# An Algebraic Approach to the Symbolic 5-Plithogenic Vector Spaces

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## Abstract

The objective of this paper is to study for the first time the concept of symbolic 5-plithogenic vector space defined over symbolic 5-plithogenic field. Many results about the algebraic properties of this class of spaces will be obtained, where we define AH-subspaces, AH-linear transformations, and kernels. Also, we study the inner products defined over symbolic 5-plithogenic vector spaces and determine the conditions of orthogonality in these spaces with many interesting examples.

**Keywords:** symbolic 5-plithogenic vector space; AH-kernel; basis; plithogenic set

## 1. Introduction

The study of algebraic structures is one of the most important branches in theoretical mathematics, due to the effective applications it provides in many other sciences [1-3]. In previous years, many types of groups have been used in the definition and study of algebraic structures and their generalizations, where we find that fuzzy and neutrosophic sets have been used by many researchers around the world in the study of more complex patterns of algebraic structures, for example vector spaces, matrices, and algebraic rings [4-9, 15-19].

The theory of plithogenic sets appeared in [10] and has aroused interest because of its possible use in generalizing classical algebraic structures similarly to refined neutrosophic algebraic structures, see [11-14]. Plithogenic sets have been widely used, by which plithogenic rings, plithogenic matrices, spaces and modules of the same type are defined [20-31].

This is such a motivation for us to make a comprehensive study on symbolic 5-plithogenic vector spaces, and their inner products to present many good results about orthogonality, functions, and AH-substructures.

## 2. Main Results

**Definition:**
Let $V$ a vector space over the field $F$, and let $5-SP_F$ be the corresponding symbolic 5-plithogenic field defined as follows:
$5-SP_F=\left\{a_0+\sum_{i=1}^{5}a_iP_i\,;a_i\in F\right\}$, we define the symbolic 5-plithogenic vector space as follows:

$$5-SP_V=\left\{t_0+\sum_{i=1}^{5}t_iP_i\,;t_i\in V\right\}$$

**Example.**
For $V=R^2$, we have:

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$$5-SP_V=\left\{(x_0,y_0)+\sum_{i=1}^{5}(x_i,y_i)P_i\,;x_i,y_i\in R\right\}$$

For example, $V_1=(2,0)+(1,1)P_1+(-3,2)P_2+(0,4)P_3+(4,0)P_4+(1,1)P_5$ is an element of $5-SP_V$.

**Definition.**

The addition on $5-SP_V$ is defined as follows:

$\left(t_0+\sum_{i=1}^{5}t_iP_i\right)+\left(s_0+\sum_{i=1}^{5}s_iP_i\right)=(t_0+s_0)+\sum_{i=1}^{5}(t_i+s_i)P_i\,;t_j,s_j\in V.$

External multiplication on $5-SP_V$ is defined as follows:

$\left(a_0+\sum_{i=1}^{5}a_iP_i\right)\times\left(t_0+\sum_{i=1}^{5}t_iP_i\right)=a_0t_0+\sum_{i,j=1}^{5}a_it_jP_{max(i,j)}$, where $a_i\in F,t_i\in V$.

**Example.**

For $5-SP_{R^2}$, consider $X=(1,1)+(1,-1)P_1+(2,0)P_2+(0,2)P_3+(1,1)P_4+(0,1)P_5$, $A=1+P_1-P_2+2P_3-P_4+P_5$, $Y=(0,0)+(0,0)P_1+(0,0)P_2+(1,5)P_3+(2,4)P_4+(1,3)P_5$

We have:

$X+Y=(1,1)+(1,-1)P_1+(2,0)P_2+(1,7)P_3+(3,5)P_4+(1,4)P_5$

$$\begin{aligned}A.X=(1,1)&+[(1,-1)+(1,1)+(1,-1)]P_1+[(2,0)+(2,0)-(2,0)-(1,1)-(1,-1)]P_2\\&+[(0,2)+(0,2)-(0,2)+2(0,2)+2(1,1)+2(1,-1)+2(2,0)]P_3\\&+[(1,1)+(1,1)-(1,1)+2(1,1)-(1,1)-(1,1)-(1,-1)-(2,0)-(0,2)]P_4\\&+[(0,1)+(0,1)-(0,1)+2(0,1)-(0,1)+(0,1)+(1,1)+(1,-1)+(0,2)+(2,0)\\&+(1,1)]P_5=(1,1)+(3,-1)P_1+(0,0)P_2+(8,6)P_3+(-2,2)P_4+(5,6)P_5\end{aligned}$$

**Theorem1.**

$(5-SP_V,+,.)$ Is module over $5-SP_F$.

