



On neutrosophic \mathcal{N} -deductive systems of Hilbert algebras

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Abstract

The notion of neutrosophic \mathcal{N} -deductive systems of Hilbert algebras is introduced, and several properties are investigated. Conditions for neutrosophic \mathcal{N} -structures to be neutrosophic \mathcal{N} -deductive systems of Hilbert algebras are provided. Relations between neutrosophic \mathcal{N} -deductive systems and their level subsets are considered. The Cartesian product of neutrosophic \mathcal{N} -structures is also supplied. Finally, we also find the property of the homomorphic pre-image of neutrosophic \mathcal{N} -deductive systems.

Keywords: Hilbert algebra; neutrosophic \mathcal{N} -structure; neutrosophic \mathcal{N} -deductive system; homomorphic pre-image.

1 Introduction

In 1965, Zadeh²⁰ proposed the degree of membership/truth (t) and defined the fuzzy set. In 1986, Atanassov¹ established the intuitionistic fuzzy set as an extension of fuzzy sets based on the degree of nonmembership/falsehood (f). Smarandache proposed the term “neutrosophic” because “neutrosophic” etymologically comes from “neutrosophic” [French neuter, Latin neuter, neutral, and Greek Sophia, skill/wisdom] which means knowledge of neutral thought, and this third/neutral represents the main distinction between “fuzzy/intuitionistic” logic/set and “neutrosophic” logic/set, that is, the included middle component, that is, the neutral/indeterminate/unknown part (besides the truth/membership and falsehood/non-membership components that both appear in fuzzy logic/set). In 1995 (published in 1998), Smarandache introduced the degree of indeterminacy/neutrality (i) as an independent component and characterized the neutrosophic set as (truth, indeterminacy, falsehood). Smarandache^{17,18} invented the neutrosophic set notion, which expands on the classic set and fuzzy set, intuitionistic fuzzy set, and interval-valued intuitionistic fuzzy set. Neutrosophic set theory is utilized in several aspects (see <http://fs.gallup.unm.edu/neutrosophy.htm>).

The concept of Hilbert algebras was introduced in the early 1950s by Henkin⁹ for some investigations of implication in intuitionistic and other non-classical logic. In the 1960s, these algebras were studied especially by Diego⁵ from an algebraic point of view. Diego⁵ proved that Hilbert algebras form a locally finite variety. Busneag^{2,3} and Jun¹¹ investigated Hilbert algebras and identified filters that construct deductive systems. Dudek⁶ investigated fuzzy subalgebras and deductive systems in Hilbert algebras.

The negative structure of sets is continually defined and investigated. Jun et al.¹² introduced a new negative-valued function and constructed \mathcal{N} -structures in 2009. In 2017, Jun et al.^{13,19} investigated neutrosophic \mathcal{N} -structures for BCK/BCI-algebras and neutrosophic commutative \mathcal{N} -ideals in BCK-algebras. Jun et al.¹⁴ investigated neutrosophic positive implicative \mathcal{N} -ideals in BCK-algebras in 2018. Rangsuk et al.¹⁶ introduced

the notions of (special) neutrosophic \mathcal{N} -UP-subalgebras, (special) neutrosophic \mathcal{N} -near UP-filters, (special) neutrosophic \mathcal{N} -UP-filters, (special) neutrosophic \mathcal{N} -UP-ideals, and (special) neutrosophic \mathcal{N} -strong UP-ideals of UP-algebras in 2019. In 2024, Iampan and Rajesh¹⁰ introduced the notions of neutrosophic \mathcal{N} -subalgebras and neutrosophic \mathcal{N} -ideals of Hilbert algebras.

In this paper, the notion of neutrosophic \mathcal{N} -deductive systems of Hilbert algebras is introduced, and several properties are investigated. Conditions for neutrosophic \mathcal{N} -structures to be neutrosophic \mathcal{N} -deductive systems of Hilbert algebras are provided. Relations between neutrosophic \mathcal{N} -deductive systems and their level subsets are considered. The Cartesian product of neutrosophic \mathcal{N} -structures is also supplied. Finally, we also find the property of the homomorphic pre-image of neutrosophic \mathcal{N} -deductive systems.

2 Preliminaries

Before we get started, let's define a Hilbert algebra.

Definition 2.1.⁵ A Hilbert algebra is a triplet with the formula $X = (X, *, 1_X)$, where X is a nonempty set, $*$ is a binary operation, and 1_X is a fixed member of X that is true according to the axioms mentioned below:

- (1) $(\forall x, y \in X)(x * (y * x) = 1_X)$,
- (2) $(\forall x, y, z \in X)((x * (y * z)) * ((x * y) * (x * z)) = 1_X)$,
- (3) $(\forall x, y \in X)(x * y = 1_X, y * x = 1_X \Rightarrow x = y)$.

In,⁶ the following conclusion was established.

Lemma 2.2. Let $X = (X, *, 1_X)$ be a Hilbert algebra. Then

- (1) $(\forall x \in X)(x * x = 1_X)$,
- (2) $(\forall x \in X)(1_X * x = x)$,
- (3) $(\forall x \in X)(x * 1_X = 1_X)$,
- (4) $(\forall x, y, z \in X)(x * (y * z) = y * (x * z))$,
- (5) $(\forall x, y, z \in X)((x * z) * ((z * y) * (x * y)) = 1_X)$.

In a Hilbert algebra $X = (X, *, 1_X)$, the binary relation \leq is defined by

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x * y = 1_X),$$

which is a partial order on X with 1_X as the largest element.

Definition 2.3.⁷ A nonempty subset D of a Hilbert algebra $X = (X, *, 1_X)$ is called a *deductive system* of X if

- (1) $1_X \in D$,
- (2) $(\forall x, y \in X)(x * y \in D, x \in D \Rightarrow y \in D)$.

A *fuzzy set*²⁰ in a nonempty set X is defined to be a function $\mu : X \rightarrow [0, 1]$, where $[0, 1]$ is the unit closed interval of real numbers.

Definition 2.4. A fuzzy set μ in a Hilbert algebra $X = (X, *, 1_X)$ is said to be a *fuzzy deductive system* of X if

- (1) $(\forall x \in X)(\mu(1_X) \geq \mu(x))$,
- (2) $(\forall x, y \in X)(\mu(y) \geq \min\{\mu(x * y), \mu(x)\})$.

Definition 2.5.¹ A neutrosophic set in a nonempty set X is defined to be a structure

$$A := \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in X\}, \tag{1}$$

where $T_A : X \rightarrow [0, 1]$ is a truth membership function, $I_A : X \rightarrow [0, 1]$ is an indeterminate membership function, and $F_A : X \rightarrow [0, 1]$ is a false membership function. The neutrosophic set in (1) is simply denoted by $A = (X, T_A, I_A, F_A)$.

Definition 2.6.¹⁵ We denote the family of all functions from a nonempty set X to the closed interval $[-1, 0]$ of the real line by $\mathcal{F}(X, [-1, 0])$. An element of $\mathcal{F}(X, [-1, 0])$ is called a *negative-valued function* from X to $[-1, 0]$ (briefly, \mathcal{N} -function on X). An ordered pair of a nonempty set X and an \mathcal{N} -function on X is called an *\mathcal{N} -fuzzy structure*. A neutrosophic \mathcal{N} -structure $X_{\mathcal{N}}$ over a nonempty universe of discourse X is defined to be the structure $(X, T_{\mathcal{N}}, I_{\mathcal{N}}, F_{\mathcal{N}})$, where $T_{\mathcal{N}}, I_{\mathcal{N}}$, and $F_{\mathcal{N}}$ are \mathcal{N} -functions on X , which are called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function on X , respectively.

