# An Effective Algorithm for Solving Weak Fuzzy Complex Diophantine Equations in Two Variables

**Mayada Abualhomos[1], Wael Mahmoud M. Salameh[2,*], Malik Bataineh[3], Mowafaq Omar Al-Qadri[4], Ayman Alahmade[5], Abdallah Al-Husban[6]**

[1.]Applied Science Private University Amman,11931, Jordan,

[2]Faculty of Information Technology, Abu Dhabi University, Abu Dhabi, UAE

[3]Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid, Jordan

[4]Department of Mathematics, Jerash University, Jerash 26150, Jordan

[5]Department of Mathematics, College of Science and Art, AlUla branch, Taibah University, Medina, Saudi Arabia

[6]Department of Mathematics, Faculty of Science and Technology, Irbid National University, P.O. Box: 2600 Irbid, Jordan

Emails: abuhomos@asu.edu.jo; wael.salameh@adu.ac.ae; msbataineh@just.edu.jo; m.alqadri@jpu.edu.jo; aaahmdi@taibahu.edu.sa; dralhosban@inu.edu.jo

**Abstract**

Weak fuzzy complex numbers are defined as $a + b\varepsilon$, with $\varepsilon^2 belongs\ to ]0,1[$ as an extension of real numbers with $a, b \in R$. This paper is dedicated to studying weak fuzzy complex linear Diophantine equations in two weak fuzzy complex variables, by transforming the weak fuzzy complex Diophantine equation to a classical equivalent Diophantine system and going directly from the solutions of classical system into the desired equation. Algorithms for generating solutions of the previous equation will be presented in terms of theorems with many related examples that clarify the validity of our work.

**Keywords:** Weak fuzzy complex integer; Diophantine system; Linear Diophantine equation.

## 1. Introduction and Preliminaries.

The main purpose of expanding the set of real numbers was the study of geometric extensions resulting from various kinds of algebraic extensions associated with real numbers. The generalization of real numbers has attracted the interest of many researchers and mathematicians, where we find many extended sets proposed by geometrical aspects such as:
Complex numbers: $\{a + bi; a, b \in R;\ i^2 = -1\}$, Dual numbers: $\{a + bJ; a, b \in R;\ J^2 = 0\}$, Split-Complex numbers: $\{a + bJ; a, b \in R;\ J^2 = 1\}$, Neutrosophic numbers: $\{a + bI; a, b \in R;\ I^2 = I\}$, and Weak fuzzy Complex numbers: $\{a + b\varepsilon; a, b \in R;\ \varepsilon^2 = t \in ]0,1[\}$[1-5, 14].
The concept of weak fuzzy complex integers is considered a generalization of classical integers, with the difference that the set of weak fuzzy complex integers does not form a ring, which makes it harder to handle.
This type of numbers has been used in the description of solutions of vector equations defined by norms in Euclidean spaces [11], and also in the study of Pythagorean triples and quaternions with many associated Diophantine equations [10,12]. And it is remarkable the rapid development in the study of these numbers, as they were used in the computer, and neutralized through the Python language [13], and Matrix theory [7].



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A linear Diophantine equation in two variables is defined as follows $ax + by = c$, where $z, b, c \in Z$, with $x, y$ as integer variables. This kind of Diophantine equation can be extended to the set of weak fuzzy complex integers and it remains an open research question is how can we solve such non-classical class Diophantine equations?.
We find it appropriate here to note that many efforts were made to find algorithms for many types of Diophantine equations in many different rings, such as Pell's, Fermat's, and neutrosophic Diophantine equations [6, 8, 9].
The weak fuzzy complex Diophantine equations were studied for the first time in [10, 12], where many solutions for Pythagoras non-linear Diophantine equations were generated and handled.

This motivates us to study the linear Diophantine equation in two weak fuzzy complex integer variables, where solutions and algorithms will be discussed and presented in terms of theorems and many related examples.

**Definition: [5]**
The *set of Weak Fuzzy Complex numbers* is defined as follows, where '$J$' is the Weak Fuzzy Complex operator ( $J \notin \boldsymbol{R}$ ):

$$\boldsymbol{F}_J = \{x_0 + x_1 J \,;\, x_0, x_1 \in \boldsymbol{R}, J^2 = t \in ]0,1[\,\}$$

Let $X = x_0 + x_1 J$, $Y = y_0 + y_1 J \in \boldsymbol{F}_J$, where $x_0, x_1, y_0, y_1 \in \boldsymbol{R}$

- Addition: $+Y = (x_0 + y_0) + (x_1 + y_1)J$ .
- Multiplication $X.Y = (x_0 y_0 + x_1 y_1 t) + (x_0 y_1 + x_1 y_0)J$ .

**Definition: [3, 10]**
Let $C_w = \{a + b\varepsilon; a, b \in R;\ \varepsilon^2 = t \in ]0,1[\}$ be the ring of the weak fuzzy complex numbers, we define:
$Z_w = \{x + y\varepsilon; x, y \in Z;\ \varepsilon^2 = t \in ]0,1[\}$ to be the set of all weak fuzzy integers.
**Remark. [10]**
$Z_w$ is not a subring of $C_w$.
For example, for $\varepsilon^2 = t = \frac{1}{2}$, we have:
$X = 3 + 5\varepsilon, Y = 1 + \varepsilon \in Z_w$, and $X.Y = 3 + 3\varepsilon + 5\varepsilon + 5\varepsilon^2 = \frac{11}{2} + 8\varepsilon \notin Z_w$.
The previous remark is what makes the Diophantine equations difficult to solve in weak fuzzy complex integer set, because they do not have the algebraic structure of the ring, and therefore classical methods cannot be used in the solution

