. We also give numerous instances of D −paracompact spaces, highlighting their applicability in different topological spaces.

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1 Overview of the Historical developments and progress

Soon after the notion of paracompact spaces was introduced, several articles were published that examine the idea from various angles and tools. In the year 1944, Dieudonné

[10] first suggested the notion of paracompact space as a concept that is the extension of compactness. Later, in 1969, Single and Arya [20] provided a new type of Para compactness, named nearly Paracompactness, weaker than the usual Para compactness, that determines its basic topological properties. Based on that, in 1978, Steen and Seebach [21] introduced the notion of metacompact in the topological space (X, τ). In 1982, Tong [22] introduced the notion of difference sets, often known as $D-$sets. Also, in 1984, Pareek [17] contributed the concept of $D-$paracompact and studied their properties and relations with other topological spaces. In 1998, Mukherjee and Debray [15] used paracompact to give, the same definition. It has been investigated in terms of a certain type of cover, called regular even cover. While, in 2006, the topologists Al-Ghour [3] defined the concepts of $\omega-$paracompactness and countable $\omega-$paracompactness as generalizations of paracompactness. He explained and characterized each of them, which he studied dealt with subspaces, products, and mappings of each. Also, in 2006, Al-Zoubi [6] as a generalization of paracompact spaces, he uses the semi$-$open sets to develop the class of $S-$paracompact space and defined $S-$paracompact space, he also investigated their fundamental characteristics. The investigations are made into the connections between $S-$paracompact spaces and other well-known spaces. Later, in 2007, Al-Zoubi and Al-Ghour [5] provided and studied P_3-paracompact, a weaker variant of paracompactness. With it and its interactions with other spaces, they obtained a variety of characterizations, properties, instances, and counterexamples. In 2013, Demir and Ozbakir [9] defined the concepts of $\beta-$paracompact spaces and $\beta-$expandable spaces as a weak variant of paracompact and expandable spaces, respectively. These spaces' basic characteristics were also provideded. Additionally, it is established that every $\beta-$paracompact space is a $\beta-$expandable space, and the connections between these spaces and a few previously researched spaces are investigated. In 2019, Turanli and Ozbakir [23] introduced and investigated β_1-L-paracompact space, which is a stronger variant of $L-$paracompact space established on an ideal space. After that, they looked into connections between β_1-L-paracompact spaces and other paracompactnesses. Additionally, they discovered numerous characteristics, instances, and counterexamples of β_1-L-paracompactness. In 2021, Al-Ghour [2] introduced both concepts of $\sigma-\omega-$paracompactness and feebly $\omega-$paracompactness, with $\omega-$paracompactness being a weaker version of $\sigma-\omega-$paracompactness. Furthermore, in 2021, Oudetallah, et al. [16] introduced the notions of $D-$metacompactness spaces and studied their properties. The present research is mainly a continuation of the study of $D-$paracompact spaces from new viewpoints. We have aimed to eventually arrive at new concepts as well as develop severa characteristic theorems and instances, all of which we present in this research.

2 Preliminary and Basic Notions

Within this section, we include some essential symbols, such as $\mathbb{R}$ to denote the collection of real numbers. Also, ϑ_u, ϑ_{cof}, $\vartheta_{l.r}$, ϑ_{ind}, and ϑ_{dis} will denote the standard, co$-$finite, left$-$ray, indiscrete, and discrete topologies, respectively. Now, we provide some basic definitions as well as the major results that are required.

