On the Geometry of Weak Fuzzy Complex Numbers and Applications to the Classification of Some A-Curves

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Abstract

The concept of A-curves is considered as a novel application of real field extensions in solving some algebraic vectorial equations defined by Euclidean norms. In this paper, we present a novel insight through the classification of A-curves by illustrating many new semi-module isomorphisms between the direct product of weak fuzzy complex numbers with itself and the direct product of classical Euclidean vector spaces multiplied by itself. These isomorphisms will give us a full classification of A-curves that are related to weak fuzzy complex ring. Also, we provide many examples to explain the contribution of our work.

Keywords: weak fuzzy complex number; weak fuzzy complex vector space; A-curve; vectorial equation.

1. Introduction

Algebraic extensions of real numbers have always been the focus of attention of researchers from everywhere in the world, where in previous research we find concepts dedicated to the construction of extended algebraic groups and rings of real numbers, such as the split-complex numbers, weak fuzzy numbers, neutrosophic numbers, and even n-cyclic numbers [1-2, 8-9].

The weak fuzzy numbers were defined as a new ring expansion of the real numbers [3], defined by the following formula: $\boldsymbol{F}_J = \{x_0 + x_1 J\, ;\ x_0\, , x_1 \in \boldsymbol{R}\, , J^2 = t \in]0,1[\ \}$

Let $X = x_0 + x_1 J, Y = y_0 + y_1 J \in \boldsymbol{F}_J$, where $x_0\, , x_1, y_0, y_1 \in \boldsymbol{R}$

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Addition operation: $X + Y = (x_0 + y_0) + (x_1 + y_1)J$.
Multiplication operation: $X.Y = (x_0 y_0 + x_1 y_1 t) + (x_0 y_1 + x_1 y_0)J$.

These numbers have been studied by many researchers, where in [10,12] we find outstanding efforts to find weak fuzzy Pythagorean triples and quadruples, and solutions to the nonlinear Diophantine equations associated with them.In [13], the computer was programmed to deal with these numbers through the Python environment as a direct application in Computer Science.

Also in [4-5], we find an algebraic study of spaces related to weak fuzzy numbers and applications of inner products to them. In [11], the researchers studied some vector equations defined by the Euclidean norms in Euclidean inner product spaces, where these equations resulted in various algebraic surfaces called A-curves.
As an open research question, the researchers wondered about the possibility of classifying these surfaces by developing the tools used in the study of weak fuzzy numbers.
All this prompted us to present a novel insight through the classification of A-curves by illustrating many new semi-module isomorphisms between the direct product of weak fuzzy complex numbers with itself and the direct product of classical Euclidean vector spaces multiplied by itself. These isomorphisms will give us a full classification of A-curves that are related to weak fuzzy complex ring.