For simplicity, use $X_{\mathcal{N}}$ instead of the neutrosophic \mathcal{N} -structure $(X, T_{\mathcal{N}}, I_{\mathcal{N}}, F_{\mathcal{N}})$.¹²

Definition 2.7.¹⁶ Let $X_{\mathcal{N}}$ be a neutrosophic \mathcal{N} -structure over a nonempty set X . The neutrosophic \mathcal{N} -structure $\overline{X_{\mathcal{N}}} = (X, \overline{T_{\mathcal{N}}}, \overline{I_{\mathcal{N}}}, \overline{F_{\mathcal{N}}})$ defined by

$$(\forall x \in X) \begin{pmatrix} \overline{T_{\mathcal{N}}}(x) = -1 - T_{\mathcal{N}}(x) \\ \overline{I_{\mathcal{N}}}(x) = -1 - I_{\mathcal{N}}(x) \\ \overline{F_{\mathcal{N}}}(x) = -1 - F_{\mathcal{N}}(x) \end{pmatrix} \tag{2}$$

is called the *complement* of $X_{\mathcal{N}}$ in X .

3 Main results

Unless otherwise indicated, X will refer to a Hilbert algebra $(X, *, 1_X)$.

Definition 3.1.¹⁰ A neutrosophic \mathcal{N} -structure $X_{\mathcal{N}}$ over X is called a neutrosophic \mathcal{N} -ideal of X if

$$(\forall x \in X) \begin{pmatrix} T_{\mathcal{N}}(1_X) \leq T_{\mathcal{N}}(x) \\ I_{\mathcal{N}}(1_X) \geq I_{\mathcal{N}}(x) \\ F_{\mathcal{N}}(1_X) \leq F_{\mathcal{N}}(x) \end{pmatrix}, \tag{3}$$

$$(\forall x, y \in X) \begin{pmatrix} T_{\mathcal{N}}(x * y) \leq T_{\mathcal{N}}(y) \\ I_{\mathcal{N}}(x * y) \geq I_{\mathcal{N}}(y) \\ F_{\mathcal{N}}(x * y) \leq F_{\mathcal{N}}(y) \end{pmatrix}, \tag{4}$$

$$(\forall x, y_1, y_2 \in X) \begin{pmatrix} T_{\mathcal{N}}((y_1 * (y_2 * x)) * x) \leq \max\{T_{\mathcal{N}}(y_1), T_{\mathcal{N}}(y_2)\} \\ I_{\mathcal{N}}((y_1 * (y_2 * x)) * x) \geq \min\{I_{\mathcal{N}}(y_1), I_{\mathcal{N}}(y_2)\} \\ F_{\mathcal{N}}((y_1 * (y_2 * x)) * x) \leq \max\{F_{\mathcal{N}}(y_1), F_{\mathcal{N}}(y_2)\} \end{pmatrix}. \tag{5}$$

Definition 3.2. A neutrosophic \mathcal{N} -structure $X_{\mathcal{N}}$ over X is called a neutrosophic \mathcal{N} -deductive system of X if it satisfies (3) and

$$(\forall x, y \in X) \begin{pmatrix} T_{\mathcal{N}}(y) \leq \max\{T_{\mathcal{N}}(x * y), T_{\mathcal{N}}(x)\} \\ I_{\mathcal{N}}(y) \geq \min\{I_{\mathcal{N}}(x * y), I_{\mathcal{N}}(x)\} \\ F_{\mathcal{N}}(y) \leq \max\{F_{\mathcal{N}}(x * y), F_{\mathcal{N}}(x)\} \end{pmatrix}. \tag{6}$$

Proposition 3.3. If $\{X_{\mathcal{N}}^i \mid i \in \Delta\}$ is a family of neutrosophic \mathcal{N} -deductive systems of X , then $\bigwedge_{i \in \Delta} X_{\mathcal{N}}^i$ is a neutrosophic \mathcal{N} -deductive system of X .

Proof. Let $\{X_{\mathcal{N}}^i \mid i \in \Delta\}$ be a family of neutrosophic \mathcal{N} -deductive systems of X . Let $x \in X$. Then

$$\begin{aligned} \left(\bigwedge_{i \in \Delta} T_{\mathcal{N}_i}\right)(1_X) &= \sup_{i \in \Delta} \{T_{\mathcal{N}_i}(1_X)\} \leq \sup_{i \in \Delta} \{T_{\mathcal{N}_i}(x)\} = \left(\bigwedge_{i \in \Delta} T_{\mathcal{N}_i}\right)(x), \\ \left(\bigwedge_{i \in \Delta} I_{\mathcal{N}_i}\right)(1_X) &= \inf_{i \in \Delta} \{I_{\mathcal{N}_i}(1_X)\} \geq \inf_{i \in \Delta} \{I_{\mathcal{N}_i}(x)\} = \left(\bigwedge_{i \in \Delta} I_{\mathcal{N}_i}\right)(x), \\ \left(\bigwedge_{i \in \Delta} F_{\mathcal{N}_i}\right)(1_X) &= \sup_{i \in \Delta} \{F_{\mathcal{N}_i}(1_X)\} \leq \sup_{i \in \Delta} \{F_{\mathcal{N}_i}(x)\} = \left(\bigwedge_{i \in \Delta} F_{\mathcal{N}_i}\right)(x). \end{aligned}$$

Let $x, y \in X$. Then

$$\begin{aligned} \left(\bigwedge_{i \in \Delta} T_{\mathcal{N}_i}\right)(y) &= \sup_{i \in \Delta} \{T_{\mathcal{N}_i}(y)\} \\ &\leq \sup_{i \in \Delta} \{\max\{T_{\mathcal{N}_i}(x * y), T_{\mathcal{N}_i}(x)\}\} \\ &\leq \max\{\sup_{i \in \Delta} \{T_{\mathcal{N}_i}(x * y)\}, \sup_{i \in \Delta} \{T_{\mathcal{N}_i}(x)\}\} \\ &= \max\{\left(\bigwedge_{i \in \Delta} T_{\mathcal{N}_i}\right)(x * y), \left(\bigwedge_{i \in \Delta} T_{\mathcal{N}_i}\right)(x)\}, \\ \left(\bigwedge_{i \in \Delta} I_{\mathcal{N}_i}\right)(y) &= \inf_{i \in \Delta} \{I_{\mathcal{N}_i}(y)\} \\ &\geq \inf_{i \in \Delta} \{\min\{I_{\mathcal{N}_i}(x * y), I_{\mathcal{N}_i}(x)\}\} \\ &\geq \min\{\inf_{i \in \Delta} \{I_{\mathcal{N}_i}(x * y)\}, \inf_{i \in \Delta} \{I_{\mathcal{N}_i}(x)\}\} \\ &= \min\{\left(\bigwedge_{i \in \Delta} I_{\mathcal{N}_i}\right)(x * y), \left(\bigwedge_{i \in \Delta} I_{\mathcal{N}_i}\right)(x)\}, \\ \left(\bigwedge_{i \in \Delta} F_{\mathcal{N}_i}\right)(y) &= \sup_{i \in \Delta} \{F_{\mathcal{N}_i}(y)\} \\ &\leq \sup_{i \in \Delta} \{\max\{F_{\mathcal{N}_i}(x * y), F_{\mathcal{N}_i}(x)\}\} \\ &\leq \max\{\sup_{i \in \Delta} \{F_{\mathcal{N}_i}(x * y)\}, \sup_{i \in \Delta} \{F_{\mathcal{N}_i}(x)\}\} \\ &= \max\{\left(\bigwedge_{i \in \Delta} F_{\mathcal{N}_i}\right)(x * y), \left(\bigwedge_{i \in \Delta} F_{\mathcal{N}_i}\right)(x)\}. \end{aligned}$$

Hence, $\bigwedge_{i \in \Delta} X_{\mathcal{N}}^i$ is a neutrosophic \mathcal{N} -deductive system of X . □

Proposition 3.4. Every neutrosophic \mathcal{N} -ideal of X is a neutrosophic \mathcal{N} -deductive system of X .