## 2. Main Results

**Definition.**
The weak fuzzy complex linear Diophantine equation with two variables is defined as follows:
$AX + BY = C$, where$A = a_1 + a_2\varepsilon, B = b_1 + b_2\varepsilon, C = c_1 + c_2\varepsilon, X = x_1 + x_2\varepsilon, Y = y_1 + y_2\varepsilon$, with $a_i, b_i, c_i, x_i, y_i \in Z$ for $1 \le i \le 2$.
**Remark.**
The weak fuzzy complex linear Diophantine equation with two variables $AX + BY = C$ can be written as follows:

$$\begin{cases} a_1x_1 + a_2x_2t + b_1y_1 + b_2y_2t = c_1 \\ a_1x_2 + a_2x_1 + b_1y_2 + b_2y_1 = c_2 \end{cases}$$

**Proof.**
$AX = (a_1 + a_2\varepsilon)(x_1 + x_2\varepsilon) = (a_1x_1 + a_2x_2t) + \varepsilon(a_1x_2 + a_2x_1); \varepsilon^2 = t \in ]0,1[$
$BY = (b_1 + b_2\varepsilon)(y_1 + y_2\varepsilon) = (b_1y_1 + b_2y_2t) + \varepsilon(b_1y_2 + b_2y_1)$
So that, the equation $AX + BY = C$ is equivalent to:

$$\begin{cases} a_1x_1 + a_2x_2t + b_1y_1 + b_2y_2t = c_1 \\ a_1x_2 + a_2x_1 + b_1y_2 + b_2y_1 = c_2 \end{cases}$$

Why weak fuzzy complex linear Diophantine equations are harder than classical linear Diophantine equations?
The weak fuzzy complex linear Diophantine equations with two variables $AX + BY = C$ is equivalent to two classical equations:

$$\begin{cases} a_1x_1 + a_2x_2t + b_1y_1 + b_2y_2t = c_1 \quad (1) \\ a_1x_2 + a_2x_1 + b_1y_2 + b_2y_1 = c_2 \quad (2) \end{cases}$$

## 3. The different types of weak fuzzy complex linear Diophantine equations with two variables.



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The weak fuzzy complex linear Diophantine equations $AX + BY = C$ is called of type (1) if $\sqrt{t} \in Q$. It is denoted by $T_1$.
The equation $AX + BY = C$ is called of type (2) if $\sqrt{t} \in R - Q$. It is denoted by $T_2$. If $t \in Q, \sqrt{t} \in R - Q$, it is called of type (3).

**Theorem.**

Let $AX + BY = C$ be the weak fuzzy complex linear Diophantine equations defined previously, then it is equivalent to:

$$\begin{cases}(a_1 + a_2\sqrt{t})(x_1 + x_2\sqrt{t}) + (b_1 + b_2\sqrt{t})(y_1 + y_2\sqrt{t}) = c_1 + c_2\sqrt{t} & (I)\\(a_1 - a_2\sqrt{t})(x_1 - x_2\sqrt{t}) + (b_1 - b_2\sqrt{t})(y_1 - y_2\sqrt{t}) = c_1 - c_2\sqrt{t} & (II)\end{cases}$$

**Proof.**

The equation $AX + BY = C$ is equivalent to:

$$\begin{cases}a_1x_1 + a_2x_2t + b_1y_1 + b_2y_2t = c_1 & (1)\\ a_1x_2 + a_2x_1 + b_1y_2 + b_2y_1 = c_2 & (2)\end{cases}$$

We multiply equation (2) by $\sqrt{t}$ and add it to (1):

$$a_1x_1 + a_2x_2t + b_1y_1 + b_2y_2t + \sqrt{t}(a_1x_2 + a_2x_1 + b_1y_2 + b_2y_1) = c_1 + c_2\sqrt{t}$$

So that:

$(a_1 + a_2\sqrt{t})(x_1 + x_2\sqrt{t}) + (b_1 + b_2\sqrt{t})(y_1 + y_2\sqrt{t}) = c_1 + c_2\sqrt{t}$ [equation$(I)$].

We multiply equation (2) by $\sqrt{t}$ and subtract (2) from (1):

$$a_1x_1 + a_2x_2t + b_1y_1 + b_2y_2t - \sqrt{t}(a_1x_2 + a_2x_1 + b_1y_2 + b_2y_1) = c_1 - c_2\sqrt{t}$$

So that:

$(a_1 - a_2\sqrt{t})(x_1 - x_2\sqrt{t}) + (b_1 - b_2\sqrt{t})(y_1 - y_2\sqrt{t}) = c_1 - c_2\sqrt{t}$ [equation$(I)$], so that the proof is complete.

**Example.**

For $\varepsilon^2 = t = \frac{1}{3}$, the equation:

$(2 + \varepsilon)X + (1 - 3\varepsilon)Y = 4 + 5\varepsilon$ is a weak fuzzy complex linear Diophantine of type (2), that is because $\sqrt{t} \notin Q$.
It is equivalent to the following system:

$$\begin{cases}\left(2 + \frac{1}{\sqrt{3}}\right)\left(x_1 + \frac{x_2}{\sqrt{3}}\right) + \left(1 - \frac{3}{\sqrt{3}}\right)\left(y_1 + \frac{y_2}{\sqrt{3}}\right) = 4 + \frac{5}{\sqrt{3}} & (I)\\\left(2 - \frac{1}{\sqrt{3}}\right)\left(x_1 - \frac{x_2}{\sqrt{3}}\right) + \left(1 + \frac{3}{\sqrt{3}}\right)\left(y_1 - \frac{y_2}{\sqrt{3}}\right) = 4 - \frac{5}{\sqrt{3}} & (II)\end{cases}$$