2. Main discussion

Definition:
The ring of weak fuzzy complex numbers is defined as follows.
$W = \{a + bJ;\ J^2 = t \in]0,1[;\ a, b \in R\}$
Addition on W is defined as follows:
$(a + bJ) + (c + dJ) = (a + c) + (b + d)J,$
Multiplication on W is defined as follows:
$(a + bJ)(c + dJ) = (ac + bdt) + (ad + bc)J;\ J^2 = t \in]0,1[.$
Definition:
Let $W^2 = \{(x, y); x, y \in W\}$, define
$(+)\ W^2 \times W^2 \to W^2;\ (x_0, x_1) + (y_0, y_1) = (x_0 + y_0, x_1 + y_1)$
$(.)\ W^2 \times W^2 \to W^2;\ (x_0, x_1).(y_0, y_1) = (x_0 y_0, x_1 y_1)$
$(\times)\ W \times W^2 \to W^2;\ (a + bJ) \times (x_0, x_1) = ((a + bJ)x_0, (a + bJ)x_1),$
Where $x_0, x_1, y_0, y_1 \in W$, $a, b \in R$
It is clear that: $(W^2, +, .)$ is a commutative ring, and $(W^2, +, \times)$ is a module over W, it is called the weak fuzzy complex Euclidean module with two dimensions.
Definition:
Let $W^3 = \{(x, y, z); x, y, z \in W\}$, define:
$(+)\ W^3 \times W^3 \to W^3;\ (x_0, x_1, x_2) + (y_0, y_1, y_3) = (x_0 + y_0, x_1 + y_1, x_2 + y_2)$
$(.)\ W^3 \times W^3 \to W^3;\ (x_0, x_1, x_3).(y_0, y_1, y_3) = (x_0 y_0, x_1 y_1, x_2 y_2)$
$(\times)\ W \times W^3 \to W^3;\ (a + bJ) \times (x_0, x_1, x_2) = ((a + bJ)x_0, (a + bJ)x_1, (a + bJ)x_2)$
It is clear that: $(W^3, +, .)$ is a commutative ring, and $(W^3, +, \times)$ is a module over W, it is called the weak fuzzy complex Euclid module with three dimensions.
Remark:
Consider $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, $\mathbb{R}^2 \times \mathbb{R}^2$, $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$, and the following algebraic operations:
$(+)\ \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2;\ (x_0, x_1) + (y_0, y_1) = (x_0 + y_0, x_1 + y_1)$
$(.)\ \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2;\ (x_0, x_1).(y_0, y_1) = (x_0 y_0, x_1 y_1)$
$(\times)\ \mathbb{R}^2 \times (\mathbb{R}^2 \times \mathbb{R}^2) \to \mathbb{R}^2 \times \mathbb{R}^2;\ (\mathrm{a}, \mathrm{b}) \times ((\mathrm{x}, \mathrm{y}), (\mathrm{z}, \mathrm{t})) = ((\mathrm{ax}, \mathrm{by}), (\mathrm{az}, \mathrm{bt})),$
$(+)\ (\mathbb{R}^2 \times \mathbb{R}^2) \times (\mathbb{R}^2 \times \mathbb{R}^2) \to \mathbb{R}^2 \times \mathbb{R}^2;$
$((x_0, x_1), (y_0, y_1)) + ((x_0', x_1'), (y_0', y_1')) = ((x_0 + x_0', x_1 + x_1'), (y_0 + y_0', y_1 + y_1'))$
$(.)\ (\mathbb{R}^2 \times \mathbb{R}^2) \times (\mathbb{R}^2 \times \mathbb{R}^2) \to \mathbb{R}^2 \times \mathbb{R}^2;$
$((x_0, x_1), (y_0, y_1)).((x_0', x_1'), (y_0', y_1')) = ((x_0 x_0', x_1 x_1'), (y_0 y_0', y_1 y_1'))$
It is known that $(\mathbb{R}^2, +, .)$ is a commutative ring, $(\mathbb{R}^2 \times \mathbb{R}^2, +, .)$ is a commutative ring, $(\mathbb{R}^2 \times \mathbb{R}^2, +, \times)$ is a module over $\mathbb{R}^2$.
Also, $(+)\ (\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2) \times (\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2) \to \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2;$
$((x_0, x_1), (y_0, y_1), (z_0, z_1)) + ((x_0', x_1'), (y_0', y_1'), (z_0', z_1')) = ((x_0 + x_0', x_1 + x_1'), (y_0 + y_0', y_1 + y_1'), (z_0 + z_0', z_1 + z_1'))$
$(.)$: $(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2) \times (\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2) \to \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2;$
$((x_0, x_1), (y_0, y_1), (z_0, z_1)).((x_0', x_1'), (y_0', y_1'), (z_0', z_1')) = ((x_0 x_0', x_1 x_1'), (y_0 y_0', y_1 y_1'), (z_0 z_0', z_1 z_1'))$
$(\times)\ \mathbb{R}^2 \times (\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2) \to \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2;$

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$(a,b) \times \left((x_0,x_1),(y_0,y_1),(z_0,z_1)\right) =$
$((ax_0,\mathrm{b}x_1),(ay_0,by_1),(az_0,bz_1))$
It is clear that $(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2,+,.)$ is a commutative ring, $(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2,+,\times)$ is a module over $\mathbb{R}^2$.