**Definition.**

Let $w_j;0\le j\le 5$ be subspaces of $V$, then:

$W=\left\{x_0+\sum_{i=1}^{5}x_iP_i\,;x_i\in w_i\right\}$ is called symbolic 5-plithogenic AH-subspace.

If $w_j=w_s$ for all $0\le j,s\le 5$, then $W$ is called AHS-subspace.

**Theorem2.**

Let $W=w_0+\sum_{i=1}^{5}w_iP_i$ be an AHS-subspace of $5-SP_V$, hence $W$ is submodule of $5-SP_V$.

**Definition.**

Let $L_j:V\to T$ be linear transformation between $V,T;0\le j\le 5$, we define the AH-linear transformation s follows:

$L=L_0+\sum_{i=1}^{5}L_iP_i:5-SP_V\to 5-SP_T$ such that:

$L\left(x_0+\sum_{i=1}^{5}x_iP_i\right)=L_0(x_0)+\sum_{i=1}^{5}L_i(x_i)P_i.$

If $L_j=L_k$ for all $0\le j,k\le 5$, then $L$ is called AHS-linear transformation.

**Definition.**

Let $L=L_0+\sum_{i=1}^{5}L_iP_i:5-SP_V\to 5-SP_T$ be an AHS-linear transformation, we define:

1). $AH-ker(L)=ker(L_0)+\sum_{i=1}^{5}ker(L_i)P_i.$

2). $AH-Im(L)=Im(L_0)+\sum_{i=1}^{5}Im(L_i)P_i$

If $L$ is an AHS-liner transformation, then we get the AHS-kernel and AHS image.

**Theorem3.**

Let $L$ be an AHS-liner transformation such that $L:5-SP_V\to 5-SP_T$, then $L$ is a module homomorphism.

**Theorem4.**

Let $L$ be an AH-liner transformation, then:

1). $AH-ker(L)$ is n AH-subspace of $5-SP_V$.

2). $AH-Im(L)$ is n AH-subspace of $5-SP_T$.

**Theorem5.**

Let $L$ be an AHS-liner transformation, then:

1). $AHS-ker(L)$ is n AH-submodule of $5-SP_V$.

2). $AH-Im(L)$ is n AH-submodule of $5-SP_T$.

**Definition.**

Let $g:V\times V\to R$ be a real inner product defined on $V$, we define the corresponding symbolic 5-plithogenic real inner product as follows:

$G:5-SP_V\times 5-SP_V\to 5-SP_R$ such that:

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$$\begin{aligned} G&\left[\left(x_0+\sum_{i=1}^{5}x_iP_i\right),\left(y_0+\sum_{i=1}^{5}y_iP_i\right)\right] \\ &= g(x_0,y_0)+\left[g\left(\sum_{i=0}^{1}x_i,\sum_{i=0}^{1}y_i\right)-g(x_0,y_0)\right]P_1 \\ &+\left[g\left(\sum_{i=0}^{2}x_i,\sum_{i=0}^{2}y_i\right)-g\left(\sum_{i=0}^{1}x_i,\sum_{i=0}^{1}y_i\right)\right]P_2+\left[g\left(\sum_{i=0}^{3}x_i,\sum_{i=0}^{3}y_i\right)-g\left(\sum_{i=0}^{2}x_i,\sum_{i=0}^{2}y_i\right)\right]P_3 \\ &+\left[g\left(\sum_{i=0}^{4}x_i,\sum_{i=0}^{4}y_i\right)-g\left(\sum_{i=0}^{3}x_i,\sum_{i=0}^{3}y_i\right)\right]P_4+\left[g\left(\sum_{i=0}^{5}x_i,\sum_{i=0}^{5}y_i\right)-g\left(\sum_{i=0}^{4}x_i,\sum_{i=0}^{4}y_i\right)\right]P_5 \end{aligned}$$

**Definition.**

Let $X = x_0 + \sum_{i=1}^{5} x_iP_i \in 5 - SP_V$, then we define:

1). $X \perp Y$ if and only if $G(X,Y) = 0$

2). $\|X\|^2 = G(X,X)$

**Theorem6.**

Let $G: 5 - SP_V \times 5 - SP_V \to 5 - SP_R$ be a symbolic 5-plithogenic real inner product, then:

1). $X \perp Y$ if and only if $x_0 \perp y_0, \sum_{i=0}^{k} x_i \perp \sum_{i=0}^{k} y_i$.