Proof. Let $X_{\mathcal{N}}$ be a neutrosophic \mathcal{N} -ideal of X . Let $x, y \in X$. Then, by (5), we have

$$\begin{aligned} T_{\mathcal{N}}(y) &= T_{\mathcal{N}}(1_X * y) = T_{\mathcal{N}}(((x * y) * (x * y)) * y) \leq \max\{T_{\mathcal{N}}(x * y), T_{\mathcal{N}}(x)\}, \\ I_{\mathcal{N}}(y) &= I_{\mathcal{N}}(1_X * y) = I_{\mathcal{N}}(((x * y) * (x * y)) * y) \geq \min\{I_{\mathcal{N}}(x * y), I_{\mathcal{N}}(x)\}, \\ F_{\mathcal{N}}(y) &= F_{\mathcal{N}}(1_X * y) = F_{\mathcal{N}}(((x * y) * (x * y)) * y) \leq \max\{F_{\mathcal{N}}(x * y), F_{\mathcal{N}}(x)\}. \end{aligned}$$

Hence, $X_{\mathcal{N}}$ is a neutrosophic \mathcal{N} -deductive system of X . □

Theorem 3.5. Let $X_{\mathcal{N}}$ be a neutrosophic \mathcal{N} -deductive system of X . Then the following hold:

$$(\forall x, y \in X) \left(\begin{array}{l} T_{\mathcal{N}}(x * y) = T_{\mathcal{N}}(1_X) \Rightarrow T_{\mathcal{N}}(x) \geq T_{\mathcal{N}}(y) \\ I_{\mathcal{N}}(x * y) = I_{\mathcal{N}}(1_X) \Rightarrow I_{\mathcal{N}}(x) \leq I_{\mathcal{N}}(y) \\ F_{\mathcal{N}}(x * y) = F_{\mathcal{N}}(1_X) \Rightarrow F_{\mathcal{N}}(x) \geq F_{\mathcal{N}}(y) \end{array} \right), \tag{7}$$

$$(\forall x, y \in X) \left(x \leq y \Rightarrow \begin{cases} T_{\mathcal{N}}(x) \geq T_{\mathcal{N}}(y) \\ I_{\mathcal{N}}(x) \leq I_{\mathcal{N}}(y) \\ F_{\mathcal{N}}(x) \geq F_{\mathcal{N}}(y) \end{cases} \right), \tag{8}$$

$$(\forall x, y, z \in X) \left(x \leq y * z \Rightarrow \begin{cases} T_{\mathcal{N}}(z) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\} \\ I_{\mathcal{N}}(z) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\} \\ F_{\mathcal{N}}(z) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\} \end{cases} \right), \tag{9}$$

$$(\forall x, y, z \in X) \left(\begin{array}{l} T_{\mathcal{N}}(x * y) \leq \max\{T_{\mathcal{N}}(x * z), T_{\mathcal{N}}(z * y)\} \\ I_{\mathcal{N}}(x * y) \geq \min\{I_{\mathcal{N}}(x * z), I_{\mathcal{N}}(z * y)\} \\ F_{\mathcal{N}}(x * y) \leq \max\{F_{\mathcal{N}}(x * z), F_{\mathcal{N}}(z * y)\} \end{array} \right). \tag{10}$$

Proof. Let $x, y \in X$ be such that $T_{\mathcal{N}}(x * y) = T_{\mathcal{N}}(1_X)$, $I_{\mathcal{N}}(x * y) = I_{\mathcal{N}}(1_X)$, and $F_{\mathcal{N}}(x * y) = F_{\mathcal{N}}(1_X)$. Then, by (6) and (3), we have

$$\begin{aligned} T_{\mathcal{N}}(y) &\leq \max\{T_{\mathcal{N}}(x * y), T_{\mathcal{N}}(x)\} = \max\{T_{\mathcal{N}}(1_X), T_{\mathcal{N}}(x)\} = T_{\mathcal{N}}(x), \\ I_{\mathcal{N}}(y) &\geq \min\{I_{\mathcal{N}}(x * y), I_{\mathcal{N}}(x)\} = \min\{I_{\mathcal{N}}(1_X), I_{\mathcal{N}}(x)\} = I_{\mathcal{N}}(x), \\ F_{\mathcal{N}}(y) &\leq \max\{F_{\mathcal{N}}(x * y), F_{\mathcal{N}}(x)\} = \max\{T_{\mathcal{N}}(1_X), T_{\mathcal{N}}(x)\} = T_{\mathcal{N}}(x). \end{aligned}$$

Therefore, (7) is proved.

(8) is a direct result of (7).

Let $x, y, z \in X$ be such that $x \leq y * z$. By (8), we have $T_{\mathcal{N}}(x) \geq T_{\mathcal{N}}(y * z)$, $I_{\mathcal{N}}(x) \leq I_{\mathcal{N}}(y * z)$, and $F_{\mathcal{N}}(x) \geq F_{\mathcal{N}}(y * z)$. By (6), we have

$$\begin{aligned} \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(y)\} &\geq \max\{T_{\mathcal{N}}(y * z), T_{\mathcal{N}}(y)\} \geq T_{\mathcal{N}}(z), \\ \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(y)\} &\leq \min\{I_{\mathcal{N}}(y * z), I_{\mathcal{N}}(y)\} \leq I_{\mathcal{N}}(z), \\ \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(y)\} &\geq \max\{F_{\mathcal{N}}(y * z), F_{\mathcal{N}}(y)\} \geq F_{\mathcal{N}}(z). \end{aligned}$$

Therefore, (9) is proved.