Theorem:
The ring (W, +, .) is isomorphic to $(\mathbb{R}^2,+,.)$.
Proof:
Define: $T_1: W \to \mathbb{R}^2$ such that:
$T_1(x_0,x_1J) = \left(x_0 - x_1\sqrt{t}, x_0 + x_1\sqrt{t}\right); J^2 = t \in]0,1[$
It is clear that T_1 is well defined mapping, and for $X = x_0 + x_1J, Y = y_0 + y_1J \in W$, we have:
$X + Y = (x_0 + y_0) + (x_1 + y_1)J, X.Y = (x_0y_0 + x_1y_1t) + J(x_0y_1 + x_1y_0),$
$T_1(X+Y) = (C_0,C_1); C_0 = (x_0 + y_0) - \sqrt{t}(x_1 + y_1) = (x_0 - \sqrt{t}x_1) + ((y_0 - \sqrt{t}y_1),$
$C_1 = (x_0 + y_0) + \sqrt{t}(x_1 + y_1) = (x_0 + \sqrt{t}x_1) + ((y_0 + \sqrt{t}y_1)$, thus
$T_1(X+Y) = T_1(X) + T_1(Y)$.
$T_1(XY) = (D_0,D_1);\ D_0 = x_0y_0 + x_1y_1t - \sqrt{t}(x_0y_1 + x_1y_0) = (x_0 - \sqrt{t}x_1)(y_0 - \sqrt{t}y_1),\ D_1 = x_0y_0 + x_1y_1t + \sqrt{t}(x_0y_1 + x_1y_0) = (x_0 + \sqrt{t}x_1)(y_0 - \sqrt{t}y_1)$, thus $T_1(XY) = T_1(X)T_1(Y)$.
Assume that $T_1(X) = 0$, then $\begin{cases} x_0 - x_1\sqrt{t} = 0 \\ x_0 + x_1\sqrt{t} = 0 \end{cases}$ so $x_0 = x_1 = 0$
And $ker(T_1) = \{0\}$, which means that T_1 is injective.
$Im(T_1) = \mathbb{R}^2$, hence T_1 is surjective and then it is a bijection.
This implies the proof.
Remark:
$T_1{}^{-1}\mathbb{R}^2 \to W; T_1{}^{-1}(x,y) = \frac{1}{2}(x+y) + \frac{1}{2\sqrt{t}}J(y-x); J^2 = t \in]0,1[$.

Definition:
Let $X = x_0 + x_1J, Y = y_0 + y_1J \in W$, we say that $X \geq Y$ if and only if
$$\begin{cases} x_0 - \sqrt{t}x_1 \geq y_0 - \sqrt{t}y_1 \\ x_0 + \sqrt{t}x_1 \geq y_0 + \sqrt{t}y_1 \end{cases}$$

Example:
Consider X=4+J, y=1+2J with $J^2 = t = \frac{1}{4} \in]0,1[$:
$x_0 - \sqrt{t}x_1 = \frac{7}{2} \geq y_0 - \sqrt{t}y_1 = 0$
$x_0 + \sqrt{t}x_1 = \frac{9}{2} \geq y_0 + \sqrt{t}y_1 = 2$, thus $X \geq Y$.

Definition.
1] $X = x_0 + x_1J \in W$ is called positive weak fuzzy complex number if $X \geq 0$, i.e: $x_0 - \sqrt{t}x_1 \geq 0, x_0 + \sqrt{t}x_1 \geq 0$.
2] X is negative if $X \leq 0$, i.e: $x_0 - \sqrt{t}x_1 \leq 0, x_0 + \sqrt{t}x_1 \leq 0$.

