



Fermatean Neutrosophic Rough Set

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Abstract

In order to balance uncertainty, a reasonable approximation of a crisp set that yields lower as well as upper approximations of the set are done. Here, first, a special class of Fermatean Neutrosophic Set by associating the Neutrosophic Set and Fermatean Fuzzy Set is devised. A relatively new concept of Fermatean Neutrosophic Approximation Space and Fermatean Neutrosophic approximation operators are introduced. In this context, a new class of Fermatean Neutrosophic Rough set is established and a few of its characteristics are mentioned. Also, the cut sets of Fermatean Neutrosophic Rough sets, which characterize Fermatean Neutrosophic rough approximation operators are investigated.

Keywords: Fermatean Neutrosophic Set (FNS); Fermatean Neutrosophic Approximation Space (FNAS); Fermatean Neutrosophic Rough Set (FNRS); Fermatean Neutrosophic relation (FN-r).

1. Introduction

Rough set theory is a mathematical approach used for dealing with uncertainty and vagueness in data analysis and decision-making. It was introduced by Polish mathematician Zdzisław Pawlak [1] in the early 1980s as a tool for analyzing imprecise, incomplete, or uncertain information. The mathematical structure of rough sets comprises an equivalence couple of root elements: the crisp entries and an equivalence algebraic relation. It extends to the features that have distinct values. Therefore, before being employed, continuous-valued attributes must be discretized. Forming equivalence classes within the presented data sets builds the rough set

approximations [2]. All data attribute values that form an equivalence class are indistinct, meaning the units are related to the characteristics that describe the data.

In 2015, C. Antony Crispin Sweety and I. Arockiarani [3] introduced a novel approximation of neutrosophic sets and discussed the relations based on rough approximations and neutrosophic sets. The work provided a rough approximation of neutrosophic sets. They also introduced rough, fuzzy neutrosophic sublattices, ideals, and other concepts.

L. A. Zadeh introduced the concept of fuzzification in 1965 [4]. Atanassov proposed an alternative intuitionistic set [5], which defines an element with the membership and non-membership. From a logical perspective, the neutrosophic logic established by Florentin Smarandache [6-7] is equipped to talk about uncertainty, imperfections, and missing and inconsistent information in the actual world.

Said Broumi and Florentin Smarandache established some similarity measurements in neutrosophic sets [8]. In 2013, Yager proposed a novel perception called Pythagorean fuzzy sets [9], which are constrained to have the square sum of their grades as less than or equal to one.

Numerous methods and strategies have been suggested to address the unpredictable nature of real-time situations. Neutrosophic sets and their combinations are one approach for handling imprecision. With a broader range of applications, these sets generalize intuitionistic fuzzy sets, fuzzy sets, and their extensions, inspiring the development of several theories. The notion of Fermatean Neutrosophic Sets (FNS) and its algebraic features, as well as set theory, were first presented by Antony Crispin Sweety in 2021 [10].

Fermatean neutrosophic graphs are defined by Broumi, S., Sundareswaran, R., Shanmugapriya, M., Bakali, A., & Talea, M. [11] Additionally, several operations on Fermatean neutrosophic graphs are shown, and the notions of regular Fermatean neutrosophic graphs are introduced.[12]

Pythagorean neutrosophic matrices, Fermatean fuzzy matrices, and intuitionistic fuzzy matrices are generalizations of a specific instance of neutrosophic matrices defined by Broumi [13] called Fermatean neutrosophic matrices.

Fermatean fuzzy sets are suggested by Senapati, T., and Yager, R. R. [14] Fermatean fuzzy sets are contrasted with Pythagorean and intuitionistic fuzzy sets. Furthermore, pay attention to the Fermatean fuzzy sets' complement operator. Subsequently, develop a Fermatean fuzzy TOPSIS approach with an example to solve the multiple criterion choice task.

Introduce the Interval-Valued Fermatean Neutrosophic Set (IVFNS) in [15], which addresses partial ignorance for multi-decision processes separately in true, false, or unsure regions. Additionally, an example has been provided to illustrate the algebraic characteristics and set theory between the interval-valued fuzzy neutrosophic [16-17].

In this frame, the concept of indiscernibility or indistinguishability is imposed on FNSs. The concepts of rough sets and FNSs are combined, and a new class of rough sets in FNAS is introduced.

The structure of the paper is as follows:

- (i) A FNS is defined as the truth, falsity, and indeterminacy membership function satisfies.
- (ii) Fermatean neutrosophic rough relations and the FNAS are defined. The notion of FNRSs is discussed. Further, Fermatean Neutrosophic Rough Approximation Operators (FNRAO) and the properties of FNRS s are studied.

2. Preliminaries

Definition 2.1. [9]

Let X be a non-empty set, and $I \in [0,1]$. A Pythagorean fuzzy set A is of the form $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle | x \in X \}$ where the degree of membership function is $\mu_A : X \rightarrow [0,1]$ and the degree of non-membership function is $\nu_A : X \rightarrow [0,1]$ for each element $x \in X$ to the set A , and $(\mu_A(x))^2 + (\nu_A(x))^2 \leq 1$ for each $x \in X$.

Definition 2.2. [6]

Let X be a non-empty set (universe). A is a neutrosophic set on X is of the form:

$$A = \{ \langle x, T_A(x), I_A(x), F_A(x) \rangle | x \in X \}$$

Where $T_A(x), I_A(x), F_A(x) \in [0,1]$, $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 2$, for every x in X . $T_A(x)$ is the degree of membership, $I_A(x)$ is the degree of indeterminacy and $F_A(x)$ is the degree of non-membership. Here $T_A(x)$ and $F_A(x)$ are dependent components and $I_A(x)$ is an independent component.

Definition 2.3. [18]

Consider X be a set that is not empty (universe). FNS with T and F are dependent neutrosophic components A on X is an object of the form

$$A = \{ \langle x, T_A(x), I_A(x), F_A(x) \rangle | x \in X \}$$

Where $T_A(x), I_A(x), F_A(x) \in [0,1]$, $0 \leq (T_A(x))^2 + (I_A(x))^2 + (F_A(x))^2 \leq 2$, for every $x \in X$ The degree of membership is $T_A(x)$, the degree of indeterminacy is $I_A(x)$ and the degree of non-

membership is $F_A(x)$. Here $T_A(x)$ and $F_A(x)$ are dependent components and $I_A(x)$ is an independent.

3. Fermatean Neutrosophic Rough Set

In this section, we introduce Fermatean neutrosophic approximation space and Fermatean neutrosophic approximation operators induced from the same. Further we define a new type of set called Fermatean neutrosophic rough set and investigate some of its properties.

Definition 3.1

Let a non-empty set P be a universe. A FNS in A_p on P is of the form

$$A_p = \left\{ \langle p : T_{A_p}(p), I_{A_p}(p), F_{A_p}(p) \rangle / p \in P \right\}$$

where T_A is truth, I_A is indeterminate, F_A is false degree membership. where

$$T_{A_p}(x), I_{A_p}(x), F_{A_p}(x) | x \in X \rightarrow [0,1], 0 \leq [T_{A_p}(p)]^3 + [I_{A_p}(p)]^3 + [F_{A_p}(p)]^3 \leq 2.$$

Definition 3.2

A PN-r on P_N is a Fermatean neutrosophic subset

$$R_{P_N} = \left\{ \langle p, q \rangle, T_{R_{P_N}}(p, q), I_{R_{P_N}}(p, q), F_{R_{P_N}}(p, q) / p, q \in P_N S \right\}$$

$$T_{R_{P_N}} : P_N \times P_N \rightarrow [0,1]; I_{R_{P_N}} : P_N \times P_N \rightarrow [0,1]; F_{R_{P_N}} : P_N \times P_N \rightarrow [0,1]$$

satisfies

$$0 \leq T_{R_{P_N}}^3(p, q) + I_{R_{P_N}}^3(p, q) + F_{R_{P_N}}^3(p, q) \leq 2 \text{ for all } (x, y) \in P_N \times P_N.$$

then the family of all PN-r on P_N by $P(P_N \times P_N)$

Definition 3.3 A FN-r on E_p is a Fermatean neutrosophic subset

$$R_{E_p} = \left\{ \langle p, q \rangle, T_{R_{E_p}}(p, q), I_{R_{E_p}}(p, q), F_{R_{E_p}}(p, q) / p, q \in E_p \right\}$$

$$T_{R_{E_p}} : E_p \times E_p \rightarrow [0,1]; I_{R_{E_p}} : E_p \times E_p \rightarrow [0,1]; F_{R_{E_p}} : E_p \times E_p \rightarrow [0,1]$$

satisfies

$$0 \leq T_{R_{EP}}^3(p, q) + I_{R_{EP}}^3(p, q) + F_{R_{EP}}^3(p, q) \leq 2 \text{ for all } (p, q) \in E_p \times E_p.$$

then the family of all FN-r on E_p by $P_N(E_p \times E_p)$.