Definition 2.1 [10] *If any open cover of the topological space* (W,ϑ) *has an open locally−finite refinement, then the space is called paracompact space.*

Definition 2.2 [11] *If any countable open cover of the topological space* (W,ϑ) *has an open locally−finite refinement, then the space is called countably paracompact.*

Definition 2.3 [23] *If any open cover of the topological space* (W,ϑ) *has an open locally−countable refinement, then the space is called paralindelöf space.*

Definition 2.4 [22] *A subset* $W_1 \subseteq (W,\vartheta)$ *is* D *−set if there are open sets* A *and* B *such that* $A \neq W$ *and* $W_1 = A - B$*. Furthermore, the subset* W_1 *is* D *−set that generated by* A *and* B*.*

Definition 2.5 [18] *A cover* $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ *of the space* (W,ϑ) *is called* D *−cover if any* E_ρ *is* D *−set, for every* $\rho \in \Lambda$*.*

Definition 2.6 [4] *If any* D *−cover of the topological space* (W,ϑ) *has an open locally countable refinement, then the space is called* D *−paralindelöf space.*

Definition 2.7 [18] *If any* D *−cover of the topological space* (W,ϑ) *has a countable subcover, then the space is called* D *−lindelõf.*

Definition 2.8 [18] *If any* D *−cover of the topological space* (W,ϑ) *has a finite subcover, then the space is called* D *−compact.*

Definition 2.9 [14] *A subset* $W_1 \subseteq (W,\vartheta)$ *is called* D *−dense set, if for any* $y \in W$*, we have* $D_y \cap W_1 \neq \emptyset$*, and each* D *−set containing* y*.*

Definition 2.10 [14] *The space* (W,ϑ) *is called* D *−separable, if the space has a subset* H *which is* D *−dense countable.*

Definition 2.11 [19] *The space* (W,ϑ) *is called locally−indiscrete if any set* ϑ_n *−open is* ϑ_n *−clopen, where* $n = 1,2$*.*

Definition 2.12 [12] *If any cover of the topological space* (W,ϑ) *has a countable subcover, then the space is called lindelöf space.*

Definition 2.13 [14] *Let* (W,ϑ) *and* (Q,ι) *be any topological spaces. If the inverse image of any open subset in* Q *of* $\varphi: W \longrightarrow Q$ *is open in* W*, then the function* φ *is called continuous function. That is, if* $x \in \iota$*, then* $\varphi^{-1}(x) \in \vartheta$ *is its inverse image.*

Definition 2.14 [8] *Let* (W,ϑ) *and* (Q,ι) *be any topological spaces. If the image* $\varphi(A)$ *is open in* Q *for each open set* A *in* W*, then the function* $\varphi: W \longrightarrow Q$ *is called open.*

Definition 2.15 [12] *Let* (W, ϑ) *and* (Q, ι) *be any topological spaces. If the image* $\varphi(F)$ *is closed in* Q *for each closed set* F *in* W*, then the function* $\varphi: W \longrightarrow Q$ *is called closed.*

Definition 2.16 [7] *If the function* $\varphi: (W, \vartheta) \longrightarrow (Q, \iota)$ *is continuous,* D *−compact and closed for each* $q \in Q$*, then the function* φ *is called* D *−perfect.*

Theorem 2.17 [18] *Any open cover is* D *−cover.*

3 New Results of D −Paracompact Spaces

This section presents the new notion of D −paracompact spaces, and explores their connections to other spaces, and introduces a new set of properties.

Definition 3.1 *A topological space* (W, ϑ) *is called* D *−paracompact if every* D *−cover of the space* (W, ϑ) *has an open locally−finite refinement.*

Theorem 3.2 *Every* D *−paracompact space* (W, ϑ) *is paracompact.*

Proof. *Suppose that* (W, ϑ) *is a* D *−paracompact space and* $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ *is an open−cover of* (W, ϑ)*. By theorem 2.17, the cover* $\tilde{E}$ *is a* D *−cover, and it has an open locally−finite refinement.*

Example 3.3 *Let* $(\mathbb{R}, \vartheta_u)$ *is a* D *−paracompact topological space. Then by applying the Theorem 3.2 we get that the space* $(\mathbb{R}, \vartheta_u)$ *is paracompact.*

The explanation for why the previous theorem's converse may not be true is illustrated by the example that follows.