**Example.**

For $\varepsilon^2 = t = \frac{1}{16}$, the equation:

$(3 + 5\varepsilon)X + (1 + \varepsilon)Y = 4 + 5\varepsilon$ is a weak fuzzy complex linear Diophantine of type (1), that is because $\sqrt{t} = \frac{1}{4} \in Q$.
It is equivalent to the following system:

$$\begin{cases}\left(3 + \frac{5}{4}\right)\left(x_1 + \frac{x_2}{4}\right) + \left(1 + \frac{1}{4}\right)\left(y_1 + \frac{y_2}{4}\right) = 2 + \frac{3}{4} & (I)\\\left(3 - \frac{5}{4}\right)\left(x_1 - \frac{x_2}{4}\right) + \left(1 - \frac{1}{4}\right)\left(y_1 - \frac{y_2}{4}\right) = 2 - \frac{3}{4} & (II)\end{cases}$$

**Solving equations of type (1).**

**Theorem.**

Let $AX + BY = C$ be a weak fuzzy complex linear Diophantine equation of type (1) with $\varepsilon^2 = t \in ]0,1[$ and $\sqrt{t} = \frac{m}{n} \in Q$, $gcd(m, n) = 1$, then $AX + BY = C$ is solvable if and only if:

$$\begin{cases}gcd(na_1 + ma_2, nb_1 + mb_2) \setminus n^2c_1 + mc_2\\gcd(na_1 - ma_2, nb_1 - mb_2) \setminus n^2c_1 - mc_2\end{cases}$$

**Proof.**

As we proved, $AX + BY = C$ is equivalent to:

$$\begin{cases}(a_1 + a_2\sqrt{t})(x_1 + x_2\sqrt{t}) + (b_1 + b_2\sqrt{t})(y_1 + y_2\sqrt{t}) = c_1 + c_2\sqrt{t}\\(a_1 - a_2\sqrt{t})(x_1 - x_2\sqrt{t}) + (b_1 - b_2\sqrt{t})(y_1 - y_2\sqrt{t}) = c_1 - c_2\sqrt{t}\end{cases}$$

Since $\sqrt{t} = \frac{m}{n} \in Q$ with $gcd(m, n) = 1$, we get:

$$\begin{cases}\left(a_1 + \frac{m}{n}a_2\right)\left(x_1 + \frac{m}{n}x_2\right) + \left(b_1 + \frac{m}{n}b_2\right)\left(y_1 + \frac{m}{n}y_2\right) = c_1 + \frac{m}{n}c_2 \ (I)\\\left(a_1 - \frac{m}{n}a_2\right)\left(x_1 - \frac{m}{n}x_2\right) + \left(b_1 - \frac{m}{n}b_2\right)\left(y_1 - \frac{m}{n}y_2\right) = c_1 - \frac{m}{n}c_2 \ (II)\end{cases}$$



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We multiply both equations by $n^2 \neq 0$:

$$\begin{cases} (na_1 + ma_2)(nx_1 + mx_2) + (nb_1 + mb_2)(ny_1 + my_2) = n^2c_1 + mc_2 \ (\acute{I}) \\ (na_1 - ma_2)(nx_1 - mx_2) + (nb_1 - mb_2)(ny_1 - my_2) = n^2c_1 - mc_2 \ (\acute{I}\acute{I}) \end{cases}$$

The original equation $AX + BY = C$ is solvable if and only if the system $(\acute{I}), (\acute{I}\acute{I})$ is solvable, which is equivalent to:

$$\begin{cases} gcd(na_1 + ma_2, nb_1 + mb_2) \setminus n^2c_1 + mc_2 \\ gcd(na_1 - ma_2, nb_1 - mb_2) \setminus n^2c_1 - mc_2 \end{cases}$$

Thus the proof is complete.

**Example.**

For $\varepsilon^2 = t = \frac{1}{4}, \sqrt{t} = \frac{1}{2} \in Q$, consider the following weak fuzzy linear Diophantine equation of type(1) $(2 + \varepsilon)X + (1 - 4\varepsilon)Y = -1 + 4\varepsilon; \varepsilon^2 = t = \frac{1}{4}$

We have $a_1 = 2, a_2 = 1, b_1 = 1, b_2 = -4, c_1 = -1, c_2 = 4, \sqrt{t} = \frac{1}{2} = \frac{m}{n}$, we can write the equation as follows:

$$\begin{cases} \left(2 + \frac{1}{2}\right)\left(x_1 + \frac{1}{2}x_2\right) + (1 - 2)\left(y_1 + \frac{1}{2}y_2\right) = -1 + 2 \\ \left(2 - \frac{1}{2}\right)\left(x_1 - \frac{1}{2}x_2\right) + (1 + 2)\left(y_1 - \frac{1}{2}y_2\right) = -1 - 2 \end{cases}$$

Hence:

$$\begin{cases} \frac{5}{2}\left(x_1 + \frac{1}{2}x_2\right) - \left(y_1 + \frac{1}{2}y_2\right) = 1 \\ \frac{3}{2}\left(x_1 - \frac{1}{2}x_2\right) + 3\left(y_1 - \frac{1}{2}y_2\right) = -3 \end{cases}$$

Multiply each equation by $n^2 = 4$:

$$\begin{cases} 5(2x_1 + x_2) - 2(2y_1 + y_2) = 4 \quad (1) \\ 3(2x_1 - x_2) + 6(2y_1 - y_2) = -12 \quad (2) \end{cases}$$

$gcd(5, -2) = 1 \setminus 4, gcd(3,6) = 3 \setminus -12$, thus it is solvable weak fuzzy complex liner Diophantine equation.