Theorem:
$(\geq)$ is a partial order relation.
Proof:
Let $\mathrm{X} = \mathrm{x}_0 + \mathrm{x}_1\mathrm{J}, \mathrm{Y} = \mathrm{y}_0 + \mathrm{y}_1\mathrm{J}, \mathrm{Z} = \mathrm{z}_0 + \mathrm{z}_1\mathrm{J} \in \mathrm{W}$,
$X \geq X$ that is because: $\begin{cases} x_0 - x_1\sqrt{t} \geq x_0 - \sqrt{t}x_1 \\ x_0 + x_1\sqrt{t} \geq x_0 + \sqrt{t}x_1 \end{cases}$
If $X \geq Y$ and $Y \geq X$, then: $\begin{cases} x_0 - x_1\sqrt{t} \geq y_0 - \sqrt{t}y_1 \\ x_0 + x_1\sqrt{t} \geq y_0 + \sqrt{t}y_1 \end{cases}$ and
$\begin{cases} y_0 - \sqrt{t}y_1 \geq x_0 - \sqrt{t}x_1 \\ y_0 + \sqrt{t}y_1 \geq x_0 + \sqrt{t}y_1 \end{cases}$, this implies that:
$\begin{cases} x_0 - x_1\sqrt{t} \geq y_0 - \sqrt{t}y_1 \\ x_0 + x_1\sqrt{t} \geq y_0 + \sqrt{t}y_1 \end{cases}$, thus $x_0 = y_0, x_1 = y_1$ and X=Y,
If $X \geq Y$ and $Y \geq Z$, then:
$$\begin{cases} x_0 - x_1\sqrt{t} \geq y_0 - y_1\sqrt{t} \geq z_0 - z_1\sqrt{t} \\ x_0 + x_1\sqrt{t} \geq y_0 + y_1\sqrt{t} \geq z_0 + z_1\sqrt{t} \end{cases}$$
Thus $X \geq Z$, and the proof is complect.

Definition:
We define the absolute value of $X = x_0 + x_1\mathrm{J} \in \mathrm{W}$ as follows,
$|X| = \frac{1}{2}\left[\left|x, +\sqrt{t}\,x_1\right| + \left|x_0 - \sqrt{t}x_1\right|\right] + \frac{1}{2\sqrt{t}}J\left[\left|x, +\sqrt{t}\,x_1\right| - \left|x_0 - \sqrt{t}x_1\right|\right]$

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Example:

For $J^2 = t = \frac{1}{9}$, consider: $X = -3 + 6J$, we have:

$$\begin{cases} x_0 - \sqrt{t}x_1 = -3 - \frac{6}{3} = -5 \\ x_0 + \sqrt{t}x_1 = -3 + \frac{6}{3} = -1 \end{cases}$$

$$\Rightarrow |X| = \tfrac{1}{2}(|-5| + |-1|) + \tfrac{1}{(2)(\frac{1}{3})}J[|-1| - |-5|] = 3 + \tfrac{3}{2}J(-4) = 3 - 6J.$$

Remark:

$|X| \geq 0$ according to the partial order relation defined on W for all $X \in$ W

Theorem:

1] $(W^2, +, .) \cong (\mathbb{R}^2 \times \mathbb{R}^2, +, .)$

2] $(W^3, +, .) \cong (\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2, +, .)$

Proof:

1] Define $T_2: W^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ such that:

$T_2(X, Y) = (T_1(x), T_1(y))$; x, y ∈ W

T_2 is well defined, that is because:

If $(x_0, y_0) = (x_1, y_1)$; $x_0, x_1, y_0, y_1 \in$ W, then:

$\begin{cases} x_0 = x_1 \\ y_0 = y_1 \end{cases}$ so that $(T_1(x_0), T_1(y_0)) = (T_1(x_1), T_1(y_1))$

Hence, $T_2(x_0, y_0) = T_2(x_1, y_1)$.

$ker(T_2) = \{(x, y) \in W^2; T_2(x, y) = ((0,0), (0,0))\}$, thus

$\begin{cases} T_1(x) = (0,0) \\ T_1(y) = (0,0) \end{cases}$ which implies that X=Y=0 and $ker(T_2) = \{(0,0)\}$.