2). $\|X\| \geq 0$, and

$\|X\| = \|x_0\| + [\|\sum_{i=0}^{1} x_i\| - \|x_0\|]P_1 + [\|\sum_{i=0}^{2} x_i\| - \|\sum_{i=0}^{1} x_i\|]P_2 + \left[\left\|\sum_{i=0}^{3} x_i\right\| - \|\sum_{i=0}^{2} x_i\|\right]P_3 + \left[\|\sum_{i=0}^{4} x_i\| - \left\|\sum_{i=0}^{3} x_i\right\|\right]P_4 + \left[\left\|\sum_{i=0}^{5} x_i\right\| - \|\sum_{i=0}^{4} x_i\|\right]P_5$

3). For $A = a_0 + \sum_{i=1}^{5} a_iP_i$, we have $G(X,Y) = AG(X,Y)$ and $A.\|X\| = |A|.\|X\|$.

4). $\|X + Y\| \leq \|X\| + \|Y\|$.

**Theorem7.**

Let $G: 5 - SP_V \times 5 - SP_V \to 5 - SP_R$ be a symbolic 5-plithogenic real inner product, then:

1). If $X \perp Y$, then $\|X + Y\|^2 = \|X\|^2 + \|Y\|^2$.

2). $|G(X,Y)| \leq \|X\|.\|Y\|$.

Now, we show the proofs of theorems.

**Proof of theorem1.**

Let $X = x_0 + \sum_{i=1}^{5} x_iP_i, Y = y_0 + \sum_{i=1}^{5} y_iP_i, Z = z_0 + \sum_{i=1}^{5} z_iP_i \in 5 - SP_V$.

$$X + Y = (x_0 + y_0) + \sum_{i=1}^{5}(x_i + y_i)P_i = (y_0 + x_0) + \sum_{i=1}^{5}(y_i + x_i)P_i = Y + X$$

$X.1 = X, -X = -x_0 + \sum_{i=1}^{5}(-x_i)P_i$ such that $X + (-X) = O$.

$X + O = O + X = X$.

Let $A = a_0 + \sum_{i=1}^{5} a_iP_i, B = b_0 + \sum_{i=1}^{5} b_iP_i \in 5 - SP_R$, then:

$$\begin{aligned}(A + B).X &= \left[(a_0 + b_0) + \sum_{i=1}^{5}(a_i + b_i)\,P_i\right]\left(x_0 + \sum_{i=1}^{5} x_iP_i\right) = (a_0 + b_0)x_0 + \sum_{i,j=1}^{5}(a_i + b_i)x_j\,P_{max(i,j)} \\ &= AX + BX\end{aligned}$$

$$\begin{aligned}A.(X + Y) &= \left(a_0 + \sum_{i=1}^{5} a_iP_i\right)\left[(x_0 + y_0) + \sum_{i=1}^{5}(x_i + y_i)\,P_i\right] = a_0(x_0 + y_0) + \sum_{i,j=1}^{5} a_i\big(x_j + y_j\big)\,P_{max(i,j)} \\ &= A.X + A.Y\end{aligned}$$

$$(A.B)X = \left(a_0b_0 + \sum_{i,j=1}^{5} a_ib_j\,P_{max(i,j)}\right)\left(x_0 + \sum_{i=1}^{5} x_iP_i\right) = a_0b_0x_0 + \sum_{i,j,k=1}^{5} a_ib_j\,x_kP_{max(i,j)} = A(B.X)$$

Thus, the proof is complete.

**Proof of theorem2.**

Let $X = x_0 + \sum_{i=1}^{5} x_iP_i, Y = y_0 + \sum_{i=1}^{5} y_iP_i \in W$, and $A = a_0 + \sum_{i=1}^{5} a_iP_i \in 5 - SP_R$, then:

$X - Y = (x_0 - y_0) + \sum_{i=1}^{5}(x_i - y_i)\,P_i; x_j - y_j \in w_j; 0 \leq j \leq 5$, hence $X - Y \in W$.

$A.X = a_0x_0 + \sum_{i,j=1}^{5} a_ix_j\,P_{mx(i,j)} \in W$, that is because $a_ix_j \in w_j$.