**Remark.**

Let $AX + BY = C$ be a weak fuzzy complex liner Diophantine equation with two variables, we put $Z_1 = nx_1 + mx_2, Z_2 = nx_1 - mx_2, T_1 = ny_1 + my_2, T_1 = ny_1 - my_2$, then $AX + BY = C$ can be written as follows:

$$\begin{cases} (na_1 + ma_2)Z_1 + (nb_1 + mb_2)T_1 = n^2c_1 + mc_2 (I) \\ (na_1 - ma_2)Z_2 + (nb_1 - mb_2)T_2 = n^2c_1 - mc_2 \ (II) \end{cases}$$

Let $(Z_1^*, T_1^*), (Z_2^*, T_2^*)$ be solutions for $(I)$ and $(II)$ respectively.

Then $x_1 = \frac{Z_1^* + Z_2^*}{2n}, x_2 = \frac{Z_1^* - Z_2^*}{2m}, y_1 = \frac{T_1^* + T_2^*}{2n}, x_1 = \frac{T_1^* - T_2^*}{2m}$

So that, the solutions $(Z_1^*, T_1^*), (Z_2^*, T_2^*)$ of the classical system $(I), (II)$ give solutions to $AX + BY = C$ if and only if;

$2n \setminus Z_1^* + Z_2^*, 2m \setminus Z_1^* - Z_2^*, 2n \setminus T_1^* + T_2^*, 2m \setminus T_1^* - T_2^*$

**Algorithm for solving type (1) equations:**

To solve $AX + BY = C$ with $\sqrt{t} \in Q$ and $t = \frac{m}{n}, gcd(m, n) = 1$, follow these steps:

**Step 1.**

Write the equivalent classical system:

$$\begin{cases} (na_1 + ma_2)Z_1 + (nb_1 + mb_2)T_1 = n^2c_1 + mc_2 (I) \\ (na_1 - ma_2)Z_2 + (nb_1 - mb_2)T_2 = n^2c_1 - mc_2 \ (II) \end{cases}$$

**Step 2.**

Find one solution $(Z_1^*, T_1^*)$ for $(I)$, and one solution $(Z_2^*, T_2^*)$ for $(II)$.

**Step 3.**

Find all solutions of $(I)$ as follows:

$$\begin{cases} Z_1 = Z_1^* + k_1 \dfrac{nb_1 + mb_2}{g_1} \\ T_1 = T_1^* - k_1 \dfrac{na_1 + ma_2}{g_1} \end{cases}; \ k_1 \in Z, g_1 = gcd(na_1 + ma_2, nb_1 + mb_2)$$

Find all solutions of $(II)$ as follows:

$$\begin{cases} Z_2 = Z_2^* + k_2 \dfrac{nb_1 - mb_2}{g_1} \\ T_2 = T_2^* - k_2 \dfrac{na_1 - ma_2}{g_1} \end{cases}; \ k_2 \in Z, g_2 = gcd(na_1 - ma_2, nb_1 - mb_2)$$