Let $x, y, z \in X$. By Lemma 2.2 (5), we have $x * z \leq (z * y) * (x * y)$. By (9), we have

$$\begin{aligned} T_{\mathcal{N}}(x * y) &\leq \max\{T_{\mathcal{N}}(x * z), T_{\mathcal{N}}(z * y)\}, \\ I_{\mathcal{N}}(x * y) &\geq \min\{I_{\mathcal{N}}(x * z), I_{\mathcal{N}}(z * y)\}, \\ F_{\mathcal{N}}(x * y) &\leq \max\{F_{\mathcal{N}}(x * z), F_{\mathcal{N}}(z * y)\}. \end{aligned}$$

Therefore, (10) is proved. □

Theorem 3.6. Let $X_{\mathcal{N}}$ be an \mathcal{N} -structure of X . Then $X_{\mathcal{N}}$ is a neutrosophic \mathcal{N} -deductive system of X if and only if

$$(\forall x, y, z \in X) \left(x * y \geq z \Rightarrow \begin{cases} T_{\mathcal{N}}(y) \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(z)\} \\ I_{\mathcal{N}}(y) \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(z)\} \\ F_{\mathcal{N}}(y) \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(z)\} \end{cases} \right). \tag{11}$$

Proof. Assume that (11) is true. Let $x \in X$. Since $x * 1_X = 1_X \geq x$, we have

$$\begin{aligned} T_{\mathcal{N}}(1_X) &\leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(x)\} = T_{\mathcal{N}}(x), \\ I_{\mathcal{N}}(1_X) &\geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(x)\} = I_{\mathcal{N}}(x), \\ F_{\mathcal{N}}(1_X) &\leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(x)\} = F_{\mathcal{N}}(x). \end{aligned}$$

Let $x, y \in X$. Since $x * y \geq x * y$, we have

$$\begin{aligned} T_{\mathcal{N}}(y) &\leq \max\{T_{\mathcal{N}}(x * y), T_{\mathcal{N}}(x)\}, \\ I_{\mathcal{N}}(y) &\geq \min\{I_{\mathcal{N}}(x * y), I_{\mathcal{N}}(x)\}, \\ F_{\mathcal{N}}(y) &\leq \max\{F_{\mathcal{N}}(x * y), F_{\mathcal{N}}(x)\}. \end{aligned}$$

Hence, $X_{\mathcal{N}}$ is a neutrosophic \mathcal{N} -deductive system of X .

Conversely, assume that $X_{\mathcal{N}}$ is a neutrosophic \mathcal{N} -deductive system of X . Let $x, y, z \in X$ be such that $x * y \geq z$. It follows from (8) that $T_{\mathcal{N}}(x * y) \leq T_{\mathcal{N}}(z)$, $I_{\mathcal{N}}(x * y) \geq I_{\mathcal{N}}(z)$, and $F_{\mathcal{N}}(x * y) \leq F_{\mathcal{N}}(z)$. By (6), we have

$$\begin{aligned} T_{\mathcal{N}}(y) &\leq \max\{T_{\mathcal{N}}(x * y), T_{\mathcal{N}}(x)\} \leq \max\{T_{\mathcal{N}}(x), T_{\mathcal{N}}(z)\}, \\ I_{\mathcal{N}}(y) &\geq \min\{I_{\mathcal{N}}(x * y), I_{\mathcal{N}}(x)\} \geq \min\{I_{\mathcal{N}}(x), I_{\mathcal{N}}(z)\}, \\ F_{\mathcal{N}}(y) &\leq \max\{F_{\mathcal{N}}(x * y), F_{\mathcal{N}}(x)\} \leq \max\{F_{\mathcal{N}}(x), F_{\mathcal{N}}(z)\}. \end{aligned}$$

Therefore, (11) is proved. □

Definition 3.7. Let $X_{\mathcal{N}}$ be a neutrosophic \mathcal{N} -structure over a nonempty set X . The neutrosophic \mathcal{N} -structures $\oplus X_{\mathcal{N}}$, $\otimes X_{\mathcal{N}}$, and $\odot X_{\mathcal{N}}$ are defined as $\oplus X_{\mathcal{N}} = (X, T_{\mathcal{N}}, \overline{T_{\mathcal{N}}}, F_{\mathcal{N}})$, $\otimes X_{\mathcal{N}} = (X, \overline{I_{\mathcal{N}}}, I_{\mathcal{N}}, F_{\mathcal{N}})$, and $\odot X_{\mathcal{N}} = (X, \overline{I_{\mathcal{N}}}, I_{\mathcal{N}}, \overline{I_{\mathcal{N}}})$.

Theorem 3.8. If $X_{\mathcal{N}}$ is a neutrosophic \mathcal{N} -deductive system of X , then $\oplus X_{\mathcal{N}}$, $\otimes X_{\mathcal{N}}$, and $\odot X_{\mathcal{N}}$ are neutrosophic \mathcal{N} -deductive systems of X .

Proof. Assume that $X_{\mathcal{N}}$ is a neutrosophic \mathcal{N} -deductive system of X . Let $x \in X$. Then $\overline{T_{\mathcal{N}}}(1_X) = -1 - T_{\mathcal{N}}(1_X) \geq -1 - T_{\mathcal{N}}(x) \geq \overline{T_{\mathcal{N}}}(x)$ and $\overline{I_{\mathcal{N}}}(1_X) = -1 - I_{\mathcal{N}}(1_X) \leq -1 - I_{\mathcal{N}}(x) \leq \overline{I_{\mathcal{N}}}(x)$. Let $x, y \in X$. Then

$$\begin{aligned} \overline{T_{\mathcal{N}}}(y) &= -1 - T_{\mathcal{N}}(y) \\ &\geq -1 - \max\{T_{\mathcal{N}}(x * y), T_{\mathcal{N}}(x)\} \\ &= \min\{-1 - T_{\mathcal{N}}(x * y), -1 - T_{\mathcal{N}}(x)\} \\ &= \min\{\overline{T_{\mathcal{N}}}(x * y), \overline{T_{\mathcal{N}}}(x)\}, \\ \overline{I_{\mathcal{N}}}(y) &= -1 - I_{\mathcal{N}}(y) \\ &\leq -1 - \min\{I_{\mathcal{N}}(x * y), I_{\mathcal{N}}(y)\} \\ &= \max\{-1 - I_{\mathcal{N}}(x * y), -1 - I_{\mathcal{N}}(y)\} \\ &= \max\{\overline{I_{\mathcal{N}}}(x * y), \overline{I_{\mathcal{N}}}(y)\}. \end{aligned}$$

Hence, $\oplus X_{\mathcal{N}}$, $\otimes X_{\mathcal{N}}$, and $\odot X_{\mathcal{N}}$ are neutrosophic \mathcal{N} -deductive systems of X . □

Theorem 3.9. If $X_{\mathcal{N}}$ is a neutrosophic \mathcal{N} -deductive system of X , then the sets $X_{T_{\mathcal{N}}} = \{x \in X \mid T_{\mathcal{N}}(x) = T_{\mathcal{N}}(1_X)\}$, $X_{I_{\mathcal{N}}} = \{x \in X \mid I_{\mathcal{N}}(x) = I_{\mathcal{N}}(1_X)\}$, and $X_{F_{\mathcal{N}}} = \{x \in X \mid F_{\mathcal{N}}(x) = F_{\mathcal{N}}(1_X)\}$ are deductive systems of X .