Example 3.4 *The topological space* $(\mathbb{R}, \vartheta_{cof})$ *is paracompact but not a* D *−paracompact. This is because for all* $x \in \mathbb{R}$*, each set of the form* $\mathbb{R} - x$ *is open in a topological space* $(\mathbb{R}, \vartheta_{cof})$*. Now, for all* $a, b \in \mathbb{R}$*, if* $A = \mathbb{R} - \{a\}$ *and* $B = \mathbb{R} - \{b\}$ *be any two open sets in* $(\mathbb{R}, \vartheta_{cof})$*, then* $D = A - B = \{a\}$ *is a* D *−set but not an open set. This is because a* D *−cover* $\widetilde{D} = \{\{a\} : a \in \mathbb{R}\}$ *has not an open locally−finite refinement. Because, if* $\widetilde{D}$ *has an open locally−finite refinement* $\{\{a_1, a_2, \dots, a_n\}$*, then we have* $\mathbb{R} \subseteq \cup_{i=1}^{n} a_i$*, which implies that* $\mathbb{R}$ *is a finite set. Which is a contradiction.*

The following example illustrates the contrapositive of the above theorem.

Example 3.5 *The topological space* $(\mathbb{R}, \vartheta_{l.r})$ *is not paracompact, which is not* D *−paracompact space.*

The next theorem has the purpose to indicate that, in under conditions, the converses of the theorem above could be held.

Theorem 3.6 *Every topological space* (W, ϑ) *that is a locally−indiscrete paracompact is* D *−paracompact.*

Proof. *Let* $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ *be any* D *−cover of* (W, ϑ)*. Then* $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ *is an open cover, which has an open locally−finite refinement. Hence, the result.*

Example 3.7 ***(i)*** *Notice that the space* $(\mathbb{R}, \vartheta_{ind})$ *is locally−indiscrete and paracompact, then it is* D *−paracompact space.*
(ii) *Let* $W = \mathbb{R}$ *and* $\vartheta = \{\phi, \mathbb{R}, \mathbb{R} - \{5\}, \{5\}\}$*. Then* (W, ϑ) *is locally−indiscrete paracompact space. Thus, it must be* D *−paracompact.*

Theorem 3.8 *Given that* $A \subseteq W$ *of the topological space* (W, ϑ)*, then* (A, ϑ_A) *is* D *−paracompact if and only if any* D *−cover of* A *by* D *−sets in* W *has an open locally−finite refinement.*

Proof. $\Rightarrow$) *Let* (A, ϑ_A) *be* D *−paracompact space and* $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ *be any* D *−cover of* A *in* W*. Let* $E_\rho^* = E_\rho \cap A$*, for all* $\rho \in \Lambda$ *is a* D *−set in* A*. Then,* $\tilde{E}^* = \{E_\rho^* : \rho \in \Lambda\}$ *must be* D *−cover of* A *by* D *−sets in* A*. Since* (A, ϑ_A) *is* D *−paracompact space, then* $\tilde{E}^*$ *has an open locally−finite refinement says* $\{E_{\rho 1}^*, E_{\rho 2}^*, \dots, E_{\rho n}^*\}$*. Thus, the family* $\{E_{\rho 1}, E_{\rho 2}, \dots, E_{\rho n}\}$ *must be an open locally−finite refinement of* $\tilde{E}$ *in* W *for* A*, which* $E^* = E \cap A$*. Hence, we get the result.*
$\Leftarrow$) *Suppose that any* D *−cover of* A *by* D *−sets in* W *has an open locally−finite refinement. Let* $\tilde{A} = \{A_\rho : \rho \in \Lambda\}$ *is a $D-$cover of* A *by* D *−sets in* W*. In consequence, there is* D *−set of* E_ρ *in* W *such as* $A_\rho = E_\rho \cap A$ *for all* $\rho \in \Lambda$*. So,* $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ *where* $\tilde{E}$ *is a* D *−cover of* A *by* D *−sets in* W*. Based on the assumption that* $\tilde{E}$ *has an open locally−finite refinement* $\{E_{\rho 1}, E_{\rho 2}, \dots, E_{\rho n}\}$*. Since* $A_\rho \subseteq E_\rho$*, for all* $\rho \in \Lambda$*, then the family* $\{E_{\rho 1}, E_{\rho 2}, \dots, E_{\rho n}\}$ *must be an open locally−finite refinement of* $\tilde{A}$ *for* A*. Hence, the result.*

Observe the next corollaries such that each open cover is a D −cover.