**Example on theorem2.**

Let $V = R^2$, then $V_1 = \{(0,a); a \in R\}$ is a subspace of $V$.

This means that:

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$W = V_1 + V_1P_1 + V_1P_2 + V_1P_3 + V_1P_4 + V_1P_5 = \{(0, a_0) + (0, a_1)P_1 + (0, a_2)P_2 + (0, a_3)P_3 + (0, a_4)P_4 + (0, a_5)P_5; a_i \in R\}$ is a submodule of $5 - SP_{R^2}$.

**Proof of theorem3.**

Let $L = l_0 + \sum_{i=1}^{5} l_i P_i$ be an AHS-linear transformation, let $X = x_0 + \sum_{i=1}^{5} x_i P_i$, $Y = y_0 + \sum_{i=1}^{5} y_i P_i \in 5 - SP_V$, $A = a_0 + \sum_{i=1}^{5} a_i P_i \in 5 - SP_R$, then:

$$L(X+Y) = L\left[(x_0 + y_0) + \sum_{i=1}^{5} (x_i + y_i)\, P_i\right] = L_0(x_0 + y_0) + \sum_{i=1}^{5} L_0(x_i + y_i)\, P_i$$

$$= \left[L_0(x_0) + \sum_{i=1}^{5} L_0(x_i)\, P_i\right] + \left[L_0(y_0) + \sum_{i=1}^{5} L_0(y_i)\, P_i\right] = L(X) + L(Y)$$

$$L(A.X) = L\left[(a_0x_0) + \sum_{i,j=1}^{5} a_i x_j\, P_{max(i,j)}\right] = L_0(a_0x_0) + \sum_{i=1}^{5} L_0\left(a_i x_j\right) P_{max(i,j)}$$

$$= a_0 L_0(x_0) + \sum_{i=1}^{5} a_i L_0\left(x_j\right) P_{max(i,j)} = A.L(X)$$

**Proof of theorem4.**

1). Since $ker(L_i)$ is a subspace of $V$, then:

$AH - ker(L) = ker(L_0) + \sum_{i=1}^{5} ker(L_i)\, P_i$ is an AH-subspace of $5 - SP_V$.

2). Since $Im(L_i)$ is a subspace of $T$, then:

$AH - Im(L) = Im(L_0) + \sum_{i=1}^{5} Im(L_i)\, P_i$ is an AH-subspace of $5 - SP_T$.

**Proof of theorem5.**

It holds directly from theorem4 and theorem2.

**Example.**

Let $V = R^2, T = R^2, 5 - SP_V, 5 - SP_T$ are the corresponding symbolic 5-plithogenic spaces.

$$\begin{cases} L_0: V \to T; L_0(x, y) = (3x, -x) \\ L_1: V \to T; L_1(x, y) = (x - y, 0) \end{cases}$$

Hence $L: 5 - SP_V \to 5 - SP_T$ such that:

$$L\left[(x_0, y_0) + \sum_{i=1}^{5} (x_i, y_i)\, P_i\right]$$
$$= L_0(x_0, y_0) + L_0(x_1, y_1)P_1 + L_1(x_2, y_2)P_2 + L_1(x_3, y_3)P_3 + L_0(x_4, y_4)P_4 + L_1(x_5, y_5)P_5$$
$$= (3x_0, -x_0) + L_0(3x_1, -x_1)P_1 + L_1(x_2 - y_2, 0)P_2 + L_1(x_3 - y_3, 0)P_3 + L_0(3x_4, -x_4)P_4 + L_1(x_5 - y_5, 0)P_5$$

$L$ is an AH-linear transformation.

$ker(L_0) = \{(0, t); t \in R\}, ker(L_1) = \{(s, s); s \in R\}$.

$AH - ker(L) = \{(0, t_0) + (0, t_1)P_1 + (t_2, t_2)P_2 + (t_3, t_3)P_3 + (0, t_4)P_4 + (t_5, t_5)P_5; t_i \in R\}$

**Remark.**

We say that $A = a_0 + \sum_{i=1}^{5} a_i\, P_i \geq B = b_0 + \sum_{i=1}^{5} b_i\, P_i; a_i, b_i \in R$ if and only if $a_0 \geq b_0, \sum_{i=0}^{k} a_i \geq \sum_{i=0}^{k} b_i\,; 1 \leq k \leq 5$

**Proof of theorem6.**

1). $G(X, Y) = 0$ if and only if:

$$\begin{cases} G(x_0, y_0) = 0 \\ \sum_{i=0}^{k} (x_0, y_0) = 0; 1 \leq k \leq 5 \end{cases}$$

Hence $x_0 \perp y_0, \sum_{i=0}^{k} x_i \perp \sum_{i=0}^{k} y_i\,; 1 \leq k \leq 5$.