Proof. Clearly, $1_X \in X_{T_{\mathcal{N}}} \cap X_{I_{\mathcal{N}}} \cap X_{F_{\mathcal{N}}}$. Let $x, y \in X$ be such that $x * y, x \in X_{T_{\mathcal{N}}}$. Then $T_{\mathcal{N}}(x) = T_{\mathcal{N}}(1_X) = T_{\mathcal{N}}(x * y)$. By (6) and (3), we have $T_{\mathcal{N}}(y) \leq \max\{T_{\mathcal{N}}(x * y), T_{\mathcal{N}}(x)\} = T_{\mathcal{N}}(1_X) \leq T_{\mathcal{N}}(y)$, whence $T_{\mathcal{N}}(y) = T_{\mathcal{N}}(1_X)$. This means that $y \in X_{T_{\mathcal{N}}}$. Hence, $X_{T_{\mathcal{N}}}$ is a deductive system of X . Let $x, y \in X$ be such that $x * y, x \in X_{I_{\mathcal{N}}}$. Then $I_{\mathcal{N}}(x) = I_{\mathcal{N}}(1_X) = I_{\mathcal{N}}(x * y)$. By (6) and (3), we have $I_{\mathcal{N}}(y) \geq \min\{I_{\mathcal{N}}(x * y), I_{\mathcal{N}}(x)\} = I_{\mathcal{N}}(1_X) \geq I_{\mathcal{N}}(y)$, whence $I_{\mathcal{N}}(y) = I_{\mathcal{N}}(1_X)$. This means that $y \in X_{I_{\mathcal{N}}}$. Hence, $X_{I_{\mathcal{N}}}$ is a deductive system of X . Let $x, y \in X$ be such that $x * y, x \in X_{F_{\mathcal{N}}}$. Then $F_{\mathcal{N}}(x) = F_{\mathcal{N}}(1_X) = F_{\mathcal{N}}(x * y)$. By (6) and (3), we have $F_{\mathcal{N}}(y) \leq \max\{F_{\mathcal{N}}(x * y), F_{\mathcal{N}}(x)\} = F_{\mathcal{N}}(1_X) \leq F_{\mathcal{N}}(y)$, whence $F_{\mathcal{N}}(y) = F_{\mathcal{N}}(1_X)$. This means that $y \in X_{F_{\mathcal{N}}}$. Hence, $X_{F_{\mathcal{N}}}$ is a deductive system of X . □

For any numbers $a^+, a^-, b^+, b^-, c^+, c^- \in [-1, 0]$ such that $a^+ > a^-, b^+ > b^-, c^+ > c^-$ and a nonempty subset S of X , define a neutrosophic \mathcal{N} -structure

$$X^S \left[\begin{matrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{matrix} \right] = \left(X, T_{\mathcal{N}}^S \left[\begin{matrix} a^- \\ a^+ \end{matrix} \right], I_{\mathcal{N}}^S \left[\begin{matrix} b^+ \\ b^- \end{matrix} \right], F_{\mathcal{N}}^S \left[\begin{matrix} c^- \\ c^+ \end{matrix} \right] \right)$$

over X , where

$$\begin{aligned} T_{\mathcal{N}}^S \left[\begin{matrix} a^- \\ a^+ \end{matrix} \right] (x) &= \begin{cases} a^- & \text{if } x \in S \\ a^+ & \text{otherwise,} \end{cases} \\ I_{\mathcal{N}}^S \left[\begin{matrix} b^+ \\ b^- \end{matrix} \right] (x) &= \begin{cases} b^+ & \text{if } x \in S \\ b^- & \text{otherwise,} \end{cases} \\ F_{\mathcal{N}}^S \left[\begin{matrix} c^- \\ c^+ \end{matrix} \right] (x) &= \begin{cases} c^- & \text{if } x \in S \\ c^+ & \text{otherwise.} \end{cases} \end{aligned}$$

Lemma 3.10. ¹⁰ If the constant 1_X of X is in a nonempty subset S of X , then the neutrosophic \mathcal{N} -structure $X^S \left[\begin{matrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{matrix} \right]$ over X satisfies (3).

Lemma 3.11. ¹⁰ If the neutrosophic \mathcal{N} -structure $X^S \left[\begin{matrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{matrix} \right]$ over X satisfies (3), then the constant 1_X of X is in a nonempty subset S of X .

Theorem 3.12. The neutrosophic \mathcal{N} -structure $X^S \left[\begin{matrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{matrix} \right]$ in X is a neutrosophic \mathcal{N} -deductive system of X if and only if a nonempty subset S of X is a deductive system of X .

Proof. Assume that $X^S \begin{bmatrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{bmatrix}$ is a neutrosophic \mathcal{N} -deductive system of X .

Since $X^S \begin{bmatrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{bmatrix}$ satisfies (3), it follows from Lemma 3.11 that $1_X \in S$. Next, let $x, y \in X$ be such that $x * y, x \in S$. Then $T_{\mathcal{N}}^S \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x) = a^- = T_{\mathcal{N}}^S \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x * y)$. Hence, by (6), we have

$$\begin{aligned} T_{\mathcal{N}}^S \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (y) &\leq \left\{ T_{\mathcal{N}}^S \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x * y), T_{\mathcal{N}}^S \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x) \right\} \\ &= \min\{a^-, a^-\} \\ &= a^- \\ &\leq T_{\mathcal{N}}^S \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (y) \end{aligned}$$

and so $T_{\mathcal{N}}^S \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (y) = a^-$. Thus, $y \in S$. Hence, S is a deductive system of X .

Conversely, assume that S is a deductive system of X . Since $1_X \in S$, it follows from Lemma 3.10 that $X^S \begin{bmatrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{bmatrix}$ satisfies (3). Next, let $x, y \in X$.

Case 1: Suppose $x * y \in S$ and $y \in S$. Then

$$\begin{aligned} T_{\mathcal{N}}^S \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x * y) &= a^- = T_{\mathcal{N}}^S \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (y), \\ I_{\mathcal{N}}^S \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (x * y) &= b^+ = I_{\mathcal{N}}^S \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (y), \\ F_{\mathcal{N}}^S \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (x * y) &= c^- = F_{\mathcal{N}}^S \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (y). \end{aligned}$$

Since S is a deductive system of X , we have $y \in S$ and so

$$T_{\mathcal{N}}^S \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (y) = a^-, I_{\mathcal{N}}^S \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (y) = b^+, F_{\mathcal{N}}^S \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (y) = c^-.$$

Then

$$\begin{aligned} T_{\mathcal{N}}^S \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (y) &= a^- \leq a^- = \max\{a^-, a^-\} = \max \left\{ T_{\mathcal{N}}^S \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x * y), T_{\mathcal{N}}^S \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x) \right\}, \\ I_{\mathcal{N}}^S \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (y) &= b^+ \geq b^+ = \min\{b^+, b^+\} = \min \left\{ I_{\mathcal{N}}^S \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (x * y), I_{\mathcal{N}}^S \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (x) \right\}, \\ F_{\mathcal{N}}^S \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (y) &= c^- \leq c^- = \max\{c^-, c^-\} = \max \left\{ F_{\mathcal{N}}^S \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (x * y), F_{\mathcal{N}}^S \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (x) \right\}. \end{aligned}$$

Case 2: Suppose $x * y \notin S$ or $x \notin S$. Then

$$\begin{aligned} T_{\mathcal{N}}^S \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x * y) &= a^+ \text{ or } T_{\mathcal{N}}^S \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x) = a^+, \\ I_{\mathcal{N}}^S \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (x * y) &= b^- \text{ or } I_{\mathcal{N}}^S \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (x) = b^-, \\ F_{\mathcal{N}}^S \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (x * y) &= c^+ \text{ or } F_{\mathcal{N}}^S \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (x) = c^+. \end{aligned}$$

Thus,

$$\max \left\{ T_{\mathcal{N}}^S \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x * y), T_{\mathcal{N}}^S \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x) \right\} = a^+,$$

$$\min \left\{ I_{\mathcal{N}}^S \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (x * y), I_{\mathcal{N}}^S \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (x) \right\} = b^-,$$

$$\max \left\{ F_{\mathcal{N}}^S \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (x * y), F_{\mathcal{N}}^S \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (x) \right\} = c^+.$$