Definition 3.4 consider a not empty set P_N . An arbitrary PN-r R_{P_N} (Satisfying Definition 3.2 or 3.3) over $(P_N \times P_N)$ then the pair (P_N, R_{P_N}) is called FNAS. For any $M \in P(P_N)$, define the upper and lower approximations with respect to (P_N, R_{P_N}) , denoted by $\underline{R}_{P_N}(M)$ and $\overline{R}_{P_N}(M)$ respectively.

$$\underline{R}_{P_N}(M) = \left\{ \left\langle m, T_{\underline{R}_{P_N}(M)}(m), I_{\underline{R}_{P_N}(M)}(m), F_{\underline{R}_{P_N}(M)}(m) \right\rangle / m \in P_N \right\}$$

$$\overline{R}_{P_N}(M) = \left\{ \left\langle x, T_{\overline{R}_{P_N}(M)}(m), I_{\overline{R}_{P_N}(M)}(m), F_{\overline{R}_{P_N}(M)}(m) \right\rangle / m \in P_N \right\}$$

where,

$$T_{\underline{R}_{P_N}(M)}(m) = \bigwedge_{n \in P_N} \left[F_{R_{P_N}}(m, n) \vee T_M(n) \right]$$

$$I_{\underline{R}_{P_N}(M)}(m) = \bigwedge_{n \in P_N} \left[1 - I_{R_{P_N}}(m, n) \vee I_M(n) \right]$$

$$F_{\underline{R}_{P_N}(M)}(m) = \bigvee_{n \in P_N} \left[T_{R_{P_N}}(m, n) \wedge F_M(n) \right]$$

$$T_{\overline{R}_{P_N}(M)}(m) = \bigvee_{n \in P_N} \left[T_{R_{P_N}}(m, n) \wedge T_M(n) \right]$$

$$I_{\overline{R}_{P_N}(M)}(m) = \bigvee_{n \in P_N} \left[I_{R_{P_N}}(m, n) \wedge I_M(n) \right]$$

$$F_{\overline{R}_{P_N}(M)}(m) = \bigwedge_{n \in P_N} \left[F_{R_{P_N}}(m, n) \vee F_M(n) \right]$$

The pair $(\underline{R}_{P_N}(M), \overline{R}_{P_N}(M))$ is called FNRS of M with respect to (P_N, R_{P_N}) and $\underline{R}_{P_N}, \overline{R}_{P_N} : P(P_N) \rightarrow P(P_N)$ are referred to as upper and lower FNRA operators respectively.

Remark 3.4 If R_{P_N} is a Fermatean fuzzy relation on P_N then (P_N, R_{P_N}) is a Fermatean fuzzy approximation space, Fermatean neutrosophic rough operators are induced from a Fermatean fuzzy approximation space that is

$$\underline{R}_{P_N}(M) = \left\{ \left\langle m, T_{\underline{R}_{P_N}(M)}(m), I_{\underline{R}_{P_N}(M)}(m), F_{\underline{R}_{P_N}(M)}(m) \right\rangle / m \in P_N \right\} M \in P(P_N)$$

$$\overline{R}_{P_N}(M) = \left\{ \left\langle m, T_{\overline{R}_{P_N}(M)}(m), I_{\overline{R}_{P_N}(M)}(m), F_{\overline{R}_{P_N}(M)}(m) \right\rangle / m \in P_N \right\} M \in P(P_N)$$

Where

$$T_{\underline{R}_{P_N}(M)}(m) = \bigwedge_{n \in P_N} \left[\gamma_{R_{P_N}}(m, n) \vee T_M(n) \right]$$

$$I_{\underline{R}_{P_N}(M)}(m) = \bigwedge_{n \in P_N} \left[\left(\mu_{R_{P_N}}(m, n) + \gamma_{R_{P_N}}(m, n) \right) \vee I_M(n) \right]$$

$$F_{\underline{R}_{P_N}(M)}(m) = \bigvee_{n \in P_N} \left[\mu_{R_{P_N}}(m, n) \wedge F_M(n) \right]$$

$$T_{\overline{R}_{P_N}(M)}(m) = \bigvee_{n \in P_N} \left[\mu_{R_{P_N}}(m, n) \wedge T_M(n) \right]$$

$$I_{\overline{R}_{P_N}(M)}(m) = \bigvee_{n \in P_N} \left[1 - \left(\mu_{R_{P_N}}(m, n) + \nu_{R_{P_N}}(m, n) \right) \wedge I_M(n) \right]$$

$$F_{\overline{R}_{P_N}(M)}(m) = \bigwedge_{n \in P_N} \left[\gamma_{R_{P_N}}(m, n) \vee F_M(n) \right]$$

Remark 3.5 let R_{P_N} is a crisp binary relation on P_N and (P_N, R_{P_N}) is a crisp approximation space, then FNRAOs are induced from crisp approximation space, such that $X \in P(P_N)$

$$\underline{R}_{P_N}(M) = \left\{ \left\langle m, T_{\underline{R}_{P_N}(M)}(m), I_{\underline{R}_{P_N}(M)}(m), F_{\underline{R}_{P_N}(M)}(m) \right\rangle / m \in P_N \right\}$$

$$\overline{R}_{P_N}(M) = \left\{ \left\langle m, T_{\overline{R}_{P_N}(M)}(m), I_{\overline{R}_{P_N}(M)}(m), F_{\overline{R}_{P_N}(M)}(m) \right\rangle / m \in P_N \right\}$$

Where,

$$T_{\underline{R}_{P_N}(M)}(m) = \bigwedge_{n \in [m]_{R_{P_N}}} T_M(n) \quad I_{\underline{R}_{P_N}(M)}(m) = \bigwedge_{n \in [m]_{R_{P_N}}} I_M(n) \quad F_{\underline{R}_{P_N}(M)}(m) = \bigvee_{n \in [m]_{R_{P_N}}} F_M(n)$$

$$T_{\overline{R}_{P_N}(M)}(m) = \bigvee_{n \in [m]_{R_{P_N}}} T_M(n) \quad I_{\overline{R}_{P_N}(M)}(m) = \bigvee_{n \in [m]_{R_{P_N}}} I_M(n) \quad F_{\overline{R}_{P_N}(M)}(m) = \bigwedge_{n \in [m]_{R_{P_N}}} F_M(n)$$

Theorem 3.6 Let (P_N, R_{P_N}) be a FNAS. Then the lower and upper FNRAOs generated by

(P_N, R_{P_N}) meet the following conditions. $\forall A, B \in P(P_N), \forall \xi, \psi, \zeta \in [0, \sqrt{2}]$ with $\xi^2 + \psi^2 + \zeta^2 \leq 2$

$$\begin{aligned}
 (EL1) \underline{R}_{P_N}(A) &= \sim \underline{R}_{P_N}(\sim A) & (EU1) \overline{R}_{P_N}(A) &= \sim \overline{R}_{P_N}(\sim A) \\
 (EL2) \underline{R}_{P_N}(A \cup \overline{(\xi, \psi, \zeta)}) &= \underline{R}_{P_N}(A) \cup \overline{(\xi, \psi, \zeta)} & (EU2) \overline{R}_{P_N}(A \cap \overline{(\xi, \psi, \zeta)}) &= \overline{R}_{P_N}(A) \cap \overline{(\xi, \psi, \zeta)} \\
 (EL3) \underline{R}_{P_N}(A \cap B) &= \underline{R}(A) \cap \underline{R}(B) & (EU3) \overline{R}_{P_N}(A \cup B) &= \overline{R}_{P_N}(A) \cup \overline{R}_{P_N}(B) \\
 (EL4) A \subseteq B &\Rightarrow \underline{R}_{P_N}(A) \subseteq \underline{R}_{P_N}(B) & (EU4) A \subseteq B &\Rightarrow \overline{R}_{P_N}(A) \subseteq \overline{R}(B) \\
 (EL5) \underline{R}_{P_N}(A \cup B) &\supseteq \underline{R}_{P_N}(A) \cup \underline{R}_{P_N}(B) & (EU5) \overline{R}_{P_N}(A \cap B) &\subseteq \overline{R}_{P_N}(A) \cap \overline{R}_{P_N}(B) \\
 (EL6) R_{P_N}^1 \subseteq R_{P_N}^2 &\Rightarrow \underline{R}_{P_N}^1(A) \supseteq \underline{R}_{P_N}^2(A) & (EU6) R_{P_N}^1 \subseteq R_{P_N}^2 &\Rightarrow \overline{R}_{P_N}^1(A) \subseteq \overline{R}_{P_N}^2(A)
 \end{aligned}$$

Proof.