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**Step 4.**
Find the solution of $AX + BY = C$ as follows:
$$x_1 = \frac{1}{2n}(Z_1 + Z_2) = \frac{1}{2n}\left(Z_1{}^* + Z_2{}^* + k_1\frac{nb_1 + mb_2}{g_1} + k_2\frac{nb_1 - mb_2}{g_1}\right)$$
$$x_2 = \frac{1}{2m}(Z_1 - Z_2) = \frac{1}{2m}\left(Z_1{}^* - Z_2{}^* + k_1\frac{nb_1 + mb_2}{g_1} - k_2\frac{nb_1 - mb_2}{g_1}\right)$$
$$y_1 = \frac{1}{2n}(T_1 + T_2) = \frac{1}{2n}\left(T_1{}^* + T_2{}^* - k_1\frac{na_1 + ma_2}{g_1} - k_2\frac{na_1 - ma_2}{g_1}\right)$$
$$y_2 = \frac{1}{2m}(T_1 - T_2) = \frac{1}{2m}\left(T_1{}^* - T_2{}^* - k_1\frac{na_1 + ma_2}{g_1} + k_2\frac{na_1 - ma_2}{g_1}\right)$$
Where $k_1, k_2 \in Z$, and $2n \setminus Z_1 + Z_2, 2m \setminus Z_1 - Z_2, 2n \setminus T_1 + T_2, 2m \setminus T_1 - T_2$.
**Example.**
Consider the equation $(2+\varepsilon)X + (1-4\varepsilon)Y = -1 + 4\varepsilon; \varepsilon^2 = t = \frac{1}{4}, \sqrt{t} = \frac{1}{2}, m = 1, n = 2$.
The equivalent system is:
$$\begin{cases} 5Z_1 - 2T_1 = 4 \quad (I) \\ 3Z_2 + 6T_2 = -12 \quad (II) \end{cases}$$
$Z_1{}^* = 2, T_1{}^* = 3$ is a solution of $(I)$.
$Z_2{}^* = 0, T_2{}^* = -2$ is a solution of $(II)$.
The solutions of $(I)$ are:
$$\begin{cases} Z_1 = 2 + k_1\frac{-2}{1} = 2 - 2k_1 \\ T_1 = 3 - k_1\frac{5}{1} = 3 - 5k_1 \end{cases}; k_1 \in Z$$
The solutions of $(I)$ are:
$$\begin{cases} Z_2 = 0 + k_2\frac{6}{3} = 2k_2 \\ T_1 = -2 - k_2\frac{3}{3} = -2 - k_2 \end{cases}; k_2 \in Z$$
We have:
$$\begin{cases} Z_1 + Z_2 = 2 - 2k_1 + 2k_2 \\ Z_1 - Z_2 = 2 - 2k_1 - 2k_2 \\ T_1 + T_2 = 1 - 5k_1 - k_2 \\ T_1 - T_2 = 5 - 5k_1 + k_2 \end{cases}; k_1, k_2 \in Z$$
$$X = x_1 + x_2\varepsilon = \frac{1}{2n}(2 - 2k_1 + 2k_2) + \frac{1}{2m}(2 - 2k_1 - 2k_2)\varepsilon = \frac{1 - k_1 + k_2}{n} + \varepsilon\frac{1 - k_1 - k_2}{m}$$
$$= \frac{1 - k_1 + k_2}{2} + \varepsilon(1 - k_1 - k_2); 2 \setminus 1 - k_1 + k_2$$
$Y = y_1 + y_2\varepsilon = \frac{1}{4}(1 - 5k_1 - k_2) + \frac{1}{2}(5 - 5k_1 + k_2)\varepsilon; 4 \setminus 1 - 5k_1 - k_2$ and $2 \setminus 5 - 5k_1 + k_2$
$(X, Y)$ is the solutions of $AX + BY = C$ under the provided conditions.
To get a special solution of $AX + BY = C$, we must give $k_1, k_2$ suitable values under conditions:
$$\begin{cases} 2 \setminus 1 - k_1 + k_2 \\ 2 \setminus 5 - 5k_1 + k_2 \\ 4 \setminus 1 - 5k_1 - k_2 \end{cases}$$
For this goal, we can put $k_1 = -2, k_2 = -1$.
We can see that $1 - k_1 + k_2 = 2,\ 5 - 5k_1 + k_2 = 14,\ 1 - 5k_1 - k_2 = 12$.
The corresponding solution is:
$$X = \frac{2}{2} + \varepsilon(4) = 1 + 4\varepsilon, Y = \frac{12}{4} + \frac{12}{2}\varepsilon = 3 + 7\varepsilon$$
Another solution can be obtained by putting $k_1 = 4, k_2 = 1, 1 - k_1 + k_2 = 1 - 4 + 1 = -2, 5 - 5k_1 + k_2 = 5 - 10 + 1 = -14, 1 - 5k_1 - k_2 = 1 - 20 - 1 = -20$.
The corresponding solution is:
$$X = \frac{-2}{2} + \varepsilon(1 - 4 - 1) = -1 - 4\varepsilon, Y = \frac{-20}{4} + \left(\frac{-14}{2}\right)\varepsilon = -5 - 7\varepsilon$$
And so on.
**Solving equations of type (2).**
**Theorem.**



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Let $AX+BY=C$ be weak fuzzy linear Diophantine equation in two variables with $A=a_1+a_2\varepsilon, B=b_1+b_2\varepsilon, C=c_1+c_2\varepsilon, X=x_1+x_2\varepsilon, Y=y_1+y_2\varepsilon$, with $\varepsilon^2=t\in ]0,1[, \sqrt{t}\in R-Q$.
Let $d_1=gcd(a_1,b_1), d_2=gcd(a_2,b_2)$, then:
$AX+BY=C$ is solvable if and only if:

1. $d_1 \setminus c_1$.
2. $gcd(d_2a_2b_1-b_2a_1d_2, a_1b_2d_1-d_1b_1a_2) \setminus d_1d_2(c_2-a_2x_1{}^*-b_2y_1{}^*)$, where $(x_1{}^*, y_1{}^*)$ is a solution of $a_1x_1+b_1y_1=c_1$

**Proof.**
The equation $AX+BY=C$ is equivalent to:
$$\begin{cases} a_1x_1+b_1y_1+t(a_2x_2+b_2y_2)=c_1 & (1) \\ a_1x_2+a_2x_1+b_1y_2+b_2y_1=c_2 & (2) \end{cases}$$
Since $t\in R-Q$ and $a_2x_2+b_2y_2\in Z$, we get $a_2x_2+b_2y_2=0$ and $a_1x_1+b_1y_1=c_1$.
The equation $a_1x_1+b_1y_1=c_1$ is solvable if and only if $d_1=gcd(a_1,b_1)\setminus c_1$.
The solution of the system $\begin{cases} a_1x_1+b_1y_1=c_1 \\ a_2x_2+b_2y_2=0 \end{cases}$ are:
$$\begin{cases} x_1=x_1{}^*+k_1\frac{b_1}{d_1} \\ y_1=y_1{}^*-k_1\frac{a_1}{d_1} \\ x_2=k_2\frac{b_2}{d_2} \\ y_2=-k_2\frac{a_2}{d_2} \end{cases}$$ with $k_1,k_2\in Z, (x_1{}^*, y_1{}^*)$ is a solution of $a_1x_1+b_1y_1=c_1$.
For $x_i{}^*, y_i{}^*; 1\le i\le 2$ to represent a solution of the original Diophantine equation, they should be a solution to equation (2).
Thus,
$$a_1\left(k_2\frac{b_2}{d_2}\right)+a_2\left(x_1{}^*+k_1\frac{b_1}{d_1}\right)+b_1\left(-k_2\frac{a_2}{d_2}\right)+b_2\left(y_1{}^*-k_1\frac{a_1}{d_1}\right)=c_2$$
By multiplying both side with $d_1d_2$, we get:
$k_1(a_2b_1d_2-b_2a_1d_2)+k_2(a_1b_2d_1-b_1a_2d_1)=d_1d_2(c_2-a_2x_1{}^*-b_2y_1{}^*)$, which is a linear Diophantine equation in two variables $k_1,k_2$.
This equation has solutions if and only if:
$gcd(a_2b_1d_2-b_2a_1d_2, a_1b_2d_1-d_1b_1a_2)\setminus d_1d_2(c_2-a_2x_1{}^*-b_2y_1{}^*)$
Thus, our proof is complete.