Therefore,

$$T_{\mathcal{N}}^S \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (y) \leq a^+ = \max \left\{ T_{\mathcal{N}}^S \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x * y), T_{\mathcal{N}}^S \begin{bmatrix} a^- \\ a^+ \end{bmatrix} (x) \right\},$$

$$I_{\mathcal{N}}^S \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (y) \geq b^- = \min \left\{ I_{\mathcal{N}}^S \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (x * y), I_{\mathcal{N}}^S \begin{bmatrix} b^+ \\ b^- \end{bmatrix} (x) \right\},$$

$$F_{\mathcal{N}}^S \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (y) \leq c^+ = \max \left\{ F_{\mathcal{N}}^S \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (x * y), F_{\mathcal{N}}^S \begin{bmatrix} c^- \\ c^+ \end{bmatrix} (x) \right\}.$$

Hence, $X^S \begin{bmatrix} a^-, & b^+, & c^- \\ a^+, & b^-, & c^+ \end{bmatrix}$ is a neutrosophic \mathcal{N} -deductive system of X . □

Definition 3.13. Let f be an \mathcal{N} -function on a nonempty set X . For any $t \in [-1, 0]$, the sets $U(f : t) = \{x \in X \mid f(x) \geq t\}$ is called an upper t -level subset of f , $L(f : t) = \{x \in X \mid f(x) \leq t\}$ is called a lower t -level subset of f , and $E(f : t) = \{x \in X \mid f(x) = t\}$ is called an equal t -level subset of f .

Theorem 3.14. A neutrosophic \mathcal{N} -structure $X_{\mathcal{N}}$ over X is a neutrosophic \mathcal{N} -deductive system of X if and only if for all $a, b, c \in [-1, 0]$, the sets $L(T_{\mathcal{N}} : a), U(I_{\mathcal{N}} : b)$, and $L(F_{\mathcal{N}} : c)$ are either empty or deductive systems of X .

Proof. Assume that $X_{\mathcal{N}}$ is a neutrosophic \mathcal{N} -deductive system of X . Let $a, b, c \in [-1, 0]$ be such that $L(T_{\mathcal{N}} : a), U(I_{\mathcal{N}} : b)$, and $L(F_{\mathcal{N}} : c)$ are nonempty. Let $x \in L(T_{\mathcal{N}} : a), y \in U(I_{\mathcal{N}} : b)$, and $z \in L(F_{\mathcal{N}} : c)$. By (3), we have $T_{\mathcal{N}}(1_X) \leq T_{\mathcal{N}}(x) \leq a, I_{\mathcal{N}}(1_X) \geq I_{\mathcal{N}}(y) \geq b$, and $F_{\mathcal{N}}(1_X) \leq F_{\mathcal{N}}(z) \leq c$. Thus, $1_X \in L(T_{\mathcal{N}} : a) \cap U(I_{\mathcal{N}} : b) \cap L(F_{\mathcal{N}} : c)$. Let $x, y \in X$ be such that $x * y, x \in L(T_{\mathcal{N}} : a)$. Then $T_{\mathcal{N}}(x * y) \leq a$ and $T_{\mathcal{N}}(x) \leq a$, so a is an upper bound of $\{T_{\mathcal{N}}(x * y), T_{\mathcal{N}}(x)\}$. By (6), we have $T_{\mathcal{N}}(y) \leq \max\{T_{\mathcal{N}}(x * y), T_{\mathcal{N}}(x)\} \leq a$. Thus, $y \in L(T_{\mathcal{N}} : a)$. Let $x, y \in X$ be such that $x * y, x \in U(I_{\mathcal{N}} : b)$. Then $I_{\mathcal{N}}(x * y) \geq b$ and $I_{\mathcal{N}}(x) \geq b$, so b is a lower bound of $\{I_{\mathcal{N}}(x * y), I_{\mathcal{N}}(x)\}$. By (6), we have $I_{\mathcal{N}}(y) \geq \min\{I_{\mathcal{N}}(x * y), I_{\mathcal{N}}(x)\} \geq b$. Thus, $y \in U(I_{\mathcal{N}} : b)$. Let $x, y \in X$ be such that $x * y, x \in L(F_{\mathcal{N}} : c)$. Then $F_{\mathcal{N}}(x * y) \leq c$ and $F_{\mathcal{N}}(x) \leq c$, so c is an upper bound of $\{F_{\mathcal{N}}(x * y), F_{\mathcal{N}}(x)\}$. By (6), we have $F_{\mathcal{N}}(y) \leq \max\{F_{\mathcal{N}}(x * y), F_{\mathcal{N}}(x)\} \leq c$. Thus, $y \in L(F_{\mathcal{N}} : c)$. Hence, $L(T_{\mathcal{N}} : a), U(I_{\mathcal{N}} : b)$, and $L(F_{\mathcal{N}} : c)$ are deductive systems of X .

Conversely, assume that for all $a, b, c \in [-1, 0]$, the sets $L(T_{\mathcal{N}} : a), U(I_{\mathcal{N}} : b)$, and $L(F_{\mathcal{N}} : c)$ are either empty or deductive systems of X . Let $x \in X$. Then $x \in L(T_{\mathcal{N}} : T_{\mathcal{N}}(x)) \neq \emptyset, x \in U(I_{\mathcal{N}} : I_{\mathcal{N}}(x)) \neq \emptyset$, and $x \in L(F_{\mathcal{N}} : F_{\mathcal{N}}(x)) \neq \emptyset$. By the assumption, we have $L(T_{\mathcal{N}} : T_{\mathcal{N}}(x)), U(I_{\mathcal{N}} : I_{\mathcal{N}}(x))$, and $L(F_{\mathcal{N}} : F_{\mathcal{N}}(x))$ are deductive systems of X . Then $1_X \in L(T_{\mathcal{N}} : T_{\mathcal{N}}(x)), 1_X \in U(I_{\mathcal{N}} : I_{\mathcal{N}}(x))$, and $1_X \in L(F_{\mathcal{N}} : F_{\mathcal{N}}(x))$. Thus, $T_{\mathcal{N}}(1_X) \leq T_{\mathcal{N}}(x), I_{\mathcal{N}}(1_X) \geq I_{\mathcal{N}}(x)$, and $F_{\mathcal{N}}(1_X) \leq F_{\mathcal{N}}(x)$. Let $x, y \in X$. Then $T_{\mathcal{N}}(x * y), T_{\mathcal{N}}(x) \in [-1, 0]$. Choose $a = \max\{T_{\mathcal{N}}(x * y), T_{\mathcal{N}}(x)\}$. Thus, $T_{\mathcal{N}}(x * y) \leq a$ and $T_{\mathcal{N}}(x) \leq a$, so $x * y, x \in L(T_{\mathcal{N}} : a) \neq \emptyset$. By the assumption, we have $L(T_{\mathcal{N}} : a)$ is a deductive system of X and so $y \in L(T_{\mathcal{N}} : a)$. Thus, $T_{\mathcal{N}}(y) \leq a = \max\{T_{\mathcal{N}}(x * y), T_{\mathcal{N}}(x)\}$. Let $x, y \in X$. Then $I_{\mathcal{N}}(x * y), I_{\mathcal{N}}(x) \in [-1, 0]$. Choose $b = \min\{I_{\mathcal{N}}(x * y), I_{\mathcal{N}}(x)\}$. Thus, $I_{\mathcal{N}}(x * y) \geq b$ and $I_{\mathcal{N}}(x) \geq b$, so $x * y, x \in U(I_{\mathcal{N}} : b) \neq \emptyset$. By the assumption, we have $U(I_{\mathcal{N}} : b)$ is a deductive system of X and so $y \in U(I_{\mathcal{N}} : b)$. Thus, $I_{\mathcal{N}}(y) \geq b = \min\{I_{\mathcal{N}}(x * y), I_{\mathcal{N}}(x)\}$. Let $x, y \in X$. Then $F_{\mathcal{N}}(x * y), F_{\mathcal{N}}(x) \in [-1, 0]$. Choose $c = \max\{F_{\mathcal{N}}(x * y), F_{\mathcal{N}}(x)\}$. Thus, $F_{\mathcal{N}}(x * y) \leq c$ and $F_{\mathcal{N}}(x) \leq c$, so $x * y, x \in L(F_{\mathcal{N}} : c) \neq \emptyset$. By the assumption, we have $L(F_{\mathcal{N}} : c)$ is a deductive system of X and so $y \in L(F_{\mathcal{N}} : c)$. Thus, $F_{\mathcal{N}}(y) \leq c = \max\{F_{\mathcal{N}}(x * y), F_{\mathcal{N}}(x)\}$. Hence, $X_{\mathcal{N}}$ is a neutrosophic \mathcal{N} -deductive system of X . □