Only the conditions of lower FNRAO $\underline{R}_{P_N}(A)$ are shown. The upper FNRAO $\overline{R}_{P_N}(A)$ can be proven in a comparable way.

(EL1) By Definition 3.3,

$$\begin{aligned}
 \sim \underline{R}_{P_N}(\sim A) &= \left\{ \left\langle x, F_{\underline{R}_{P_N}(\sim A)}(x), 1 - I_{\underline{R}_{P_N}(\sim A)}(x), T_{\underline{R}_{P_N}(\sim A)}(x) \right\rangle / x \in P_N \right\} \\
 &= \left\{ \left\langle x, \bigvee_{y \in P_N} [T_{\underline{R}_{P_N}}(x, y) \wedge F_{(-A)}(y)], \bigwedge_{y \in P_N} [1 - I_{\underline{R}_{P_N}}(x, y) \vee I_{(-A)}(y)], \bigwedge_{y \in P_N} [F_{\underline{R}_{P_N}}(x, y) \vee T_{(-A)}(y)] \right\rangle \right\} \\
 &= \left\{ \left\langle x, \bigvee_{y \in P_N} [T_{\underline{R}_{P_N}}(x, y) \wedge T_{(A)}(y)], \bigvee_{y \in P_N} [I_{\underline{R}_{P_N}}(x, y) \wedge I_{(A)}(y)], \bigwedge_{y \in P_N} [F_{\underline{R}_{P_N}}(x, y) \vee T_{(A)}(y)] \right\rangle \right\} \\
 &= \left\{ \left\langle x, T_{\overline{R}_{P_N}(A)}(x), I_{\overline{R}_{P_N}(A)}(x), F_{\overline{R}_{P_N}(A)}(x) \right\rangle / x \in P_N \right\} = \overline{R}(A) \\
 (EL2) \underline{R}_{P_N}(A \cup B) &= \left\{ \left\langle x, T_{\underline{R}(A \cup B)}(x), I_{\underline{R}(A \cup B)}(x), F_{\underline{R}(A \cup B)}(x) \right\rangle / x \in P_N \right\} \\
 &= \left\{ \left\langle x, \bigwedge_{y \in P_N} T_{(A \cup B)}(y), \bigwedge_{y \in P_N} I_{(A \cup B)}(y), \bigvee_{y \in P_N} F_{(A \cup B)}(y) \right\rangle / x \in P_N \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= \left\{ \left\langle x, \bigwedge_{y \in P_N} [T_A(y) \wedge T_B(y)], \bigwedge_{y \in P_N} [I_A(y) \wedge I_B(y)], \bigvee_{y \in P_N} [F_A(y) \vee F_B(y)] \right\rangle / x \in P_N \right\} \\
 &= \left\{ \left\langle x, [T_{\underline{R}_{P_N}(A)}(x) \wedge T_{\underline{R}_{P_N}(B)}(x)], [I_{\underline{R}_{P_N}(A)}(x) \wedge I_{\underline{R}_{P_N}(B)}(x)], [F_{\underline{R}_{P_N}(A)}(x) \wedge F_{\underline{R}_{P_N}(B)}(x)] \right\rangle / x \in P_N \right\} \\
 &= \underline{R}(A) \cap \underline{R}(B).
 \end{aligned}$$

(EL3) By using the definition of $\underline{R}(A)$, it is simple to verify.

(EL4) It is easy to understand.

The properties of the upper FNRAO can also be demonstrated.

Remark 3.7

The properties (EL1) and (EU1) prove that FNRAOs \underline{R}_{P_N} and \overline{R}_{P_N} are complementary to one another, and the characteristics (EL2) and (EU2) imply, following characteristics (EL2)' and (EU2)'

$$(EL2)' \quad \underline{R}_{P_N}(P_N) = P_N$$

$$(EU2)' \quad \overline{R}_{P_N}(\varphi) = \varphi$$

Example 3.8 Let (P_N, R_{P_N}) be a FNRAO where $P_N = \{p_1, p_2, p_3\}$ and $R_{P_N} \in FNRA(P_N \times P_N)$ is defined as

$$R_{P_N} = \left\{ \left\langle (p_1, p_1)1.0, 0.6, 0.4 \right\rangle, \left\langle (p_1, p_2)0.9, 1, 0.4 \right\rangle, \left\langle (p_1, p_3)1.0, 0.7, 0.2 \right\rangle \right\} \\
 \left\{ \left\langle (p_2, p_1)1.0, 0.5, 0.1 \right\rangle, \left\langle (p_2, p_2)0.6, 0.8, 1 \right\rangle, \left\langle (p_2, p_3)0.9, 0.9, 0.6 \right\rangle \right\} \\
 \left\{ \left\langle (p_3, p_1)0.7, 1.0, 0.2 \right\rangle, \left\langle (p_3, p_2)0.8, 0.6, 1 \right\rangle, \left\langle (p_3, p_3)1.0, 0.7, 0.5 \right\rangle \right\}$$

If a FNS

$$X = \left\{ \langle p_1, 0.7, 0.9, 0.8 \rangle, \langle p_2, 0.8, 1, 0.6 \rangle, \langle p_3, 1.3, 0.4, 0.3 \rangle \right\}$$

Then,

$$\overline{R}_{P_N} = \left\{ \langle p_1, 1.0, 1.0, 0.3 \rangle, \langle p_2, 0.9, 0.8, 0.6 \rangle, \langle p_3, 1.0, 0.9, 0.5 \rangle \right\}$$

$$\underline{R}_{P_N} = \{ \langle p_1, 0.7, 0.7, 0.8 \rangle, \langle p_2, 0.7, 0.5, 0.8 \rangle, \langle p_3, 0.7, 0.7, 0.7 \rangle \}$$

upper and lower approximations of X respectively.

Definition 3.9 Let $X \in P(P_N)$ and $\xi, \psi, \zeta \in [0, 1]$ with $\xi^3 + \psi^3 + \zeta^3 \leq 2$ and (ξ, ψ, ζ) level set of X denoted by $X^{(\xi, \psi, \zeta)}$ is defined as

$$X^{(\xi, \psi, \zeta)} = \{ x \in P_N / T_X(x) \geq \xi, I_X(x) \geq \psi, F_X(x) \leq \zeta \}$$

We define

$$X_\xi = \{ x \in P_N / T_X(x) \geq \xi \} \text{ and } X_{\xi^+} = \{ x \in P_N / T_X(x) > \xi \} \text{ the } \xi \text{ level cut and}$$

strong ξ level cut of truth value function generated by A and

$$X_\psi = \{ x \in P_N / I_X(x) \geq \psi \} \text{ and } X_{\psi^+} = \{ x \in P_N / I_X(x) > \psi \} \text{ the } \psi \text{ level cut and}$$

strong ψ level cut of truth value function generated by A and

$$X_\zeta = \{ x \in P_N / F_X(x) \leq \zeta \} \text{ and } X^{\zeta^+} = \{ x \in P_N / F_X(x) < \zeta \} \text{ the } \zeta \text{ level cut and}$$

strong ζ level cut of truth value function generated by X .