2). $\|X\|^2 = G(X, X) = \|x_0\|^2 + \sum_{i=1}^{5} \left(\left\|\sum_{j=0}^{k} x_j\right\|^2 - \left\|\sum_{j=0}^{k-1} x_j\right\|^2\right) P_i; 1 \leq k \leq 5$

Hence:

$$\|X\| = \|x_0\| + \left[\left\|\sum_{i=0}^{1} x_i\right\| - \|x_0\|\right] P_1 + \left[\left\|\sum_{i=0}^{2} x_i\right\| - \left\|\sum_{i=0}^{1} x_i\right\|\right] P_2 + \left[\left\|\sum_{i=0}^{3} x_i\right\| - \left\|\sum_{i=0}^{2} x_i\right\|\right] P_3$$
$$+ \left[\left\|\sum_{i=0}^{4} x_i\right\| - \left\|\sum_{i=0}^{3} x_i\right\|\right] P_4 + \left[\left\|\sum_{i=0}^{5} x_i\right\| - \left\|\sum_{i=0}^{4} x_i\right\|\right] P_5 \geq 0$$

3). $G(AX,Y)=G\left[\left(a_0x_0+\sum_{i,j=1}^{5}a_ix_j\,P_{max(i,j)},y_0+\sum_{i,j=1}^{5}y_i\,P_i\right)\right]=g(a_0x_0,y_0)+[g(a_0x_0+a_1x_1,y_0)-g(a_0x_0,y_0)]P_1+[g(\sum_{i=0}^{2}a_ix_i,\sum_{i=0}^{2}y_i)-g(\sum_{i=0}^{1}a_ix_i,\sum_{i=0}^{1}y_i)]P_2+[g(\sum_{i=0}^{3}a_ix_i,\sum_{i=0}^{3}y_i)-g(\sum_{i=0}^{2}x_i,\sum_{i=0}^{2}y_i)]P_3+[g(\sum_{i=0}^{4}a_ix_i,\sum_{i=0}^{4}y_i)-g(\sum_{i=0}^{3}a_ix_i,\sum_{i=0}^{3}y_i)]P_4+\left[g\left(\sum_{i=0}^{5}a_ix_i,\sum_{i=0}^{5}y_i\right)-g(\sum_{i=0}^{4}a_ix_i,\sum_{i=0}^{4}y_i)\right]P_5=A.G(X,Y)$

$\|A.X\|^2=G(AX,AX)=A^2G(X,X)=A^2\|X\|^2$, thus $\|A.X\|=|A|.\|X\|$, where

$$|A|=|a_0|+\left[\left|\sum_{i=0}^{1}a_i\right|-|a_0|\right]P_1+\left[\left|\sum_{i=0}^{2}a_i\right|-\left|\sum_{i=0}^{1}a_i\right|\right]P_2+\left[\left|\sum_{i=0}^{3}a_i\right|-\left|\sum_{i=0}^{2}a_i\right|\right]P_3+\left[\left|\sum_{i=0}^{4}a_i\right|-\left|\sum_{i=0}^{3}a_i\right|\right]P_4+\left[\left|\sum_{i=0}^{5}a_i\right|-\left|\sum_{i=0}^{4}a_i\right|\right]P_5$$