Also, $Im(T_2) = (Im(T_1), Im(T_1)) = \mathbb{R}^2 \times \mathbb{R}^2$, thus T_2 is a bijection for $(x_0, y_0), (x_1, y_1) \in W^2$, we have:

$T_2[(x_0, y_0) + (x_1, y_1)] = T_2[(x_0 + x_1, y_0 + y_1)]$

$= (T_1(x_0 + x_1), T_1(y_0 + y_1)) = (T_1(x_0), T_1(y_0)) + (T_1(x_1), T_1(y_1))$

$= T_2(x_0, y_0) + T_2(x_1, y_1)$.

$T_2[(x_0, y_0).(x_1, y_1)] = T_2[(x_0x_1, y_0y_1)] = (T_1(x_0x_1), T_1(y_0y_1)) = (T_1(x_0), T_1(y_0)).(T_1(x_1), T_1(y_1)) = T_2(x_0, y_0).T_2(x_1, y_1)$

Thus T_2 is a ring isomorphism.

2] Define $T_3: W^3 \to \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ such that:

$T_3(X, Y, Z) = (T_1(x), T_1(y), T_1(z))$; x, y, z ∈ W

T_3 can be proved that it is a ring isomorphism by a similar argument of T_2.

Example:

Let $A = (3 + 2J, 1 - 4J) \in W^2$; $J^2 = t = \frac{1}{4} \in]0,1[$, then

$T_2(A) = (T_1(3 + 2J), T_1(1 - 4J)) = ((2,4), (3, -1))$.

Remark that $T_1(3 + 2J) = \left(3 - \left(\frac{1}{2}\right)(2), 3 + \left(\frac{1}{2}\right)(2)\right) = (2,4)$

$$T_1(1 - 4J) = \left(1 + \frac{1}{2}(4), 1 + \frac{1}{2}(-4)\right) = (3, -1)$$

Let $B = (3 + 2J, 1 - 4J, 3 + 2J) \in W^3$, then:

$$T_3(B) = (T_1(3 + 2J), T_1(1 - 4J), T_1(3 + 2J)) = ((2,4), (3, -1), (2,4))$$

Remark:

1] the inverse isomorphism of T_2 is: ${T_2}^{-1}: \mathbb{R}^2 \times \mathbb{R}^2 \to W^2$ such that:

$${T_2}^{-1}((x_0, x_1), (y_0, y_1)) = \left({T_1}^{-1}(x_0, x_1), {T_1}^{-1}(y_0, y_1)\right)$$

$$= \left(\frac{x_0 + x_1}{2} + \frac{J}{2\sqrt{t}}(x_1 - x_0), \frac{y_0 + y_1}{2} + \frac{J}{2\sqrt{t}}(y_1 - y_0)\right)$$

2] the inverse isomorphism of T_3 is: $T_3: \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to W^3$ such that:

$${T_3}^{-1}((x_0, x_1), (y_0, y_1), (z_0, z_1)) = \left({T_1}^{-1}(x_0, x_1), {T_1}^{-1}(y_0, y_1), {T_1}^{-1}(z_0, z_1)\right)$$

Definition: [15-16]

Let M be a module over the ring R, N be a module over T, then $f:: M \to N$ such that: $f(x + y) = f(x) + f(y)$, $f(a.x) = g(a)f(x)$; $g: R \to T$ is called a semi module isomorphism if and only if:

1] f is a bijection.

2] g is a ring isomorphism.

Theorem:

1] the module $(W^2, +, \times)$ is semi isomorphic to $(R^2 \times R^2, +, \times)$.

2] the module $(W^3, +, \times)$ is semi isomorphic to $(R^2 \times R^2 \times R^2, +, \times)$.

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Proof:

1] Consider $T_2: W^2 \to R^2 \times R^2, T_1: W \to R^2$, we have T_2 is a bijection that preserves addition, i.e:

$$T_2[(x_0, x_1) + (y_0, y_1)] = T_2(x_0, x_1) + T_2(y_0, y_1)$$

Also, T_1 is a ring isomorphism between W and R^2.