Definition 3.15. ¹⁰ Let $X_{\mathcal{N}} = (X, T_{\mathcal{N}X}, I_{\mathcal{N}X}, F_{\mathcal{N}X})$ and $Y_{\mathcal{N}} = (Y, T_{\mathcal{N}Y}, I_{\mathcal{N}Y}, F_{\mathcal{N}Y})$ be neutrosophic \mathcal{N} -structures of X and Y , respectively. The Cartesian product $X_{\mathcal{N}} \times Y_{\mathcal{N}} = (X \times Y, \Delta, \Theta, \Lambda)$ defined by

$$(\forall (x, y) \in X \times Y) \left(\begin{array}{l} \Delta(x, y) = \max\{T_{\mathcal{N}X}(x), T_{\mathcal{N}Y}(y)\} \\ \Theta(x, y) = \min\{I_{\mathcal{N}X}(x), I_{\mathcal{N}Y}(y)\} \\ \Lambda(x, y) = \max\{F_{\mathcal{N}X}(x), F_{\mathcal{N}Y}(y)\} \end{array} \right), \tag{12}$$

where Δ, Θ , and Λ are \mathcal{N} -functions on $X \times Y$.

Remark 3.16. Let $(X, *, 1_X)$ and $(Y, \star, 1_Y)$ be Hilbert algebras. Then $(X \times Y, \circ, (1_X, 1_Y))$ is a Hilbert algebra defined by $(x, y) \circ (u, v) = (x * u, y \star v)$ for every $x, u \in X$ and $y, v \in Y$.

Proposition 3.17. If $X_{\mathcal{N}} = (X, T_{\mathcal{N}X}, I_{\mathcal{N}X}, F_{\mathcal{N}X})$ and $Y_{\mathcal{N}} = (Y, T_{\mathcal{N}Y}, I_{\mathcal{N}Y}, F_{\mathcal{N}Y})$ are neutrosophic \mathcal{N} -deductive systems of Hilbert algebras X and Y , respectively, then the Cartesian product $X_{\mathcal{N}} \times Y_{\mathcal{N}}$ is a neutrosophic \mathcal{N} -deductive system of $X \times Y$.

Proof. Assume that $X_{\mathcal{N}} = (X, T_{\mathcal{N}X}, I_{\mathcal{N}X}, F_{\mathcal{N}X})$ and $Y_{\mathcal{N}} = (Y, T_{\mathcal{N}Y}, I_{\mathcal{N}Y}, F_{\mathcal{N}Y})$ are neutrosophic \mathcal{N} -deductive systems of Hilbert algebras X and Y , respectively. Let $(x, y) \in X \times Y$. Then

$$\begin{aligned} \Delta(1_X, 1_Y) &= \max\{T_{\mathcal{N}X}(1_X), T_{\mathcal{N}Y}(1_Y)\} \\ &\leq \max\{T_{\mathcal{N}X}(x), T_{\mathcal{N}Y}(y)\} \\ &= \Delta(x, y), \\ \Theta(1_X, 1_Y) &= \min\{I_{\mathcal{N}X}(1_X), I_{\mathcal{N}Y}(1_Y)\} \\ &\geq \min\{I_{\mathcal{N}X}(x), I_{\mathcal{N}Y}(y)\} \\ &= \Theta(x, y), \\ \Lambda(1_X, 1_Y) &= \max\{F_{\mathcal{N}X}(1_X), F_{\mathcal{N}Y}(1_Y)\} \\ &\leq \max\{F_{\mathcal{N}X}(x), F_{\mathcal{N}Y}(y)\} \\ &= \Lambda(x, y). \end{aligned}$$

Let $(x_1, x_2), (y_1, y_2) \in X \times Y$. Then

$$\begin{aligned} \Delta(y_1, y_2) &= \max\{T_{\mathcal{N}X}(y_1), T_{\mathcal{N}Y}(y_2)\} \\ &\leq \max\{\max\{T_{\mathcal{N}X}(x_1 * y_1), T_{\mathcal{N}X}(x_1)\}, \max\{T_{\mathcal{N}Y}(x_2 * y_2), T_{\mathcal{N}Y}(x_2)\}\} \\ &= \max\{\max\{T_{\mathcal{N}X}(x_1 * y_1), T_{\mathcal{N}Y}(x_2 * y_2)\}, \max\{T_{\mathcal{N}X}(x_1), T_{\mathcal{N}Y}(x_2)\}\} \\ &= \max\{\Delta(x_1 * y_1, x_2 * y_2), \Delta(x_1, x_2)\} \\ &= \max\{\Delta((x_1, x_2) \circ (y_1, y_2)), \Delta(x_1, x_2)\}, \\ \Theta(y_1, y_2) &= \min\{I_{\mathcal{N}X}(y_1), I_{\mathcal{N}Y}(y_2)\} \\ &\geq \min\{\min\{I_{\mathcal{N}X}(x_1 * y_1), I_{\mathcal{N}X}(x_1)\}, \min\{I_{\mathcal{N}Y}(x_2 * y_2), I_{\mathcal{N}Y}(x_2)\}\} \\ &= \min\{\min\{I_{\mathcal{N}X}(x_1 * y_1), I_{\mathcal{N}Y}(x_2 * y_2)\}, \min\{I_{\mathcal{N}X}(x_1), I_{\mathcal{N}Y}(x_2)\}\} \\ &= \min\{\Theta(x_1 * y_1, x_2 * y_2), \Theta(x_1, x_2)\} \\ &= \min\{\Theta((x_1, x_2) \circ (y_1, y_2)), \Theta(x_1, x_2)\}, \\ \Lambda(y_1, y_2) &= \max\{F_{\mathcal{N}X}(y_1), F_{\mathcal{N}Y}(y_2)\} \\ &\leq \max\{\max\{F_{\mathcal{N}X}(x_1 * y_1), F_{\mathcal{N}X}(x_1)\}, \max\{F_{\mathcal{N}Y}(x_2 * y_2), F_{\mathcal{N}Y}(x_2)\}\} \\ &= \max\{\max\{F_{\mathcal{N}X}(x_1 * y_1), F_{\mathcal{N}Y}(x_2 * y_2)\}, \max\{F_{\mathcal{N}X}(x_1), F_{\mathcal{N}Y}(x_2)\}\} \\ &= \max\{\Lambda(x_1 * y_1, x_2 * y_2), \Lambda(x_1, x_2)\} \\ &= \max\{\Lambda((x_1, x_2) \circ (y_1, y_2)), \Lambda(x_1, x_2)\}. \end{aligned}$$

Hence, $X_{\mathcal{N}} \times Y_{\mathcal{N}}$ is a neutrosophic \mathcal{N} -deductive system of $X \times Y$. □

The following theorem is a straightforward result of Proposition 3.17 and Theorem 3.8.