$$\begin{aligned}\|X+Y\|=&\|x_0+y_0\|+\left[\left\|\sum_{i=0}^{1}(x_i+y_i)\right\|-\|x_0+y_0\|\right]P_1+\left[\left\|\sum_{i=0}^{2}(x_i+y_i)\right\|-\left\|\sum_{i=0}^{1}(x_i+y_i)\right\|\right]P_2\\&+\left[\left\|\sum_{i=0}^{3}(x_i+y_i)\right\|-\left\|\sum_{i=0}^{2}(x_i+y_i)\right\|\right]P_3+\left[\left\|\sum_{i=0}^{4}(x_i+y_i)\right\|-\left\|\sum_{i=0}^{3}(x_i+y_i)\right\|\right]P_4\\&+\left[\left\|\sum_{i=0}^{5}(x_i+y_i)\right\|-\left\|\sum_{i=0}^{4}(x_i+y_i)\right\|\right]P_5\\&\le\|x_0\|+\|y_0\|+\left[\left\|\sum_{i=0}^{1}x_i\right\|+\left\|\sum_{i=0}^{1}y_i\right\|-\|x_0\|-\|y_0\|\right]P_1\\&+\left[\left\|\sum_{i=0}^{2}x_i\right\|+\left\|\sum_{i=0}^{2}y_i\right\|-\left\|\sum_{i=0}^{1}x_i\right\|-\left\|\sum_{i=0}^{1}y_i\right\|\right]P_2\\&+\left[\left\|\sum_{i=0}^{3}x_i\right\|+\left\|\sum_{i=0}^{3}y_i\right\|-\left\|\sum_{i=0}^{2}x_i\right\|-\left\|\sum_{i=0}^{2}y_i\right\|\right]P_3\\&+\left[\left\|\sum_{i=0}^{4}x_i\right\|+\left\|\sum_{i=0}^{4}y_i\right\|-\left\|\sum_{i=0}^{3}x_i\right\|-\left\|\sum_{i=0}^{3}y_i\right\|\right]P_4\\&+\left[\left\|\sum_{i=0}^{5}x_i\right\|+\left\|\sum_{i=0}^{5}y_i\right\|-\left\|\sum_{i=0}^{4}x_i\right\|-\left\|\sum_{i=0}^{4}y_i\right\|\right]P_5=\|X\|+\|Y\|\end{aligned}$$

**Proof of theorem7.**

1). Let $X\perp Y$, then:

$\|X+Y\|^2=G(X+Y,X+Y)=G(X,X)+G(Y,Y)+2G(X,Y)=\|X\|^2+\|Y\|^2$

2).

$$\begin{aligned}|G(X,Y)|=&|g(x_0,y_0)|+\left[\left|g\left(\sum_{i=0}^{1}x_i,\sum_{i=0}^{1}y_i\right)\right|-|g(x_0,y_0)|\right]P_1\\&+\left[\left|g\left(\sum_{i=0}^{2}x_i,\sum_{i=0}^{2}y_i\right)\right|-\left|g\left(\sum_{i=0}^{1}x_i,\sum_{i=0}^{1}y_i\right)\right|\right]P_2\\&+\left[\left|g\left(\sum_{i=0}^{3}x_i,\sum_{i=0}^{3}y_i\right)\right|-\left|g\left(\sum_{i=0}^{2}x_i,\sum_{i=0}^{2}y_i\right)\right|\right]P_3\\&+\left[\left|g\left(\sum_{i=0}^{4}x_i,\sum_{i=0}^{4}y_i\right)\right|-\left|g\left(\sum_{i=0}^{3}x_i,\sum_{i=0}^{3}y_i\right)\right|\right]P_4\\&+\left[\left|g\left(\sum_{i=0}^{5}x_i,\sum_{i=0}^{5}y_i\right)\right|-\left|g\left(\sum_{i=0}^{4}x_i,\sum_{i=0}^{4}y_i\right)\right|\right]P_5\end{aligned}$$

On the other hand, according to Cauchy-Shwartz inequality, we can write:

$$\begin{cases}|g(x_0,y_0)|\le\|x_0\|+\|y_0\|\\\left|g\left(\sum_{i=0}^{k}x_i,\sum_{i=0}^{k}y_i\right)\right|\le\left\|\sum_{i=0}^{k}x_i\right\|+\left\|\sum_{i=0}^{k}y_i\right\|;1\le k\le5\end{cases}$$

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Hence $|G(X,Y)|\|X\|.\|Y\|$.

**Example on theorem6 and theorem7.**

Let $X=(1,2)+(0,-1)P_1+(1,-1)P_2+(-1,1)P_3+(-1,1)P_4+(1,-2)P_5$

$Y=(2,-1)+(-3,2)P_1+(1,2)P_2+(1,-4)P_3+(3,1)P_4+(-4,-3)P_5$

We have:

$$\begin{cases} x_0=(1,2) \\ \sum_{i=0}^{1} x_i=(1,1) \\ \sum_{i=0}^{2} x_i=(2,0) \\ \sum_{i=0}^{3} x_i=(1,1) \\ \sum_{i=0}^{4} x_i=(0,2) \\ \sum_{i=0}^{5} x_i=(1,0) \end{cases}$$

and

$$\begin{cases} y_0=(2,-1) \\ \displaystyle\sum_{i=0}^{1} y_i=(-1,1) \\ \displaystyle\sum_{i=0}^{2} y_i=(0,3) \\ \displaystyle\sum_{i=0}^{3} y_i=(1,-1) \\ \displaystyle\sum_{i=0}^{4} y_i=(4,0) \\ \displaystyle\sum_{i=0}^{5} y_i=(0,-3) \end{cases}$$