Similarly, we can define the level cuts sets, such as

$$X^{(\xi^+, \psi^+, \zeta^+)} = \{ x \in P_N / T_X(x) > \xi, I_X(x) > \psi, F_X(x) < \zeta \} \text{ is } (\xi^+, \psi^+, \zeta^+),$$

$$X^{(\xi^+, \psi, \zeta)} = \{ x \in P_N / T_X(x) > \xi, I_X(x) \geq \psi, F_X(x) \leq \zeta \} \text{ is } (\xi^+, \psi, \zeta)$$

$$X^{(\xi, \psi^+, \zeta)} = \{ x \in P_N / T_X(x) \geq \xi, I_X(x) > \psi, F_X(x) \leq \zeta \} \text{ is } (\xi, \psi^+, \zeta)$$

$$X^{(\xi, \psi, \zeta^+)} = \{ x \in P_N / T_X(x) \geq \xi, I_X(x) \geq \psi, F_X(x) < \zeta \} \text{ is } (\xi, \psi, \zeta^+)$$

level cut set of X respectively. Likewise, other level cuts can also be defined.

Theorem 3.10 The level cut sets of FNSs meet the following criteria: $\forall A, B \in P(P_N)$, $\xi, \psi, \zeta \in [0, 1]$ with $\xi^3 + \psi^3 + \zeta^3 \leq 2$, $\xi_1, \psi_1, \zeta_1 \in [0, 1]$ with $\xi_1^3 + \psi_1^3 + \zeta_1^3 \leq 2$ and $\xi_2, \psi_2, \zeta_2 \in [0, 1]$ with $\xi_2^3 + \psi_2^3 + \zeta_2^3 \leq 2$.

$$(1) A^{(\xi, \psi, \zeta)} = A_{\xi} \cap A_{\psi} \cap A^{\zeta}$$

$$(2) (A')_{\xi} = A'^{\xi+} : (A')_{\psi} = A'(1-\psi+); (A')^{\zeta} = A'_{\zeta+}$$

$$(3) \left(\bigcap_{t \in J} A_t \right)_{\xi} = \bigcap_{t \in J} (A_t)_{\xi}$$

$$\left(\bigcap_{t \in J} A_t \right)_{\psi} = \bigcap_{t \in J} (A_t)_{\psi}$$

$$\left(\bigcap_{t \in J} A_t \right)^{\zeta} = \bigcap_{t \in J} (A_t)^{\zeta}$$

$$(4) \left(\bigcup_{t \in J} A_t \right)_{\xi} = \bigcup_{t \in J} (A_t)_{\xi}$$

$$\left(\bigcup_{t \in J} A_t \right)_{\psi} = \bigcup_{t \in J} (A_t)_{\psi}$$

$$\left(\bigcup_{t \in J} A_t \right)^{\zeta} = \bigcup_{t \in J} (A_t)^{\zeta}$$

$$(5) \left(\bigcup_{t \in J} A_t \right)^{(\xi, \psi, \zeta)} \supseteq \bigcup_{t \in J} (A_t)^{(\xi, \psi, \zeta)}$$

$$(6) \left(\bigcap_{t \in J} A_t \right)^{(\xi, \psi, \zeta)} \supseteq \bigcap_{t \in J} (A_t)^{(\xi, \psi, \zeta)}$$

$$(7) \text{ For } \xi_1 \geq \xi_2 \quad \psi_1 \geq \psi_2 \quad \zeta_1 \geq \zeta_2$$

$$A_{\xi_1} \subseteq A_{\xi_2}; A_{\psi_1} \subseteq A_{\psi_2}; A^{\zeta_1} \subseteq A^{\zeta_2}, A^{(\xi_1, \psi_1, \zeta_1)} \subseteq A^{(\xi_2, \psi_2, \zeta_2)}$$

Proof: (1) and (3) follow directly from Definition 3.9

$$(2) \text{ Since } A' = \{ \langle x, F_A(x), 1 - I_A(x), T_A(x) \rangle / x \in P_N \}$$

$$(A')_{\xi} = \{ x \in P_N / F_A(x) \geq \xi \}$$

By definition,

$$A'^{\xi+} = \{ x \in P_N / F_A(x) < \xi \}$$

$$A'^{\xi+} = \{ x \in P_N / F_A(x) \geq \xi \}$$

$$\Rightarrow (A)'_{\xi} = (A'^{\xi+})$$

Similarly, we can prove,

$$(A')\psi = (A'(1-\psi+)) \text{ and } (A')^{\zeta} = (A'_{\zeta+}).$$

$$(4) \bigcap_{t \in J} A_t = \left\{ \left\langle x, \bigwedge_{t \in J} T_{A_t}(x), \bigwedge_{t \in J} I_{A_t}(x), \bigvee_{t \in J} F_{A_t}(x) \right\rangle / x \in E_P \right\}$$

$$\text{We have } \left(\bigcap_{t \in J} A_t \right)_{\xi} = \left\{ x \in P_N / \bigwedge_{t \in J} T_{A_t}(x) \geq \xi \right\} = \left\{ x \in E_P / T_{A_t}(x) \geq \xi, \forall t \in J \right\} = \bigcap_{t \in J} (A_t)_{\xi}$$

Similarly,

$$\left(\bigcap_{t \in J} A_t \right) \psi = \left\{ x \in P_N / \bigwedge_{t \in J} I_{A_t}(x) \geq \psi \right\} = \left\{ x \in P_N / I_{A_t}(x) \geq \psi, \forall t \in J \right\} = \bigcap_{t \in J} (A_t) \psi$$

And

$$\left(\bigcap_{t \in J} A_t \right)^{\zeta} = \left\{ x \in P_N / \bigvee_{t \in J} F_{A_t}(x) \geq \zeta \right\} = \left\{ x \in P_N / F_{A_t}(x) \leq \zeta, \forall t \in J \right\} = \bigcap_{t \in J} (A_t)^{\zeta}$$

hence

$$\begin{aligned} \left(\bigcap_{t \in J} A_t \right)^{\xi, \psi, \zeta} &= \left(\bigcap_{t \in J} A_t \right)_{\xi} \cap \left(\bigcap_{t \in J} A_t \right) \psi \cap \left(\bigcap_{t \in J} A_t \right)^{\zeta} = \bigcap_{t \in J} \left((A_t)_{\xi} \cap (A_t) \psi \cap (A_t)^{\zeta} \right) \\ &= \bigcap_{t \in J} (A_t)^{(\xi, \psi, \zeta)} \end{aligned}$$

$$(5) \text{ Consider } \bigcup_{t \in J} (A_t) = \left\{ \left\langle x, \bigvee_{t \in J} T_{A_t}(x), \bigvee_{t \in J} I_{A_t}(x), \bigwedge_{t \in J} F_{A_t}(x) \right\rangle / x \in P_N \right\}$$

then

$$\left(\bigcup_{t \in J} A_t \right)_{\xi} = \left\{ x \in P_N / \bigvee_{t \in J} T_{A_t}(x) \geq \xi \right\} = \left\{ x \in P_N / T_{A_t}(x) \geq \xi, \forall t \in J \right\} = \bigcup_{t \in J} (A_t)_{\xi}$$

Similarly,

$$\left(\bigcup_{t \in J} A_t\right) \psi = \left\{x \in P_N / \bigvee_{t \in J} I_{A_t}(x) \geq \psi\right\} = \left\{x \in P_N / I_{A_t}(x) \geq \psi, \forall t \in J\right\} = \bigcup_{t \in J} (A_t) \psi$$

And

$$\left(\bigcup_{t \in J} A_t\right)^\zeta = \left\{x \in P_N / \bigwedge_{t \in J} F_{A_t}(x) \geq \zeta\right\} = \left\{x \in P_N / F_{A_t}(x) \leq \zeta, \forall t \in J\right\} = \bigcup_{t \in J} (A_t)^\zeta$$

then conclude

$$\begin{aligned} \left(\bigcup_{t \in J} A_t\right)^{\xi, \psi, \zeta} &= \left(\bigcup_{t \in J} A_t\right)_{\xi} \cup \left(\bigcup_{t \in J} A_t\right) \psi \cup \left(\bigcup_{t \in J} A_t\right)^\zeta = \bigcup_{t \in J} (A_t)_{\xi} \cup (A_t) \psi \cup (A_t)^\zeta \\ &= \bigcup_{t \in J} (A_t)^{(\xi, \psi, \zeta)} \end{aligned}$$

(6) For any $x \in A_{\xi_1}$, according to Definition 3.9 we have for $T_A(p) \geq \xi_1 \geq \xi_2$, we obtain $A_{\xi_1} \subseteq A_{\xi_2}$. Similarly, for $\psi_1 \geq \psi_2$ and $\zeta_1 \geq \zeta_2$ we obtain $A\psi_1 \subseteq A\psi_2$ and $A^{\zeta_1} \subseteq A^{\zeta_2}$. Hence, $A^{(\xi_1, \psi_1, \zeta_1)} \subseteq A^{(\xi_2, \psi_2, \zeta_2)}$.