Corollary 3.9 *If the space* (A, ϑ_A) *is* D *−paracompact, then any open cover of* A *by open sets in* W *has an open locally−finite refinement.*

Corollary 3.10 *A space* (A, ϑ_A) *is paracompact if any* D *−cover of* A *by* D *−sets in* W *has an open locally−finite refinement.*

Proof. *Since each* D *−paracompact space is paracompact, then the second part of Theorem 3.8 is this corollary's direct cause.*

Theorem 3.11 *Let* (W, ϑ_1) *and* (W, ϑ_2) *be any topological spaces. If* $\vartheta_1 \subseteq \vartheta_2$ *and* (W, ϑ_2) *is* D *−paracompact, then* (W, ϑ_1) *is a* D *−paracompact.*

Proof. *Let* $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ *be any* D *−cover of* (W, ϑ_1)*. Since* $\vartheta_1 \subseteq \vartheta_2$*, we have* $\tilde{E}$ *is* D *−cover of* (W, ϑ_2)*, which it is has an open locally−finite refinement. Hence, the result.*

Theorem 3.12 *If the topological space* (W, ϑ) *is* D *−paracompact, then there are closed, paracompact subspaces in* (W, ϑ).

Proof. *Let A be any closed subset of* W *and* W *be* D *−paracompact space. If* $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ *is an open cover of A by open sets in* W, *so* $\tilde{E} \cup \{W - A\}$ *must be open cover of* W. *Since* W *is a* D *−paracompact space, it has an open locally−finite refinement of* $\tilde{E}^*$. *Moreover,* $\tilde{E}^* - \{W - A\}$ *is an open locally−finite refinement of* $\tilde{E}$ *for A. Hence, we get the result.*

Theorem 3.13 *Any* D *−separable,* D *−paracompact* (W, ϑ) *must be* D *−lindelöf.*

Proof. *Let* $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ *be any* D *−cover of* W. *Believe that* $\tilde{E}$ *has no a countable subcover. Let* $\widetilde{K} = \{K_\varepsilon : \varepsilon \in \Upsilon\}$ *be an open uncountably locally−finite refinement subcover of* $\tilde{E}$. *Now, if D is a countable* D *−dense subsets of* W, *then* $K_\varepsilon \cap D \neq \emptyset$, *for all* $\varepsilon \in \Upsilon$. *Thus, the set D is uncountable because* $\widetilde{K}$ *is uncountable, that is a contradiction. Hence, we get the result.*

With the same work, we can achieve the next corollary.

Corollary 3.14 *Any* D *−separable,* D *−paracompact* (W, ϑ) *must be lindelöf.*

Definition 3.15 *The topological space* (W, ϑ) *is called countably* D *−paracompact if every countably* D *−cover of* (W, ϑ) *has an open locally−finite refinement.*

Theorem 3.16 *If any topological space* (W, ϑ) *is* D *−lindelöf and countably* D *−paracompact, then it must be* D *−paracompact.*

Proof. *Let* $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ *be any* D *−cover for* W *and let* W *is a* D *−lindelöf. Then* $\tilde{E}$ *has a countable subcover* $\widetilde{K} = \{K\}_{i=1}^{\infty}$. *Since* W *is countably* D *−paracompact, then* $\widetilde{K}$ *has an open locally−finite refinement* $\tilde{S}$ *of* $\tilde{E}$. *Thus, the space* (W, ϑ) *must be* D *−paracompact.*

With the same work, we can achieve the next corollaries.