We have:

$$\begin{cases} g(x_0,y_0)=0 \\ g\left(\displaystyle\sum_{i=0}^{1} x_i,\sum_{i=0}^{1} y_i\right)=0 \\ g\left(\displaystyle\sum_{i=0}^{2} x_i,\sum_{i=0}^{2} y_i\right)=0 \\ g\left(\displaystyle\sum_{i=0}^{3} x_i,\sum_{i=0}^{3} y_i\right)=0 \\ g\left(\displaystyle\sum_{i=0}^{4} x_i,\sum_{i=0}^{4} y_i\right)=0 \\ g\left(\displaystyle\sum_{i=0}^{5} x_i,\sum_{i=0}^{5} y_i\right)=0 \end{cases}$$

This means that $G(X,Y)=0$ and $X\perp Y$.

On the other hand:

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$$\begin{cases} \|x_0\| = \sqrt{5} \\ \left\|\sum_{i=0}^{1} x_i\right\| = \sqrt{2} \\ \left\|\sum_{i=0}^{2} x_i\right\| = 2 \\ \left\|\sum_{i=0}^{3} x_i\right\| = \sqrt{2} \\ \left\|\sum_{i=0}^{4} x_i\right\| = 2 \\ \left\|\sum_{i=0}^{5} x_i\right\| = 1 \end{cases}$$

Thus $\|X\| = \sqrt{5} + \left(\sqrt{2} - \sqrt{5}\right)P_1 + \left(2 - \sqrt{2}\right)P_2 + \left(\sqrt{2} - 2\right)P_3 + \left(2 - \sqrt{2}\right)P_4 - P_5$

$\Longrightarrow \|X\|^2 = 5 - 3P_1 + 2P_2 - 2P_3 + 2P_4 - 3P_5$

And

$$\begin{cases} \|y_0\| = \sqrt{5} \\ \left\|\sum_{i=0}^{1} y_i\right\| = \sqrt{2} \\ \left\|\sum_{i=0}^{2} y_i\right\| = 3 \\ \left\|\sum_{i=0}^{3} y_i\right\| = \sqrt{2} \\ \left\|\sum_{i=0}^{4} y_i\right\| = 4 \\ \left\|\sum_{i=0}^{5} y_i\right\| = 3 \end{cases}$$

So that:

$\|Y\| = \sqrt{5} + \left(\sqrt{2} - \sqrt{5}\right)P_1 + \left(3 - \sqrt{2}\right)P_2 + \left(\sqrt{2} - 5\right)P_3 + \left(4 - \sqrt{2}\right)P_4 - P_5$

$\Longrightarrow \|Y\|^2 = 5 - 3P_1 + 7P_2 - 7P_3 + 14P_4 - 7P_5$

$\|X\|^2 + \|Y\|^2 = 10 - 6P_1 + 9P_2 - 9P_3 + 16P_4 - 10P_5$

$$\begin{cases} \|x_0 + y_0\| = \|(3,1)\| = \sqrt{10} \\ \left\|\sum_{i=0}^{1} (x_i + y_i)\right\| = \|(0,2)\| = 2 \\ \left\|\sum_{i=0}^{2} (x_i + y_i)\right\| = \|(2,3)\| = \sqrt{13} \\ \left\|\sum_{i=0}^{3} (x_i + y_i)\right\| = \|(2,0)\| = 2 \\ \left\|\sum_{i=0}^{4} (x_i + y_i)\right\| = \|(4,2)\| = \sqrt{20} \\ \left\|\sum_{i=0}^{5} (x_i + y_i)\right\| = \|(1,3)\| = \sqrt{10} \end{cases}$$

Hence $\|X + Y\|^2 = 10 + (4 - 10)P_1 + (13 - 4)P_2 + (4 - 13)P_3 + (20 - 4)P_4 + (10 - 20)P_5 = 10 - 6P_1 + 9P_2 - 9P_3 + 16P_4 - 10P_5 = \|X\|^2 + \|Y\|^2$