**Algorithm for solving type (2) equation:**
Follow these steps:
**Step1.**
Write the equivalent system:
$$\begin{cases} a_1x_1+b_1y_1=c_1 \\ a_2x_2+b_2y_2=0 \end{cases}$$
**Step2.**
Find $d_1=gcd(a_1,b_1), d_2=gcd(a_2,b_2)$, then check if $d_1\setminus c_1$,
$gcd(a_2b_1d_2-b_2a_1d_2, a_1b_2d_1-d_1b_1a_2)\setminus d_1d_2(c_2-a_2x_1{}^*-b_2y_1{}^*)$
**Step 3.**
Find all solutions of linear Diophantine equation in two variables $k_1,k_2$:
$(a_2b_1d_2-b_2a_1d_2)k_1+(a_1b_2d_1-d_1b_1a_2)k_2=d_1d_2(c_2-a_2x_1{}^*-b_2y_1{}^*)$
**Step4.**
Get $x_1=x_1{}^*+k_1\frac{b_1}{d_1}, y_1=y_1{}^*-k_1\frac{a_1}{d_1}, x_2=k_2\frac{b_2}{d_2}, y_2=-k_2\frac{a_2}{d_2}$.

**Example.**
Consider the following weak fuzzy complex linear Diophantine equation in two variables of type (2).
$(3+\varepsilon)X+(1-\varepsilon)Y=7+5\varepsilon; \varepsilon^2=\frac{1}{\pi}=t$.
We have:
$a_1=3, a_2=1, b_1=1, b_2=-1, c_1=7, c_2=5$
The equivalent system is:
$$\begin{cases} 3x_1+y_1=7 \\ x_2-y_2=0 \end{cases}$$
$d_1=gcd(a_1,b_1)=1\setminus 7, x_1{}^*=2, y_1{}^*=1, d_2=gcd(a_2,b_2)=1$
$a_2b_1d_2-b_2a_1d_2=(1)(1)(1)-(-1)(3)(1)=1+3=4$



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$a_1b_2d_1 - d_1b_1a_2 = (3)(-1)(1) - (1)(1)(1) = -3 - 1 = -4$
$gcd(a_2b_1d_2 - b_2a_1d_2, a_1b_2d_1 - d_1b_1a_2) = gcd(4, -4) = 4$
$d_1d_2(c_2 - a_2x_1{}^* - b_2y_1{}^*) = 1\big(5 - (1)(2) - (-1)(1)\big) = 4$
It is clear that $4 \setminus 4$, so that we can find solutions for the original equation.
Consider the equation $4k_1 - 4k_2 = 4$, it is equivalent to:
$k_1 - k_2 = 1$, hence:

$$\begin{cases} k_1 = 1 + s_1\dfrac{-1}{1} = 1 - s_1 \\ k_2 = 0 - s_1\dfrac{1}{1} = -s_1 \end{cases} ; s_1 \in Z$$

Thus, $x_1 = x_1{}^* + k_1\frac{b_1}{d_1} = 2 + (1 - s_1)\frac{1}{1} = 3 - s_1, y_1 = y_1{}^* - k_1\frac{a_1}{d_1} = 1 - (1 - s_1)\frac{3}{1} = -2 + 3s_1, x_2 = k_2\frac{b_2}{d_2} = -s_1\left(\frac{-1}{1}\right) = s_1, y_2 = -k_2\frac{a_2}{d_2} = s_1\left(\frac{1}{1}\right) = s_1$
Hence,
$X = x_1 + x_2\varepsilon = (3 - s_1) + \varepsilon s_1, Y = y_1 + y_2\varepsilon = (-2 + 3s_1) + \varepsilon s_1; s_1 \in Z$
is the solution of$(3 + \varepsilon)X + (1 - \varepsilon)Y = 7 + 5\varepsilon$.

**Solving equations of type (3).**

Let $AX + BY = C$ be a weak fuzzy complex linear Diophantine equation with two variables, where $\varepsilon^2 = t \in ]0,1[, t = \frac{m}{n} \in Q, gcd(m,n) = 1, \sqrt{t} \notin Q$.
To solve the type (3) equation, we follow these steps:

**Step1.**

We write the equivalent system:

$$\begin{cases} a_1x_1 + b_1y_1 + t(a_2x_2 + b_2y_2) = c_1 \quad (1) \\ a_1x_2 + a_2x_1 + b_1y_2 + b_2y_1 = c_2 \quad (2) \end{cases} ; t = \frac{m}{n}$$

Multiply (1) by $n$, and put $Z_1 = a_1x_1 + b_1y_1, Z_2 = a_2x_2 + b_2y_2, T_1 = a_2x_1 + b_2y_2, T_2 = a_1x_2 + b_1y_2$, hence:

$$\begin{cases} nZ_1 + mZ_2 = nc_1 \quad (I) \\ T_1 + T_2 = c_2 \quad (II) \end{cases}$$

Since $gcd(m,n) = 1 \setminus nc_1$, then $(I)$ and $(II)$ are solvable.