Corollary 3.11 Assume that R_{P_N} is a FN-r in P_N ,

$$(R_{P_N})_{\xi} = \{(x, y) \in P_N \times P_N / T_{R_{P_N}}(x, y) \geq \xi\}, (R_{P_N})_{\xi}(x) = \{y \in P_N / T_{R_{P_N}}(x, y) \geq \xi\}$$

$$(R_{P_N})_{\xi^+} = \{(x, y) \in P_N \times P_N / T_{R_{P_N}}(x, y) > \xi\}, (R_{P_N})_{\xi^+}(x) = \{y \in P_N / T_{R_{P_N}}(x, y) > \xi\}$$

$$(R_{P_N})\psi = \{(x, y) \in P_N \times P_N / I_{R_{P_N}}(x, y) \geq \psi\}, (R_{P_N})\psi(x) = \{y \in P_N / I_{R_{P_N}}(x, y) \geq \psi\}$$

$$(R_{P_N})\psi^+ = \{(x, y) \in P_N \times P_N / I_{R_{P_N}}(x, y) > \psi\}, (R_{P_N})\psi^+(x) = \{y \in P_N / I_{R_{P_N}}(x, y) > \psi\}$$

$$(R_{P_N})^{\zeta} = \{(x, y) \in P_N \times P_N / F_{R_{P_N}}(x, y) \leq \zeta\}, (R_{P_N})^{\zeta}(x) = \{y \in P_N / F_{R_{P_N}}(x, y) \leq \zeta\}$$

$$(R_{P_N})^{\zeta^+} = \{(x, y) \in P_N \times P_N / F_{R_{P_N}}(x, y) < \zeta\}, (R_{P_N})^{\zeta^+}(x) = \{y \in P_N / F_{R_{P_N}}(x, y) < \zeta\}$$

$$(R_{P_N})^{(\xi, \psi, \zeta)} = \{(x, y) \in P_N \times P_N / T_{R_{P_N}}(x, y) \geq \xi, I_{R_{P_N}}(x, y) \geq \psi, F_{R_{P_N}}(x, y) \leq \zeta\}$$

$$(R_{P_N})^{(\xi, \psi, \zeta)}(x) = \{y \in P_N / T_{R_{P_N}}(x, y) \geq \xi, I_{R_{P_N}}(x, y) \geq \psi, F_{R_{P_N}}(x, y) \leq \zeta\}$$

Then for all $(R_{P_N})_{\xi}, (R_{P_N})_{\xi+}, (R_{P_N})_{\psi}, (R_{P_N})_{\psi+}, (R_{P_N})_{\zeta}, (R_{P_N})_{\zeta+}, (R_{P_N})^{(\xi\psi\zeta)}$ are crisp relation in P_N and

1) Let R_{P_N} is reflexive, then the above level cuts are reflexive.

2) If R_{P_N} is symmetric, then the above level cuts are symmetric.

3) If R_{P_N} is transitive, then the above level cuts are transitive.

Theorem 3.12 Consider (P_N, R_{P_N}) be a FNAS and $A \in P(P_N)$, then the upper FNAO can be shown as a few things below $\forall x \in P_N$.

$$\begin{aligned} 1) \quad T_{\bar{R}_{P_N}(A)}(x) &= \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\bar{R}_{P_N})_{\xi}(A_{\xi})(x) \right] = \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\bar{R}_{P_N})_{\xi}(A_{\xi+})(x) \right] \\ &= \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\bar{R}_{P_N})_{\xi+}(A_{\xi})(x) \right] = \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\bar{R}_{P_N})_{\xi+}(A_{\xi+})(x) \right] \\ 2) \quad I_{\bar{R}_{P_N}(A)}(x) &= \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\bar{R}_{P_N})_{\xi}(A_{\xi})(x) \right] = \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\bar{R}_{P_N})_{\xi}(A_{\xi+})(x) \right] \\ &= \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\bar{R}_{P_N})_{\xi+}(A_{\xi})(x) \right] = \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\bar{R}_{P_N})_{\xi+}(A_{\xi+})(x) \right] \\ 3) \quad F_{\bar{R}_{P_N}(A)}(x) &= \bigwedge_{\xi \in [0,1]} \left[\xi \vee (\bar{R}_{P_N})_{\xi}(A_{\xi})(x) \right] = \bigwedge_{\xi \in [0,1]} \left[\xi \vee (\bar{R}_{P_N})_{\xi}(A_{\xi+})(x) \right] \\ &= \bigwedge_{\xi \in [0,1]} \left[\xi \vee (\bar{R}_{P_N})_{\xi+}(A_{\xi})(x) \right] = \bigwedge_{\xi \in [0,1]} \left[\xi \vee (\bar{R}_{P_N})_{\xi+}(A_{\xi+})(x) \right] \end{aligned}$$

And more over for any $\xi \in [0,1]$

$$\begin{aligned} 4) \quad \left[\bar{R}_{P_N}(A) \right]_{\xi+} &\subseteq \bar{R}_{P_N}^{\xi+}(A_{\xi+}) \subseteq \bar{R}_{P_N}^{\xi}(A_{\xi}) \subseteq \left[\bar{R}_{P_N}(A) \right]_{\xi} \\ 5) \quad \left[\bar{R}_{P_N}(A) \right]_{\xi+} &\subseteq \bar{R}_{P_N}^{\xi+}(A_{\xi+}) \subseteq \bar{R}_{P_N}^{\xi}(A_{\xi}) \subseteq \left[\bar{R}_{P_N}(A) \right]_{\xi} \\ 6) \quad \left[\bar{R}_{P_N}(A) \right]_{\xi+} &\subseteq \bar{R}_{P_N}^{\xi+}(A_{\xi+}) \subseteq \bar{R}_{P_N}^{\xi}(A_{\xi}) \subseteq \left[\bar{R}_{P_N}(A) \right]_{\xi} \\ 7) \quad \left[\bar{R}_{P_N}(A) \right]_{\xi+} &\subseteq \bar{R}_{P_N}^{\xi+}(A_{\xi+}) \subseteq \bar{R}_{P_N}^{\xi}(A_{\xi}) \subseteq \left[\bar{R}_{P_N}(A) \right]_{\xi} \\ 8) \quad \left[\bar{R}_{P_N}(A) \right]_{\xi+} &\subseteq \bar{R}_{P_N}^{\xi+}(A_{\xi+}) \subseteq \bar{R}_{P_N}^{\xi}(A_{\xi}) \subseteq \left[\bar{R}_{P_N}(A) \right]_{\xi} \\ 9) \quad \left[\bar{R}_{P_N}(A) \right]_{\xi+} &\subseteq \bar{R}_{P_N}^{\xi+}(A_{\xi+}) \subseteq \bar{R}_{P_N}^{\xi}(A_{\xi}) \subseteq \left[\bar{R}_{P_N}(A) \right]_{\xi} \end{aligned}$$

Proof:

1) For $x \in P_N$, we have

$$\begin{aligned} & \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\overline{R}_{P_N})_{\xi} (A_{\xi})(x) \right] = \text{Sup} \left\{ \xi \in [0,1] / p \in (\overline{R}_{P_N})_{\xi} (A_{\xi}) \right\} \\ & = \text{Sup} \left\{ \xi \in [0,1] / (R_{P_N})_{\xi} (x) \cap A_{\xi} \neq \varnothing \right\} \\ & = \text{Sup} \left\{ \xi \in [0,1] / \exists y \in P_N \left(y \in (R_{P_N})_{\xi} (x), y \in A_{\xi} \right) \right\} \\ & = \text{Sup} \left\{ \xi \in [0,1] / \exists y \in P_N \left[T_{R_{P_N}} (x, y) \geq \xi, T_A (y) \geq \xi \right] \right\} \\ & = \bigvee_{y \in P_N} \left[T_{R_{P_N}} (x, y) \right] \wedge T_A (y) = T_{\overline{R}_{P_N}} (x) \end{aligned}$$

$$\begin{aligned} 2) & \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\overline{R}_{P_N})_{\xi} (A_{\xi})(x) \right] = \text{Sup} \left\{ \xi \in [0,1] / x \in (\overline{R}_{P_N})_{\xi} (A_{\xi}) \right\} \\ & = \text{Sup} \left\{ \xi \in [0,1] / (R_{P_N})_{\xi} (x) \cap A_{\xi} \neq \varnothing \right\} \\ & = \text{Sup} \left\{ \xi \in [0,1] / \exists y \in P_N \left(y \in (R_{P_N})_{\xi} (x), y \in A_{\xi} \right) \right\} \\ & = \text{Sup} \left\{ \xi \in [0,1] / \exists y \in P_N \left[I_{R_{P_N}} (x, y) \geq \xi, I_A (y) \geq \xi \right] \right\} \\ & = \bigvee_{y \in P_N} \left[I_{R_{P_N}} (x, y) \right] \wedge I_A (y) = I_{\overline{R}_{P_N}} (x) \end{aligned}$$

$$\begin{aligned} 3) & \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\overline{R}_{P_N})_{\xi} (A_{\xi})(x) \right] = \text{inf} \left\{ \xi \in [0,1] / x \in (\overline{R}_{P_N})_{\xi} (A_{\xi}) \right\} \\ & = \text{inf} \left\{ \xi \in [0,1] / (R_{P_N})_{\xi} (x) \cap A_{\xi} \neq \varnothing \right\} \\ & = \text{inf} \left\{ \xi \in [0,1] / \exists y \in P_N \left(y \in (R_{P_N})_{\xi} (x), y \in A_{\xi} \right) \right\} \\ & = \text{inf} \left\{ \xi \in [0,1] / \exists y \in P_N \left[F_{R_{P_N}} (x, y) \geq \xi, F_A (y) \geq \xi \right] \right\} \\ & = \bigwedge_{y \in P_N} \left[F_{R_{P_N}} (x, y) \right] \vee F_A (y) = F_{\overline{R}_{P_N}} (x) \end{aligned}$$

Similarly

$$\begin{aligned} T_{\overline{R}_{P_N}(A)} (x) & = \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\overline{R}_{P_N})_{\xi} (A_{\xi+})(x) \right] \\ & = \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\overline{R}_{P_N})_{\xi+} (A_{\xi})(x) \right] \\ & = \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\overline{R}_{P_N})_{\xi+} (A_{\xi+})(x) \right] \\ I_{\overline{R}_{P_N}(A)} (x) & = \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\overline{R}_{P_N})_{\xi} (A_{\xi+})(x) \right] \end{aligned}$$

$$\begin{aligned}
 &= \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\bar{R}_{P_N}) \xi + (A^\xi)(x) \right] \\
 &= \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\bar{R}_{P_N}) \xi + (A^\xi +)(x) \right] \\
 F_{\bar{R}_{P_N}(A)}^-(x) &= \bigwedge_{\xi \in [0,1]} \left[\xi \vee (\bar{R}_{P_N})^\xi (A^{\xi+})(x) \right] \\
 &= \bigwedge_{\xi \in [0,1]} \left[\xi \vee (\bar{R}_{P_N})^{\xi+} (A^\xi)(x) \right] \\
 &= \bigwedge_{\xi \in [0,1]} \left[\xi \vee (\bar{R}_{P_N})^{\xi+} (A^{\xi+})(x) \right]
 \end{aligned}$$

4) Since $\bar{R}_{P_N \xi+}(A_{\xi+}) \subseteq \bar{R}_{P_N \xi+}(A_\xi) \subseteq \bar{R}_{P_N \xi}(A_\xi)$ we prove only

$$\begin{aligned}
 \left[\bar{R}_{P_N}(A) \right]_{\xi+} &\subseteq \bar{R}_{P_N \xi+}(A_{\xi+}) \text{ and } \bar{R}_{P_N \xi}(A_\xi) \subseteq \left[\bar{R}_{P_N}(A) \right]_{\xi} \text{ for any} \\
 x \in \left[\bar{R}_{P_N}(A) \right]_{\xi+}, T_{\bar{R}_{P_N}(A)} &> \xi \\
 \Rightarrow \bigvee_{y \in P_N} \left[T_{R_{P_N}}(x, y) \wedge T_A(y) \right] &> \xi \exists y' \in P_N \ni \left[T_{R_{P_N}}(x, y') \wedge T_{R_{E_P}}(y'') \right] > \xi \\
 \Rightarrow y' \in R_{P_N \xi+}(x) \text{ and } y' \in A_{\xi+} &\Rightarrow R_{P_N \xi+}(x) \cap A_{\xi+} \neq \emptyset
 \end{aligned}$$

according to the definition of upper crisp approximation operator, $x \in \bar{R}_{P_N \xi+}(A_{\xi+})$. Hence $\left[\bar{R}_{P_N}(A) \right]_{\xi+} \subseteq \bar{R}_{P_N \xi+}(A_{\xi+})$

Next, to prove $\bar{R}_{P_N \xi}(A_\xi) \subseteq \left[\bar{R}_{P_N}(A) \right]_{\xi}$ for any $x \in \bar{R}_{P_N \xi}(A_\xi)$, $R_{P_N \xi}(A_\xi)(x) = 1$, if $\exists \psi$, then

$$\begin{aligned}
 T_{\bar{R}_{P_N}(A)}^-(x) &= \bigvee_{\psi \in [0, \sqrt{2}]} \left[\psi \wedge (\bar{R}_{P_N})_{\psi}(A_{\psi})(x) \right] \geq \xi \wedge (\bar{R}_{P_N})_{\psi}(A_{\psi})(x) = \xi. \text{ We obtain} \\
 x \in \left[\bar{R}_{P_N}(A) \right]_{\xi} \bar{R}_{P_N \xi}(A_\xi) &\subseteq \left[\bar{R}_{P_N}(A) \right]_{\xi}
 \end{aligned}$$

(5) Similar to (4) It is enough to prove $\left[\bar{R}_{P_N \xi+}(A^{\xi+}) \right]_{\xi} \subseteq \bar{R}_{P_N \xi+}(A^\xi) \subseteq \left[\bar{R}_{P_N \xi}(A^\xi) \right]$

Hence,

- i. $\left[\bar{R}_{P_N}(A) \right]_{\xi+} \subseteq \bar{R}_{P_N \xi+}(A^{\xi+})$
- ii. $\left[\bar{R}_{P_N \xi}(A^\xi) \right] \subseteq \left[\bar{R}_{P_N}(A) \right]_{\xi}$

i) For $x \in \left[\bar{R}_{P_N}(A) \right]_{\xi+}$, $I_{\bar{R}_{P_N}(A)}^-(x) > \xi \Rightarrow \bigvee_{y \in P_N} \left[I_{R_{P_N}}^-(x, y) \wedge I_A(y) \right] > \xi$,

$\exists y' \in P_N \ni I_{R_{P_N}}(x, y') \wedge I_A(y') > \xi$, (ie) $I_{R_{P_N}}(x, y') \xi$ and $I_A(y') \xi \Rightarrow y' \in R_{P_N \xi+}(x)$ and $q' \in A^{\xi+}$, $y' \in R(x) \cap A^{\xi+} \Rightarrow R^{\xi+}(x) \cap A^{\xi+} \neq \emptyset$.

according to the definition of crisp approximation operator, $x \in \bar{R}_{P_N} \xi + (A\xi +)$ therefore $[\bar{R}_{P_N}(A)]^{\xi+} \subseteq \bar{R}_{P_N} \xi + (A\xi +)$. Let $x \in [\bar{R}_{P_N} \xi(A\xi)], [\bar{R}_{P_N} \xi(A\xi)](x) = 1$.