Corollary 3.17 *If any topological space* (W, ϑ) *is* D *−lindelöf and countably* D *−paracompact, then it must be paracompact.*

Corollary 3.18 *If any topological space* (W, ϑ) *is lindelöf and countably* D *−paracompact, then it must be paracompact.*

Example 3.19 *Let the space* (W, ϑ_{dis}) *be a* D *−lindelöf and a countably* D *−paracompact space. Then by applying the Theorem 3.16, we get that* (W, ϑ_{dis}) *must be* D *−paracompact space.*

Theorem 3.20 *If any topological space* (W, ϑ) *is* D *−paralindelöf and countably* D *−paracompact, then it must be a* D *−paracompact space.*

Proof. *Let* $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ *be any* D *−cover for* W *and* W *be a* D *−paralindelöf space. Then* $\tilde{E}$ *has an open locally−countable refinement* $\widetilde{K} = \{K_{\rho i}\}_{i=1}^{\infty}$.*Now, since the topological space* W *is* *countably D −paracompact, then $\tilde{K}$ has an open locally−finite refinement $\tilde{S}$ of $\tilde{E}$. Thus, the space (W, ϑ) must be D −paracompact.*

The following corollaries can be obtained with the same work.

Corollary 3.21 *If any topological space (W, ϑ) is D −paralindelöf and countably D −paracompact, then it must be a paracompact space.*

Corollary 3.22 *If any topological space (W, ϑ) is paralindelöf and countably D −paracompact, then it must be a paracompact space.*

4 Product of D −Paracompact Topological Spaces

This section presents several main theoretical results regards the concepts of maps with the product for two D −paracompact spaces.

Theorem 4.1 *Let (W, ϑ) and (Q, ι) be any topological spaces. If $\varphi: W \longrightarrow Q$ is a D −perfect function, and W is a locally−indiscrete space, then W must be D −paracompact if the space Q is so.*

Proof. *Let $\tilde{E} = \{E_\rho: \rho \in \Lambda\}$ be any D −cover of W. Since φ is D −perfect, then for any $q \in Q$, we have $\varphi^{-1}(q)$ is D −compact subsets of W. So, there is a finite subset ι_q of Λ, such that $\varphi^{-1}(q) \cup_{\rho \in \iota_q} H_\rho$, and $\tilde{E}$ is an open cover of W. That is, $P_q = Q - \varphi\left(W - \cup_{\rho \in \iota_q} H_\rho\right)$ is D −open subsets of Q and $\varphi^{-1}(P_q) \cup_{\rho \in \iota_q} H_\rho$, for any $q \in P_q$, therefore $\tilde{P} = \{P_q: q \in Q\}$ is open D −cover of Q. Since Q is D −paracompact, then we get that $\tilde{P}$ has an open locally−finite refinement $\tilde{P}^* = \{P_q^*: q \in Q\}$. Thus, the set P_q^* is D −open subsets of W. Since the function φ is D −perfect, that is means $\{\varphi^{(-1)}(P_q^*): q \in Q\}$ is an open locally−finite refinement of W. Hence, we get the result that W is D −paracompact space.*

Theorem 4.2 *Let $\varphi: W \longrightarrow Q$ as D −perfect function. Then, W is paracompact space if the space Q is D −paracompact.*

Proof. *As the proof of the above Theorem 4.1, this theory is also simply to prove.*

Theorem 4.3 *If $\varphi: W \longrightarrow Q$ be D −perfect, where Q is a countable and W is locally−indiscrete, then W must be countably D −paracompact if the space Q is so.*

Proof. *Let $\tilde{E} = \{E_\rho: \rho \in \Lambda\}$ be countable D −cover of W and φ be a D −perfect function. Then for any $q \in Q$, $\varphi^{-1}(q)$ is D −compact space subsets of W. It therefore obtains in a finite subset of Λ such as $\varphi^{-1}(q) \subseteq \cup_{i=1}^{n} A_i$. Since W is locally indiscrete, then A_i is D −open subset of W, for any $i \in \Lambda$. Presently, $B_q = Q - \varphi(W - \cup_{i=1}^{n} A_i)$ is D −set containing q. Moreover, $\varphi^{-1}(B_q) \subseteq \cup_{i=1}^{n} A_i$. Thus, $\tilde{B} = \{B_q: q \in Q\}$ presents countable D −cover for Q. Now, since Q is countably D −paracompact, then $\tilde{B}$ has an open locally−finite refinement $\tilde{B}^* =$*