Now, we must prove that $T_2[A.(x,y)] = T_1(A).T_2(x,y)$ for all $(x,y) \in W^2, A \in W$

$$T_2[A.(x,y)] = T_2(Ax, Ay) = \left(T_1(Ax), T_1(Ay)\right)$$

$$= (T_1(A)T_1(x), T_1(A)T_1(y)) = T_1(A).(T_1(x), T_1(y)) = T_1(A).T_2(x,y)$$

This means that T_2 is a semi module isomorphism.

2] It can be proved by a similar argument by considering

$T_3: W^3 \to \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ and $T_1: W \to \mathbb{R}^2$

The classification of A-curves $C_{A_t}; t \in]0,1[$

Definition: [7]

1] The A_t-Curve in $\mathbb{R}^2$ is defined by the following Vectorial equation:

$$\begin{cases} \|x\|^2 + t.\|y\|^2 = r_1 \\ x_1x_2 + y_1y_2 = r_2 \end{cases}; X = (x_1, y_1), Y = (x_2, y_2) \in \mathbb{R}^2, t \in]0,1[$$

$$r_1, r_2 \geq 0$$

2] The A_t-Curve in $\mathbb{R}^3$ is defined by the following Vectorial equation:

$$\begin{cases} \|x\|^2 + t.\|y\|^2 = r_1 \\ x_1x_2 + y_1y_2 + z_1z_2 = r_2 \end{cases}; X = (x_1, y_1, z_1), Y = (x_2, y_2, z_2) \in \mathbb{R}^3,$$

$$t \in]0,1[, \ r_1, r_2 \geq 0$$

Theorem: [7]

1] C_{A_t} in $\mathbb{R}^2$ is equivalent to a weak fuzzy complex circle: $X^2 + Y^2 = r_1 + 2r_1J; J^2 = t, X = x_1 + x_2J, \ Y = y_1 + y_2J \in W$

2] C_{A_t} in $\mathbb{R}^3$ is equivalent to a weak fuzzy complex sphere: $X^2 + Y^2 + Z^2 = r_1 + 2r_2J;$ $X = x_1 + x_2J, \ Y = y_1 + y_2J, \ Z = z_1 + z_2J \in W$

Theorem:

1] C_{A_t} in $\mathbb{R}^2$ is isomorphic to the cartesian product to two circles in the Educlidean space $\mathbb{R}^2$, i.e: for every single point $(x', y') \in C_{A_t}$ there exists only two points(up to isomorphism) and each point belongs to an ordinary Educlidean circle, if $r_1 - 2r_2\sqrt{t} > 0$.

Also, it is isomorphic to an empty set if $r_1 - 2r_2\sqrt{t} < 0$, and it is isomorphic to a single point if $r_1 = r_2 = 0$.

1] C_{A_t} in $\mathbb{R}^3$ is isomorphic to the cartesian product of two classical Educlidean spheres if $r_1 - 2r_2\sqrt{t} > 0$, and to an empty set if $r_1 - 2r_2\sqrt{t} < 0$ and to a single point if $r_1 = r_2 = 0$.

Proof:

1] the equation of C_{A_t} in $\mathbb{R}^2$ is:

$X^2 + Y^2 = r_1 + 2r_2J; X = x_1 + x_2J, \ Y = y_1 + y_2J, J^2 = t \in]0,1[$.

By applying the isomorphism T_1, we get:

$T_1(X^2) + T_1(Y^2) = T_1(r_1 + 2r_2J)$, hence:

$[T_1(x)]^2 + [T_1(Y)]^2 = T_1(r_1 + 2r_2J)$, thus:

$$\begin{cases} (x_1 - x_2\sqrt{t})^2 + (y_1 - y_2\sqrt{t})^2 = r_1 - 2r_2J \\ (x_1 + x_2\sqrt{t})^2 + (y_1 + y_2\sqrt{t})^2 = r_1 + 2r_2J \end{cases}$$