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It is clear that $\|X+Y\| \leq \|X\| + \|Y\|$. That is because:

$$\begin{cases} \sqrt{10} \leq 2\sqrt{5} \\ 2 \leq 2\sqrt{2} \\ \sqrt{13} \leq 5 \\ 2 \leq 2\sqrt{2} \\ \sqrt{20} \leq 6 \\ \sqrt{10} \leq 4 \end{cases}$$

**Example on theorem3.**

Let $X = (1,0) + (0,1)P_1 + (1,-1)P_2 + (2,-1)P_3 + (1,1)P_4 + (-3,0)P_5$

$Y = (0,2) + (1,1)P_1 + (2,1)P_2 + (1,2)P_3 + (-2,0)P_4 + (0,-3)P_5$

We have:

$$\begin{cases} x_0 = (1,0), \|x_0\| = 1 \\ \sum_{i=0}^{1} x_i = (1,1), \|x_0\| = \sqrt{5} \\ \sum_{i=0}^{2} x_i = (2,0), \|\sum_{i=0}^{2} x_i\| = 2 \\ \sum_{i=0}^{3} x_i = (4,-1), \|\sum_{i=0}^{3} x_i\| = \sqrt{17} \\ \sum_{i=0}^{4} x_i = (5,0), \|\sum_{i=0}^{4} x_i\| = 5 \\ \sum_{i=0}^{5} x_i = (2,0), \|\sum_{i=0}^{5} x_i\| = 2 \end{cases} \text{and} \begin{cases} y_0 = (0,2), \|y_0\| = 2 \\ \sum_{i=0}^{1} y_i = (1,3), \|\sum_{i=0}^{1} y_i\| = \sqrt{10} \\ \sum_{i=0}^{2} y_i = (3,4), \|\sum_{i=0}^{2} y_i\| = 5 \\ \sum_{i=0}^{3} y_i = (4,6), \|\sum_{i=0}^{3} y_i\| = \sqrt{52} \\ \sum_{i=0}^{4} y_i = (2,6), \|\sum_{i=0}^{4} y_i\| = \sqrt{40} \\ \sum_{i=0}^{5} y_i = (2,3), \|\sum_{i=0}^{5} y_i\| = \sqrt{13} \end{cases}$$

$\|X\| = 1 + (\sqrt{2}-1)P_1 + (2-\sqrt{2})P_2 + (\sqrt{17}-2)P_3 + (5-\sqrt{17})P_4 - 3P_5$

$\|Y\| = 2 + (\sqrt{10}-2)P_1 + (5-\sqrt{10})P_2 + (\sqrt{52}-5)P_3 + (\sqrt{40}-\sqrt{52})P_4 + (\sqrt{13}-\sqrt{40})P_5$

$\|X\|.\|Y\| = 2 + (\sqrt{20}-2)P_1 + (10-\sqrt{20})P_2 + (\sqrt{854}-10)P_3 + (\sqrt{200}-\sqrt{854})P_4 + (2\sqrt{13}-\sqrt{200})P_5$

$$\begin{cases} |g(x_0, y_0)| = 0 \\ \left|g\left(\sum_{i=0}^{1} x_i, \sum_{i=0}^{1} y_i\right)\right| = 4 \\ \left|g\left(\sum_{i=0}^{2} x_i, \sum_{i=0}^{2} y_i\right)\right| = 6 \\ \left|g\left(\sum_{i=0}^{3} x_i, \sum_{i=0}^{3} y_i\right)\right| = 10 \\ \left|g\left(\sum_{i=0}^{4} x_i, \sum_{i=0}^{4} y_i\right)\right| = 10 \\ \left|g\left(\sum_{i=0}^{5} x_i, \sum_{i=0}^{5} y_i\right)\right| = 4 \end{cases}$$

$|G(X,Y)| = 4P_1 + 2P_2 + 4P_3 - 6P_5 \leq \|X\|.\|Y\|$.

## 3. Conclusion

In this paper, we have studied for the first time the concept of symbolic 5-plithogenic vector spaces, where many results about the algebraic properties of this class of spaces will be obtained, where we defined AH-subspaces, AH-linear transformations, and kernels. Also, we study the inner products defined over symbolic 5-plithogenic vector spaces and determine the conditions of orthogonality in these spaces with many interesting examples.

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