$\{B_{q1}, B_{q2}, \dots, B_{qn}\}$ *and so* B_q^* *is* D *−open subset of* W*. Also, since* φ *is* D *−perfect, that is means* $\{\varphi^{-1}(B_q^*) : q \in Q\}$ *is an open locally−finite refinement of* W*. Thus, the space* W *must be countably* D *−paracompact.*

Theorem 4.4 *Let* (W, ϑ) *and* (Q, ι) *be topological spaces, such that* W *is* D *−compact and* Q *is* D *−paracompact spaces. Then* $W \times Q$ *must be* D *−paracompact space.*

Proof. *Given the truth that the projection function* $T: W \times Q \longrightarrow Q$ *is continuous and* $T^{-1}\{q\} = W \times \{q\} \simeq W$ *is* D *−compact, for any* $q \in Q$*. That is means* $T: W \times Q \longrightarrow Q$ *is* D *−perfect function. So, since* Q *is* D *−paracompact space, then* $W \times Q$ *must be also* D *−paracompact.*

Theorem 4.5 *Let* (W, ϑ) *and* (Q, ι) *be topological spaces. If* W *is paracompact, and* Q *is* D *−paracompact spaces, then the projection function* $T: (W \times Q, \vartheta \times \iota) \longrightarrow (Q, \iota)$ *must be closed.*

Proof. *Let* (W, ϑ) *be paracompact and* (Q, ι) *be a* D *−paracompact space. Then* $(W \times Q, \vartheta \times \iota)$ *is* D *−paracompact, therefore the projection function* $T: (W \times Q, \vartheta \times \iota) \longrightarrow (Q, \iota)$ *must be closed function.*

Theorem 4.6 *Let* $\varphi: (W, \vartheta) \longrightarrow (Q, \iota)$ *be a closed, continuous, and onto function such that* Q *is locally−indiscrete space. Then,* Q *is* D *−paracompact if the space* W *is so.*

Proof. *Let* $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ *be any* D *−cover of* Q*. Since* φ *is onto function and continuous, then* $\tilde{E} = \{\varphi^{-1}(E_\rho) : \rho \in \Lambda\}$ *is an open cover of* W*. Since* W *is* D *−paracompact, then there is an open locally−finite refinement of* $\tilde{E}$ *as* $\tilde{E}^* = \{\varphi^{-1}(E_\rho^*) : \rho \in \Lambda\}$*. Thus,* Q *must be a* D *−paracompact space.*

With the same work, the corollary that follows can be achieved.

Corollary 4.7 *Let* $\varphi: (W, \vartheta) \longrightarrow (Q, \iota)$ *be a closed, continuous, and onto function such that* Q *is locally−indiscrete space. Then,* Q *is* D *−paracompact if the space* W *is paracompact.*

5 Conclusions

This research highlights that D −paracompact topological spaces have a key topological characteristic. Their flexibility to provide D −covers with locally−finite refinements is demonstrated by their conclusion. The research introduced several new properties and explained examples that relate to such superimposed spaces. The study's additional objective was to draw attention to some of the new notions and characteristics of D −paracompact spaces, as well as some properties of the Cartesian multiplication of such spaces under special conditions. Furthermore, certain illustrative examples and the main characteristics of these concepts were carefully studied. We identified their principal characteristics. We talked about their main traits and demonstrated how they work together. The study also highlighted these spaces' characteristics and offered numerous instances of them. These spaces will serve as a springboard for research into the various futures that these spaces may have. In the future, these spaces might be generalized to bitopological and tritopological spaces.

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