We change the variables as follows:

$$\begin{cases} x_1 - x_2\sqrt{t} = X_1 \\ x_1 + x_2\sqrt{t} = X_2 \end{cases}, \begin{cases} y_1 - y_2\sqrt{t} = Y_1 \\ y_1 + y_2\sqrt{t} = Y_2 \end{cases}$$

Then,

$$\begin{cases} {X_1}^2 + {Y_1}^2 = r_1 - 2r_2\sqrt{t} \\ {X_2}^2 + {Y_2}^2 = r_1 + 2r_2\sqrt{t} \end{cases}$$

According to the assumption, we have $r_1, r_2 \geq 0$, this implies the following cases,

Case 1:

If $r_1 = r_2 = 0$, then C_{A_t} has only one point (0,0).

Case 2:

If $r_1 - 2r_2\sqrt{t} < 0$, then C_{A_t} is an empty set

Case 3:

If $r_1 - 2r_2\sqrt{t} > 0$, then C_{A_t} is isomorphic to the direct product of the two classical circles:

$$\begin{cases} {X_1}^2 + {Y_1}^2 = r_1 - 2r_2\sqrt{t} \\ {X_1}^2 + {Y_1}^2 = r_1 + 2r_2\sqrt{t} \end{cases}$$

2] C_{A_t} in $\mathbb{R}^3$ has the following equation:

$X^2+Y^2+Z^2=r_1+2r_2J; J^2=t\in]0,1[\ X=x_1+x_2J,\ Y=y_1+y_2J, Z=z_1+z_2J\in W.$

By using T_1, we get:

$T_1(X^2)+T_1(Y^2)+T_1(Z^2)=T_1(r_1+2r_2J)$, thus:

$$\begin{cases}(x_1-\sqrt{t}x_2)^2+(y_1-\sqrt{t}y_2)^2+(z_1-\sqrt{t}z_2)^2=r_1-2r_2\sqrt{t}\\ (x_1+\sqrt{t}x_2)^2+(y_1+\sqrt{t}y_2)^2+(z_1+\sqrt{t}z_2)^2=r_1+2r_2\sqrt{t}\end{cases}$$

If $r_1-2r_2\sqrt{t}>0$, we get two classical Euclidean spheres.

If $r_1=r_2=0$, we get a single point $(0,0)$.

If $r_1-2r_2\sqrt{t}<0$, we get an empty set

Example:

Consider the following C_{A_t}-Curve in $\mathbb{R}^2$

$\begin{cases}\|x\|^2+\|y\|^2t=4\\ x_1x_2+y_1y_2=2\end{cases}$ with $t=\frac{1}{4}\in]0,1[; X=(x_1,y_1), Y=(x_2,y_2)\in\mathbb{R}^3,$

With the following representation:

$X_J{}^2+Y_J{}^2=4+4J(C_J); X_J=x_1+x_2J, Y_J=y_1+y_2J$

We have $r_1=4, r_2=2, r_1+2r_2\sqrt{t}=6, r_1-2r_2\sqrt{t}=2>0$, Hence $C_{A\frac{1}{4}}$ is equivalent to the following classical circles:

$\begin{cases}X_1{}^2+Y_1{}^2=2(C_1)\\ X_2{}^2+Y_2{}^2=6(C_2)\end{cases}; X_1,Y_1,X_2,Y_1\in\mathbb{R}$, and $X_1=x_1-x_2\sqrt{t}$

$Y_2=y_1+y_2\sqrt{t}\ , X_2=x_1+x_2\sqrt{t}, Y_1=y_1-y_2\sqrt{t}$

For example: $(1,1)\in C_1, (\sqrt{6},0)\in C_2$, i.e:

$x_1=1, x_2=\sqrt{6}, y_1=1, y_2=0$, this generates a point in $\mathbb{R}^2\times\mathbb{R}^2$ as follows:

$$=\big((x_1,x_2),(y_1,y_2)\big)=\big((1,\sqrt{6}),(1,0)\big)$$

The inverse image $T_2{}^{-1}(L)$ generals a point on C_J as follows

$$T_2{}^{-1}(L)=\left(\frac{1+\sqrt{6}}{2}+J(\sqrt{6}-1),\frac{1}{2}-J\right)\in C_J$$