**Step2.**

Find the solutions of $(I)$ as follows:
$Z_1 = Z_1{}^* + k_1m, Z_2 = Z_2{}^* - k_1n; k_1 \in Z$
Find the solutions of $(I)I$ as follows:
$T_1 = T_1{}^* + k_2, t_2 = T_2{}^* - k_2; k_2 \in Z$

**Step3.**

Write the second equivalent Diophantine system:

$$\begin{cases} a_1x_1 + b_1y_1 = Z_1 \\ a_2x_1 + b_2y_1 = T_1 \end{cases} \quad (S_1)$$

And,

$$\begin{cases} a_2x_2 + b_2y_2 = Z_2 \\ a_1x_2 + b_1y_2 = T_2 \end{cases} \quad (S_2)$$

To solve system $(S_1)$:

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2$$

$$\Delta_{x_1} = \begin{vmatrix} Z_1 & b_1 \\ T_1 & b_2 \end{vmatrix} = Z_1b_2 - b_1T_1 = (Z_1{}^* + k_1m)b_2 - b_1(T_1{}^* + k_2) = Z_1{}^*b_2 + k_1b_2m - b_1T_1{}^* - b_1k_2; k_1, k_2 \in Z$$

$$\Delta_{y_1} = \begin{vmatrix} a_1 & Z_1 \\ a_2 & T_1 \end{vmatrix} = a_1T_1 - a_2Z_1 = a_1(T_1{}^* + k_2) - a_2(Z_1{}^* + k_1m) = a_1T_1{}^* + a_1k_2 - a_2Z_1{}^* - a_2k_1m; k_1, k_2 \in Z$$

To get an accepted solution, it should has the following properties $\Delta \setminus \Delta_{x_1}, \Delta \setminus \Delta_{y_1}$.
On the other hand, $x_1 = \frac{\Delta_{x_1}}{\Delta} = \frac{Z_1{}^*b_2 + k_1b_2m - b_1T_1{}^* - b_1k_2}{a_1b_2 - b_1a_2}, y_1 = \frac{\Delta_{y_1}}{\Delta} = \frac{a_1T_1{}^* + a_1k_2 - a_2Z_1{}^* - a_2k_1m}{a_1b_2 - b_1a_2}$
To solve system $(S_2)$:

$$\acute{\Delta} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = b_1a_2 - a_1b_2 = -\Delta$$

$$\Delta_{x_2} = \begin{vmatrix} Z_2 & b_2 \\ T_2 & b_1 \end{vmatrix} = Z_2b_1 - b_1T_2 = (Z_2{}^* - k_1n)b_1 - b_2(T_2{}^* - k_2) = Z_2{}^*b_1 - k_1b_1n - b_2T_2{}^* + b_2k_2; k_1, k_2 \in Z$$

$$\Delta_{y_2} = \begin{vmatrix} a_2 & Z_2 \\ a_1 & T_2 \end{vmatrix} = a_2T_2 - a_1Z_2 = a_2(T_2{}^* - k_2) - a_1(Z_2{}^* - k_1n) = a_2T_2{}^* - a_2k_2 - a_1Z_2{}^* + a_1k_1n; k_1, k_2 \in Z$$



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To get an accepted solution, it should has the following properties $-\Delta \setminus \Delta_{x_2}, -\Delta \setminus \Delta_{y_2}$.

On the other hand, $x_2 = \frac{\Delta_{x_2}}{-\Delta} = \frac{Z_2^* b_1 - k_1 b_1 n - b_2 T_2^* + b_2 k_2}{b_1 a_2 - a_1 b_2}, y_1 = \frac{\Delta_{y_2}}{\Delta} = \frac{a_2 T_2^* - a_2 k_2 - a_1 Z_2^* + a_1 k_1 n}{b_1 a_2 - a_1 b_2}$

**Example.**

Consider the following weak fuzzy complex linear Diophantine equation in two variables of type (3):

$(2+3\varepsilon)X + (1-\varepsilon)Y = 3 + 7\varepsilon; \varepsilon^2 = t = \frac{1}{5} \in Q, \sqrt{t} \notin Q.$

We have:

$a_1 = 2, a_2 = 3, b_1 = 1, b_2 = -1, c_1 = 3, c_2 = 7, m = 1, n = 5$

The equivalent system:

$$\begin{cases} 5Z_1 + Z_2 = 15 \quad (I) \\ T_1 + T_2 = 7 \quad (II) \end{cases}$$

$Z_1^* = 3, Z_2^* = 0, T_1^* = 3, T_2^* = 4$

Also,

$$\begin{cases} Z_1 = 3 + k_1 \\ Z_2 = -5k_1 \\ T_1 = 3 + k_2 \\ t_2 = 4 - k_2 \end{cases}; k_1, k_2 \in Z$$

The system $(S_1)$ is:

$$\begin{cases} 2x_1 + y_1 = 3 + k_1 \\ 3x_1 - y_1 = 3 + k_2 \end{cases}$$

$$\Delta = \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -5, \Delta_{x_1} = \begin{vmatrix} 3+k_1 & 1 \\ 3+k_2 & -1 \end{vmatrix} = -3 - k_1 - 3 - k_2 = -6 - k_1 - k_2$$

$$\Delta_{y_1} = \begin{vmatrix} 2 & 3+k_1 \\ 3 & 3+k_2 \end{vmatrix} = 6 + 2k_2 - 9 - 3k_1 = -3 - 3k_1 + 2k_2; k_1, k_2 \in Z$$

The accepted values of $k_1, k_2$ are exactly the values that have the property:

$-6 - k_1 - k_2 \equiv 0 (mod\ 5), -3 - 3k_1 + 2k_2 \equiv 0 (mod\ 5),$

So that: $-k_1 - k_2 \equiv 1 (mod\ 5), -3k_1 + 2k_2 \equiv 3 (mod\ 5)$

$$x_1 = \frac{-6 - k_1 - k_2}{-5} = \frac{6 + k_1 + k_2}{5}$$

$$y_1 = \frac{-3 - 3k_1 + 2k_2}{-5} = \frac{3 + 3k_1 - 2k_2}{5}$$

The system $(S_2)$ is:

$$\begin{cases} 3x_2 - y_2 = -5k_1 \\ 2x_2 + y_2 = 4 - k_2 \end{cases}$$

$$\acute{\Delta} = \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = 5, \Delta_{x_2} = \begin{vmatrix} -5k_1 & -1 \\ 4-k_2 & 1 \end{vmatrix} = -5k_1 + 4 - k_2$$

$$\Delta_{y_2} = \begin{vmatrix} 3 & -5k_1 \\ 2 & 4-k_2 \end{vmatrix} = 12 - 3k_2 + 10k_1; k_1, k_2 \in Z$$

$$x_2 = \frac{-5k_1 + 4 - k_2}{5}$$

$$y_2 = \frac{12 - 3k_2 + 10k_1}{5}$$

Under the conditions:

$-5k_1 + 4 - k_2 \equiv 0 (mod\ 5), 12 - 3k_2 + 10k_1 \equiv 0 (mod\ 5),$

So that: $k_2 \equiv 4 (mod\ 5)$

For example, if we take $k_1 = 5, k_2 = 4$, then:

$2k_1 - 3k_2 = -7 \equiv 3 (mod\ 5), -k_1 - k_2 = -9 \equiv 1 (mod\ 5), k_2 \equiv 4 (mod\ 5)$

$$\begin{cases} x_1 = \frac{6+9}{5} = 3, y_1 = \frac{3 - 8 + 15}{5} = 2 \\ x_2 = \frac{-25 + 4 - 4}{5} = -5, \quad y_2 = \frac{12 - 12 + 50}{5} = 10 \end{cases}$$

Thus $X = 3 - 5\varepsilon, Y = 2 + 10\varepsilon$ is a solution of the original equation.

## 4. Conclusion

In this paper, we have studied the linear Diophantine equation in two weak fuzzy complex integer variables, where we have presented algorithms for generating possible solutions by transforming it into a classical equivalent Diophantine system. Also, we illustrated many related examples to clarify how algorithms work in each possible case.



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**References**


[1] Deckelman, S., Robson, B. Split-Complex Numbers and Dirac Bra-kets., *Communications In Information and Systems*,**14**, (2014), 135-159.

[2] Akar, M., Yuce, S., Sahin, S., On the Dual Hyperbolic Numbers and The Complex Hyperbolic Numbers, *JCSCM*, **8**, (2018), DOI: 10.20967/jcscm.2018.01.001.

[3] Hatip, A., An Introduction To Weak Fuzzy Complex Numbers, *Galoitica Journal Of Mathematical Structures and Applications*, **3**, (2023).

[4] Ali, R., On The Weak Fuzzy Complex Inner Products On Weak Fuzzy Complex Vector Spaces, *Neoma Journal Of Mathematics and Computer Science*, (2023).

[5] Hatip, A., On The Weak Fuzzy Complex Vector Spaces, *Galoitica Journal Of Mathematical Structures And Applications*, **3**, (2023).

[6] Ali, R., A Short Note On The Solution of n-Refined Neutrosophic Linear Diophantine Equations, *International Journal Of Neutrosophic Science*, **15**, (2021).

[7] Alhasan,Y., Alfahal, A., Abdulfatah, R., Nordo, G.& Zahra, M., On Some Novel Results About Weak Fuzzy Complex Matrices. *International Journal of Neutrosophic Science*, **21**, (2023), 134-140. https://doi.org/10.54216/IJNS.210112

[8] Ahmad, K., Bal, M., and Aswad, M., A Short Note On The Solutions Of Fermat's Diophantine Equation In Some Neutrosophic Rings, *Journal Of Neutrosophic and Fuzzy Systems*, (2022).

[9] Abobala, M., Partial Foundation of Neutrosophic Number Theory, *Neutrosophic Sets and Systems*, **39** , (2021).

[10] Alfahal, A., Abobala, M., Alhasan, Y., and Abdulfatah, R., Generating Weak Fuzzy Complex and Anti Weak Fuzzy Complex Integer Solutions for Pythagoras Diophantine Equation $\boldsymbol{X2} + \boldsymbol{Y2} = \boldsymbol{Z2}$, *International Journal of Neutrosophic Science*, **22**, (2023).

[11] Alhasan, Y., Xu, L., Abdulfatah, R., & Alfahal, A., The Geometrical Characterization for The Solutions of a Vectorial Equation By Using Weak Fuzzy Complex Numbers and Other Generalizations Of Real Numbers. *International Journal of Neutrosophic Science (IJNS)*, **21**, (2023), 155-159. https://doi.org/10.54216/IJNS.210415

[12] Galarza, F., Flores, M., Rivero, D., & Abobala, M., On Weak Fuzzy Complex Pythagoras Quadruples. *International Journal of Neutrosophic Science (IJNS)*, **22**, (2023), 108-113. https://doi.org/10.54216/IJNS.220209 .

[13] Razouk,L., Mahmoud, S., & Ali, M., A Computer Program For The System Of Weak Fuzzy Complex Numbers And Their Arithmetic Operations Using Python. *Galoitica: Journal Of Mathematical Structures and Applications*, **8**, (2023), 45-51. https://doi.org/10.54216/GJMSA.080104.

[14] Merkepci, M., and Abobala, M., " On Some Novel Results About Split-Complex Numbers, The Diagonalization Problem And Applications To Public Key Asymmetric Cryptography", Journal of Mathematics, Hindawi, 2023.




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