If there exists ψ then $T_{\bar{R}_{P_N}(A)}(x) = \bigvee_{\psi \in [0, \sqrt{2}]} [\psi \wedge \bar{R}_{P_N} \psi(A\psi)(x)] \geq \xi \wedge \bar{R}_{P_N} \xi(A\xi)(x) = \xi$. We obtain $x \in [\bar{R}_{P_N}(A)]^{\xi}$ therefore $\bar{R}_{P_N} \xi(A\xi) \subseteq [\bar{R}_{P_N}(A)]^{\xi}$.

(6) Like with (4) and (5), the evidence of (6) only needs to be proven once

$$[\bar{R}_{P_N}(A)]^{\xi+} \subseteq \bar{R}_{P_N}^{\xi+}(A^{\xi+}) \text{ and } [\bar{R}_{P_N}^{\xi}(A^{\xi})] \subseteq [\bar{R}_{P_N}(A)]^{\xi}.$$

For any $x \in [\bar{R}_{P_N}(A)]^{\xi+}, F_{\bar{R}_{P_N}}(x) < \xi$ (ie) $\bigwedge_{y \in P_N} [F_{R_{P_N}}(x, y) \vee F_A(y)] < \xi$ and $\exists y' \in P_N \ni F_R(x, y') \vee F_A(y') < \xi$. Hence $F_{R_{P_N}}(x, y') < \xi, T_A(y') < \xi$ (ie) $y' \in R^{\xi+}(x)$ and $y' \in A^{\xi+}$. $R_{P_N}^{\xi+}(x) \cap A^{\xi+} \neq \emptyset$. therefore, $x \in \bar{R}_{P_N}^{\xi+}(A^{\xi+})$ and $[\bar{R}_{P_N}(A)]^{\xi+} \subseteq \bar{R}_{P_N}^{\xi+}(A^{\xi+})$.

Next for any $x \in \bar{R}_{P_N}^{\xi}(A^{\xi})$ note $\bar{R}_{P_N}^{\xi}(A^{\xi})(x) = 1$ then $F_{\bar{R}_{P_N}(A)}(x) = \bigwedge_{\psi \in [0, \sqrt{2}]} [\psi \vee \bar{R}^{\psi}(A^{\psi})(x)] \leq \xi \vee \bar{R}^{\xi}(A^{\xi})(x) = \xi$

Thus $x \in [\bar{R}_{P_N}(A)]^{\xi}$. Hence $\bar{R}_{P_N}^{\xi}(A^{\xi}) \subseteq [\bar{R}_{P_N}(A)]^{\xi}$. The proof of (7), (8), (9) can be obtained similar to (4), (5), (6).

Theorem 3.13 Let (P_N, R_{P_N}) be a FNAS and $A \in P(P_N)$ then $\forall x \in P_N$

- 1) $T_{\bar{R}_{P_N}(A)}(x) = \bigwedge_{\xi \in [0, 1]} [\xi \vee (1 - \underline{R}_{P_N}^{\xi})(A_{\xi+})(x)] = \bigwedge_{\xi \in [0, 1]} [\xi \vee (1 - R_{P_N})^{\xi}(A_{\xi})(x)]$
 $= \bigwedge_{\xi \in [0, 1]} [\xi \vee (1 - \underline{R}_{P_N})^{\xi+}(A_{\xi+})(x)] = \bigwedge_{\xi \in [0, 1]} [\xi \vee (1 - R_{P_N})^{\xi+}(A_{\xi})(x)]$
- 2) $I_{\bar{R}_{P_N}(A)}(x) = \bigwedge_{\xi \in [0, 1]} [\xi \vee (1 - \underline{R}_{P_N})(1 - \xi)(A^{\xi+})(x)] = \bigwedge_{\xi \in [0, 1]} [\xi \vee (1 - \bar{R}_{P_N})(1 - \xi)(A^{\xi})(x)]$
 $= \bigwedge_{\xi \in [0, 1]} [\xi \vee (1 - \underline{R}_{P_N})(1 - \xi+)(A^{\xi+})(x)] = \bigwedge_{\xi \in [0, 1]} [\xi \vee (1 - R_{P_N})(1 - \xi+)(A^{\xi})(x)]$
- 3) $F_{\bar{R}_{P_N}(A)}(x) = \bigvee_{\xi \in [0, 1]} [\xi \wedge (1 - \underline{R}_{P_N})(A^{\xi+})(x)] = \bigvee_{\xi \in [0, 1]} [\xi \wedge (1 - \bar{R}_{P_N})_{\xi}(A^{\xi})(x)]$
 $= \bigvee_{\xi \in [0, 1]} [\xi \wedge (1 - \underline{R}_{P_N})_{\xi+}(A^{\xi+})(x)] = \bigvee_{\xi \in [0, 1]} [\xi \wedge (1 - R_{P_N})_{\xi+}(A^{\xi})(x)]$

And more over for any $\xi \in [0, 1]$

- 4) $[\underline{R}_{P_N}(A)]_{\xi^+} \subseteq \underline{R}_{P_N}^{\xi}(A_{\xi^+}) \subseteq \underline{R}_{P_N}^{\xi^+}(A_{\xi^+}) \subseteq \underline{R}_{P_N}^{\xi^+}(A_{\xi}) \subseteq [\underline{R}_{P_N}(A)]_{\xi}$
- 5) $[\underline{R}_{P_N}(A)]_{\xi^+} \subseteq R_{P_N}(1-\xi)(A_{\xi^+}) \subseteq R_{P_N}(1-\xi^+)(A_{\xi^+}) \subseteq \bar{R}_{P_N}(1-\xi^+)(A_{\xi}) \subseteq [\underline{R}_{P_N}(A)]_{\xi}$
- 6) $[\underline{R}_{P_N}(A)]_{\xi^+}^{\xi^+} \subseteq \underline{R}_{P_N}(A_{\xi^+}) \subseteq \underline{R}_{P_N}^{\xi^+}(A_{\xi^+}) \subseteq \underline{R}_{P_N}^{\xi^+}(A_{\xi}) \subseteq [\underline{R}_{P_N}(A)]_{\xi}^{\xi^+}$
- 7) $[\underline{R}_{P_N}(A)]_{\xi} \subseteq \underline{R}_{P_N}^{\xi}(A_{\xi^+}) \subseteq \underline{R}_{P_N}^{\xi}(A_{\xi}) \subseteq \underline{R}_{P_N}^{\xi^+}(A_{\xi}) \subseteq [\underline{R}_{P_N}(A)]_{\xi}$
- 8) $[\underline{R}_{P_N}(A)]_{\xi^+} \subseteq R_{P_N}(1-\xi)(A_{\xi^+}) \subseteq R_{P_N}^{\xi}(A_{\xi}) \subseteq \bar{R}_{P_N}(1-\xi^+)(A_{\xi}) \subseteq [\underline{R}_{P_N}(A)]_{\xi}^{\xi^+}$
- 9) $[\underline{R}_{P_N}(A)]_{\xi^+}^{\xi^+} \subseteq \underline{R}_{P_N}(A_{\xi^+}) \subseteq \underline{R}_{P_N}^{\xi}(A_{\xi}) \subseteq \underline{R}_{P_N}^{\xi^+}(A_{\xi}) \subseteq [\underline{R}_{P_N}(A)]_{\xi}^{\xi^+}$