Remark that: $\left(\frac{1+\sqrt{6}}{2}+J(\sqrt{6}-1)\right)^2+(\frac{1}{2}-J)^2=4+4J$

This point gives us a solution to the original vectorial equation $\begin{cases}\|x\|^2+\|y\|^2t=4\\ x_1x_2+y_1y_2=2\end{cases}; \ t=\frac{1}{4}$

This solution $X=(x_1,y_1)=(\frac{1+\sqrt{6}}{2},\frac{1}{2}), Y=(x_2,y_2)=(\sqrt{6}-1,-1)$

Example:

Let's find a solution of the vectorial equation:

$\begin{cases}\|x\|^2+t.\|y\|^2=9\\ x_1x_2+y_1y_2+z_1z_2=6\end{cases}; X=(x_1,y_1,z_1), Y=(x_2,y_2,z_2)\in\mathbb{R}^3\ t=\frac{1}{9}\in]0,1[.$

Any solution to the previous vectorial equation is equivalent to a single point on $C_{A_{\frac{1}{9}}}$ in $\mathbb{R}^3$.

The weak fuzzy complex formula of it is:

$C_J)X_J{}^2+Y_J{}^2+Z_J{}^2=9+12J; X_J{}^2=x_1+x_2J, Y_J{}^2=y_1+y_2J, Z_J{}^2=z_1+z_2J\in W$ on the other hand, we have $r_1=9, r_2=6, r_1+2r_2\sqrt{t}=9+2(6)(\frac{1}{3})=13, r_1-2r_2\sqrt{t}=9-2(6)(\frac{1}{3})=5>0$, hence C_J can be represented by the following classical spheres:

$\begin{cases}X_1{}^2+Y_1{}^2+Z_1{}^2=5(C_1)\\ X_2{}^2+Y_2{}^2+Z_2{}^2=13(C_2)\end{cases}$, where $\begin{cases}X_1=x_1-x_2\sqrt{t}=x_1-\frac{x_2}{3}\in\mathbb{R}\\ Y_1=y_1-y_2\sqrt{t}=y_1-\frac{y_2}{3}\in\mathbb{R}\\ Z_1=z_1-z_2\sqrt{t}=z_1-\frac{z_2}{3}\in\mathbb{R}\end{cases}$

And: $\begin{cases}X_2=x_1+x_2\sqrt{t}=x_1+\frac{x_2}{3}\in\mathbb{R}\\ Y_2=y_1+y_2\sqrt{t}=y_1+\frac{y_2}{3}\in\mathbb{R}\\ Z_2=z_1+z_2\sqrt{t}=z_1+\frac{z_2}{3}\in\mathbb{R}\end{cases}$

$(2,1,0)\in C_1, (3,0,2)\in C_2$, hence:

$$L=\big((x_1,x_2),(y_1,y_2),(z_1,z_2)\big)=\big((2,3),(1,0),(0,2)\big)\in R^2\times R^2\times R^2$$

The inverse image $T_3{}^{-1}(L)=(\frac{5}{2}+\frac{3}{2}J,\frac{1}{2}-\frac{3}{2}J,1+3J)$is a point on (C_J).

Thus: $X=(x_1,y_1,z_1)=(\frac{5}{2},\frac{1}{2},1), Y=(x_2,y_2,z_2)=(\frac{3}{2},-\frac{3}{2},3)$ is a solution to the vectorial equation.

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3. Conclusion

In this article, we have studied the classification problem of A-curves generated from the Euclidean vectorial equation $\begin{cases} \|x\|^2 + t.\|y\|^2 = r \\ x_1x_2 + y_1y_2 + z_1z_2 = k \end{cases}$; by using some novel suggested semi-module homomorphisms and semi-module isomorphisms that helped in formulating weak fuzzy complex numbers as real vectors. Also, we have presented many examples to explain the contribution of our work.

In the future, we aim to classify weak fuzzy complex inner product spaces by using the same semi-homomorphisms.

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