Theorem 3.18. If $X_{\mathcal{N}} = (X, T_{\mathcal{N}X}, I_{\mathcal{N}X}, F_{\mathcal{N}X})$ and $Y_{\mathcal{N}} = (Y, T_{\mathcal{N}Y}, I_{\mathcal{N}Y}, F_{\mathcal{N}Y})$ are neutrosophic \mathcal{N} -deductive systems of Hilbert algebras X and Y , respectively, then $\oplus(X_{\mathcal{N}} \times Y_{\mathcal{N}})$, $\otimes(X_{\mathcal{N}} \times Y_{\mathcal{N}})$, and $\odot(X_{\mathcal{N}} \times Y_{\mathcal{N}})$ are neutrosophic \mathcal{N} -deductive systems of X .

Let $(X, *, 1_X)$ and $(Y, \star, 1_Y)$ be Hilbert algebras. A mapping $f : X \rightarrow Y$ of Hilbert algebras is called a homomorphism if $f(x * y) = f(x) \star f(y)$ for all $x, y \in X$. Note that if $f : X \rightarrow Y$ is a homomorphism of Hilbert algebras, then $f(1_X) = 1_Y$. Let $f : X \rightarrow Y$ be a homomorphism of Hilbert algebras. For any neutrosophic \mathcal{N} -structure $Y_{\mathcal{N}}$ over Y , we define a new neutrosophic \mathcal{N} -structure $f^{-1}(Y_{\mathcal{N}}) = (X, T_{f^{-1}(Y_{\mathcal{N}})}, I_{f^{-1}(Y_{\mathcal{N}})}, F_{f^{-1}(Y_{\mathcal{N}})})$ over X by

$$(\forall x \in X) \begin{pmatrix} T_{f^{-1}(Y_{\mathcal{N}})}(x) = T_{\mathcal{N}Y}(f(x)) \\ I_{f^{-1}(Y_{\mathcal{N}})}(x) = I_{\mathcal{N}Y}(f(x)) \\ F_{f^{-1}(Y_{\mathcal{N}})}(x) = F_{\mathcal{N}Y}(f(x)) \end{pmatrix}.$$

Theorem 3.19. Let $(X, *, 1_X)$ and $(Y, *, 1_Y)$ be Hilbert algebras. Let $f : X \rightarrow Y$ be a homomorphism and $Y_{\mathcal{N}}$ be a neutrosophic \mathcal{N} -structure over Y . If $Y_{\mathcal{N}}$ is a neutrosophic \mathcal{N} -deductive system of Y , then $f^{-1}(Y_{\mathcal{N}})$ is a neutrosophic \mathcal{N} -deductive system of X .

Proof. Assume that $Y_{\mathcal{N}}$ is a neutrosophic \mathcal{N} -deductive system of Y . Let $x \in X$. Then

$$\begin{aligned} T_{f^{-1}(Y_{\mathcal{N}})}(1_X) &= T_{\mathcal{N}Y}(f(1_X)) = T_{\mathcal{N}Y}(1_Y) \leq T_{\mathcal{N}Y}(f(x)) = T_{f^{-1}(Y_{\mathcal{N}})}(x), \\ I_{f^{-1}(Y_{\mathcal{N}})}(1_X) &= I_{\mathcal{N}Y}(f(1_X)) = I_{\mathcal{N}Y}(1_Y) \geq I_{\mathcal{N}Y}(f(x)) = I_{f^{-1}(Y_{\mathcal{N}})}(x), \\ F_{f^{-1}(Y_{\mathcal{N}})}(1_X) &= F_{\mathcal{N}Y}(f(1_X)) = F_{\mathcal{N}Y}(1_Y) \leq F_{\mathcal{N}Y}(f(x)) = F_{f^{-1}(Y_{\mathcal{N}})}(x). \end{aligned}$$

Let $x, y \in X$. Then

$$\begin{aligned} T_{f^{-1}(Y_{\mathcal{N}})}(y) &= T_{\mathcal{N}Y}(f(y)) \\ &\leq \max\{T_{\mathcal{N}Y}(f(x) * f(y)), T_{\mathcal{N}Y}(f(x))\} \\ &= \max\{T_{\mathcal{N}Y}(f(x * y)), T_{\mathcal{N}Y}(f(x))\} \\ &= \max\{T_{f^{-1}(Y_{\mathcal{N}})}(x * y), T_{f^{-1}(Y_{\mathcal{N}})}(x)\}, \\ I_{f^{-1}(Y_{\mathcal{N}})}(y) &= I_{\mathcal{N}Y}(f(y)) \\ &\geq \min\{I_{\mathcal{N}Y}(f(x) * f(y)), I_{\mathcal{N}Y}(f(x))\} \\ &= \min\{I_{\mathcal{N}Y}(f(x * y)), I_{\mathcal{N}Y}(f(x))\} \\ &= \min\{I_{f^{-1}(Y_{\mathcal{N}})}(x * y), I_{f^{-1}(Y_{\mathcal{N}})}(x)\}, \\ T_{f^{-1}(Y_{\mathcal{N}})}(y) &= T_{\mathcal{N}Y}(f(y)) \\ &\leq \max\{T_{\mathcal{N}Y}(f(x) * f(y)), T_{\mathcal{N}Y}(f(x))\} \\ &= \max\{T_{\mathcal{N}Y}(f(x * y)), T_{\mathcal{N}Y}(f(x))\} \\ &= \max\{T_{f^{-1}(Y_{\mathcal{N}})}(x * y), T_{f^{-1}(Y_{\mathcal{N}})}(x)\}. \end{aligned}$$

Hence, $f^{-1}(Y_{\mathcal{N}})$ is a neutrosophic \mathcal{N} -deductive system of X . □

4 Conclusion

In this paper, we have introduced the notion of neutrosophic \mathcal{N} -deductive systems of Hilbert algebras. Neutrosophic \mathcal{N} -deductive systems and their level subsets have an interesting, straightforward relationship. From our study, we found that the Cartesian product of neutrosophic \mathcal{N} -deductive systems of a Hilbert algebra is a neutrosophic \mathcal{N} -deductive system. Finally, we found that the homomorphic pre-image of a neutrosophic \mathcal{N} -deductive system of a Hilbert algebra is a neutrosophic \mathcal{N} -deductive system.

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