Proof. (1) and (2). For any $x \in P_N$ by the duality of upper and lower crisp approximation operators and in terms of Theorem, we have

$$T_{\bar{R}_{P_N}(A)}(x) = \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\bar{R}_{P_N})_{\xi}(A_{\xi})(x) \right]$$

$$I_{\bar{R}_{P_N}(A)}(x) = \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\bar{R}_{P_N})_{\xi}^{\xi}(A_{\xi})(x) \right]$$

$$F_{\bar{R}_{P_N}(A)}(x) = \bigwedge_{\xi \in [0,1]} \left[\xi \vee (\bar{R}_{P_N})_{\xi}^{\xi}(A_{\xi})(x) \right],$$

$$\text{then } T_{\bar{R}_{P_N}(-A)}(p) = \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\bar{R}_{P_N})_{\xi}(\sim A_{\xi})(x) \right]$$

$$= \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\bar{R}_{P_N})_{\xi}(\sim A_{\xi^+})(x) \right]$$

$$= \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\sim \underline{R}_{P_N})_{\xi}(A_{\xi^+})(x) \right]$$

$$= \bigvee_{\xi \in [0,1]} \left[\xi \wedge (1 - \underline{R}_{P_N})_{\xi^+}(\sim A_{\xi^+})(x) \right]$$

$$I_{\bar{R}_{P_N}(-A)}(x) = \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\bar{R}_{P_N})_{\xi}^{\xi}(\sim A_{\xi})(x) \right]$$

$$= \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\bar{R}_{P_N})_{\xi}^{\xi}(\sim A(1-\xi^+))(x) \right]$$

$$= \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\sim \underline{R}_{P_N})_{\xi}^{\xi^+}(A(1-\xi^+))(x) \right]$$

$$= \bigvee_{\xi \in [0,1]} \left[\xi \wedge (\sim \underline{R}_{P_N})_{\xi}^{\xi}(\sim A(1-\xi^+))(x) \right]$$

$$F_{\bar{R}_{P_N}(-A)}(x) = \bigwedge_{\xi \in [0,1]} \left[\xi \vee (\bar{R}_{P_N})_{\xi}^{\xi}(\sim A_{\xi})(x) \right]$$

$$\begin{aligned}
 &= \bigwedge_{\xi \in [0,1]} \left[\xi \vee (\bar{R}_{P_N})^\xi (\sim A_{\xi^+})(x) \right] \\
 &= \bigwedge_{\xi \in [0,1]} \left[\xi \vee (\underline{R}_{P_N})^\xi (A_{\xi^+})(x) \right] \\
 &= \bigwedge_{\xi \in [0,1]} \left[\xi \vee (\underline{R}_{P_N})^\xi (\sim A_{\xi^+})(x) \right]
 \end{aligned}$$

Thus, by fixing $\underline{R}_{P_N}(A) = \sim \bar{R}_{P_N}(\sim A)$,

$$\text{Hence } T_{\underline{R}_{P_N}(A)}(x) = T_{\bar{R}_{P_N}(\sim A)}(x) = \bigwedge_{\xi \in [0,1]} \left[\xi \vee (1 - \underline{R}_{P_N})_\xi (\sim A^{\xi^+})(x) \right]$$

$$I_{\underline{R}_{P_N}(A)}(x) = I_{\bar{R}_{P_N}(\sim A)}(x) = \bigwedge_{\xi \in [0,1]} \left[\xi \vee (1 - \underline{R}_{P_N})_\xi (\sim A(1 - \xi^+))(x) \right]$$

$$F_{\underline{R}_{P_N}(A)}(x) = F_{\bar{R}_{P_N}(\sim A)}(x) = \bigvee_{\xi \in [0,1]} \left[\xi \wedge (1 - \underline{R}_{P_N})^\xi (\sim A_{\xi^+})(x) \right]$$

Likewise

$$T_{\bar{R}_{P_N}(A)}(x) = T_{\underline{R}_{P_N}(\sim A)}(x) = \bigwedge_{\xi \in [0,1]} \left[\xi \vee (1 - \bar{R}_{P_N})_\xi (\sim A^{\xi^+})(x) \right] = T_{\underline{R}_{P_N}(\sim A)}(x)$$

$$= \bigwedge_{\xi \in [0,1]} \left[\xi \vee (1 - \bar{R}_{P_N})_{\xi^+} (\sim A^{\xi^+})(x) \right] = T_{\bar{R}_{P_N}(\sim A)}(x)$$

$$= \bigwedge_{\xi \in [0,1]} \left[\xi \vee (1 - \bar{R}_{P_N})_{\xi^+} (\sim A^\xi)(x) \right]$$

$$I_{\bar{R}_{P_N}(A)}(x) = I_{\underline{R}_{P_N}(\sim A)}(x) = \bigwedge_{\xi \in [0,1]} \left[\xi \vee (1 - \bar{R}_{P_N})_\xi (\sim A(1 - \xi))(x) \right] = I_{\bar{R}_{P_N}(\sim A)}(x)$$

$$= \bigwedge_{\xi \in [0,1]} \left[\xi \vee (1 - \bar{R}_{P_N})_\xi (\sim A(1 - \xi^+))(x) \right] = I_{\underline{R}_{P_N}(\sim A)}(x)$$

$$= \bigwedge_{\xi \in [0,1]} \left[\xi \vee (1 - \bar{R}_{P_N})_\xi (\sim A(1 - \xi))(x) \right]$$

$$F_{\bar{R}_{P_N}(A)}(x) = F_{\underline{R}_{P_N}(\sim A)}(x) = \bigvee_{\xi \in [0,1]} \left[\xi \wedge (1 - \bar{R}_{P_N})^\xi (\sim A_\xi)(x) \right] = F_{\bar{R}_{P_N}(\sim A)}(x)$$

$$= \bigvee_{\xi \in [0,1]} \left[\xi \wedge (1 - \bar{R}_{P_N})^{\xi^+} (\sim A_{\xi^+})(x) \right] = F_{\underline{R}_{P_N}(\sim A)}(x)$$

$$= \bigvee_{\xi \in [0,1]} \left[\xi \wedge (1 - \bar{R}_{P_N})^{\xi^+} (\sim A_\xi)(x) \right]$$

It is easy to prove that $\underline{R}_{P_N}^\xi(A_{\xi^+}) \subseteq \underline{R}_{P_N}^{\xi^+}(A_{\xi^+}) \subseteq \underline{R}_{P_N}^{\xi^+}(A_\xi)$. To show that $[\underline{R}_{P_N}(A)]_{\xi^+} \subseteq \underline{R}_{P_N}^{\xi^+}(A_{\xi^+})$, $\underline{R}_{P_N}^{\xi^+}(A_\xi) \subseteq [\underline{R}_{P_N}(A)]_\xi$ For any $x \in \underline{R}_{P_N}^\xi(A_{\xi^+})$,

$T_{\underline{R}_{P_N}(A)}(x) > \xi$ then we have $\bigwedge_{y \in P_N} [F_{\underline{R}_{P_N}}(x, y) \vee T_A(y)] > \xi$ then $[F_{\underline{R}_{P_N}}(x, y) \vee T_A(y)] > \xi$ for any $y \in P_N$, that is if $F_{\underline{R}_{P_N}}(x, y) \leq \xi$, then $T_A(y) > \xi$.

Alternatively, for any $y \in P_N$, if $y \in R_{P_N}^{\xi'}(x)$ then $y \in A_{\xi'}$. Therefore, $R^{\xi}(x) \subseteq A_{\xi'}$, then by the definition of lower approximation operator we have $x \in \underline{R}^{\xi}(A_{\xi'})$. Thus we conclude $[\underline{R}_{P_N}(A)]_{\xi'} \subseteq \underline{R}_{P_N}^{\xi}(A_{\xi'})$. Also, for any $x \in \underline{R}^{\xi}(A_{\xi'})$, we have $\underline{R}^{\xi}(A_{\xi'}) = 1$.

Then,

$$\begin{aligned} T_{R_{P_N}}(x) &= \bigwedge_{\xi' \in [0,1]} [\xi' \vee \underline{R}_{P_N}^{\xi'}(A_{\xi'})(x)] \\ &= \bigvee_{\xi' \in [0,1]} [\xi' \wedge \underline{R}_{P_N}^{\xi'}(A_{\xi'})(x)] \\ &\geq \xi \wedge \underline{R}_{P_N}^{\xi}(A_{\xi})(x) = \xi \end{aligned}$$

Hence $\xi \in [\underline{R}_{P_N}(A)]_{\xi}$ and $\underline{R}_{P_N}^{\xi}(A_{\xi}) \subseteq [\underline{R}_{P_N}(A)]_{\xi}$.

Similarly, we can prove (5) and (6) and hence (7), (8) and (9) can be concluded.

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