



Interval Valued Neutrosophic INK -Ideal Via INK -Algebra

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Abstract

Based on Zadeh's notion of interval-valued fuzzy set, we develop in this work the notion of interval-valued neutrosophic INK -ideals (briefly, IVN INK -ideals) in INK -algebra and evaluate some of its properties. We show that all N INK -ideals of INK -algebra A can be realized as IV level INK -ideals of INK -algebra A . We then deduce numerous related results, which are summarized in the abstract.

Keywords: Fuzzy INK -ideal; interval valued neutrosophic fuzzy set; interval valued neutrosophic INK -ideal.

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1 Introduction

BCK -algebra and BCI -algebra are two classes of abstract algebras introduced by Imai and Iséki in 1966.¹ The BCK -algebra class is recognized as a legitimate subclass of the BCI -algebra class.^{2,6} A concept known as d -algebra was presented by Neggers et al.⁷ It is a generalization of $BCH/BCI/BCK$ -algebras and generalizes several of the theorems covered in BCI -algebra. Zadeh⁹ established the idea of a fuzzy set. Fuzzy ideals and fuzzy subalgebras have since been used for various algebraic structures, including semigroups, groups, rings, BCI/BCH -algebras, and so on. An interval-valued fuzzy set, or a fuzzy set with an interval-valued membership function, is Zadeh's expansion of the fuzzy set idea in.⁹ The IV fuzzy set is the name given to this interval-valued fuzzy set. Zadeh also developed an approximation inference technique based on his IV fuzzy sets. Ideals and interval-valued fuzzy INK -subalgebras were defined by Kaviyarasu et al.⁴ In this paper, we establish the notion of interval-valued neutrosophic INK -ideals in INK -algebra (briefly, $i - v$ N INK -ideals in INK -algebra) and analyze some of their features using the notion of interval-valued fuzzy set by Zadeh. We establish the possibility of realizing any neutrosophic INK -ideals of neutrosophic INK -algebra X as IV level neutrosophic ideals of a INK -algebra X . Subsequently, we derive various results related to this that are presented in the abstract.

2 Preliminaries

Definition 2.1. ⁴ An algebra $(U, *, 0)$ is a non-null set U with a constant '0' and a single binary operation '*' is called INK -algebras if it satisfying the following conditions.

- (i) $\alpha * \alpha = 0$
- (ii) $\alpha * 0 = \alpha$ for $\alpha \in U$
- (iii) $0 * \alpha = \alpha$
- (iv) $(\beta * \alpha) * (\beta * \gamma) = (\alpha * \gamma)$ for all $\alpha, \beta, \gamma \in U$.

Definition 2.2.⁴ Let U be a *INK*-algebra and $T \subseteq U$. Then T is said to a *INK*-subalgebra (briefly, *INK*-subalg) of U , if $\alpha * \beta \in T$, for all $\alpha, \beta \in U$.

Definition 2.3.³ A mapping $f : U \rightarrow V$ of *INK*-algebras is called a *INK*-homomorphism if $f(\alpha * \beta) = f(\alpha) * f(\beta)$ for all $\alpha, \beta \in U$.

Now, we go over a few fuzzy logic concepts (see⁹).

Let U be a set. A fuzzy set τ_1 in U is characterized by a membership function $\mu_{\tau_1} : U \rightarrow [0, 1]$. Let f be a mapping from the set U to the set V and let τ_2 be a fuzzy set in V with membership function μ_{τ_2} . The inverse image of τ_2 , denoted $f^{-1}(\tau_2)$, is the fuzzy set in U with membership function $\mu_{f^{-1}(\tau_2)}$ defined by $\mu_{f^{-1}(\tau_2)}(\alpha) = \mu_{\tau_2}(f(\alpha))$ for all $\alpha \in U$. Conversely, let τ_1 be a fuzzy set in U with membership function μ_{τ_1} . Then the image of τ_1 , denoted by $f(\tau_1)$, is the fuzzy set in V such that:

$$\mu_{f(\tau_1)}(\beta) = \begin{cases} \sup_{\gamma \in f^{-1}(\beta)} \mu_{\tau_1}(\gamma) & \text{if } f^{-1}(\beta) = \{\alpha : f(\alpha) = \beta\} \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

A fuzzy set τ_1 in the *INK*-algebra U with the membership function μ_{τ_1} is said to be have the sup property if for any subset $T \subseteq U$ there exists $\alpha_0 \in T$ such that $\mu_{\tau_1}(\alpha_0) = \sup_{t \in T} \mu_{\tau_1}(t)$.

Definition 2.4.⁴ A fuzzy subset μ in a *INK*-algebra U is called a fuzzy *INK*-subalgebra (briefly, *FINK*-subalg) of U if $\mu(\alpha * \beta) \geq \min\{\mu(\alpha), \mu(\beta)\}$, for all $\alpha, \beta \in U$.

Definition 2.5.⁸ A fuzzy subset μ in a *INK*-algebra U is called a anti fuzzy subalgebra⁸ (briefly, *AFINK*-subalg) of U if $\mu(\alpha * \beta) \leq \max\{\mu(\alpha), \mu(\beta)\}$, for all $\alpha, \beta \in U$.

Definition 2.6.⁸ Let μ be a fuzzy set of a set U . For a fixed $t \in [0, 1]$, the set $\mu_t = \{\alpha \in U / \mu(\alpha)\}$ is called an upper level of μ .

Definition 2.7.⁸ Let U be a fuzzy subset of S . Then for $t \in [0, 1]$, the t -level cut of U is the set $U_t := \{x \in S | U(x) \geq t\}$.

Definition 2.8.⁸ Let U be a fuzzy subset of a *INK*-algebra S . Then for $t \in [0, 1]$, the lower t -level cut of U is the set $U^t := \{x \in S | A(x) \leq t\}$. Clearly $U^1 = X$ and $U_t \cup U^t = X$ for $t \in [0, 1]$. If $t_1 < t_2$, then $U^{t_1} \subseteq U^{t_2}$.

An interval valued neutrosophic set (briefly, *IVNS*)⁴ Λ in a non-empty set X is defined to be a structure $\Lambda = \{(\alpha, T_\Lambda(\alpha), I_\Lambda(\alpha), F_\Lambda(\alpha)) | \alpha \in X\}$, where $T_\Lambda : U \rightarrow \theta[0, 1]$, $I_\Lambda : U \rightarrow \theta[0, 1]$ and $F_\Lambda : U \rightarrow \theta[0, 1]$, which are called a truth membership function, an indeterminacy membership function and a falsity membership function respectively.

The intervals $T_\Lambda(\alpha)$, $I_\Lambda(\alpha)$ and $F_\Lambda(\alpha)$ denote the intervals of the degree of membership, indeterminacy and non-membership of the element α to the set $\theta[0, 1]$, respectively, where $T_\Lambda^-(\alpha) = [T_\Lambda^L(\alpha), T_\Lambda^U(\alpha)]$, $I_\Lambda^-(\alpha) = [I_\Lambda^L(\alpha), I_\Lambda^U(\alpha)]$ and $F_\Lambda^-(\alpha) = [F_\Lambda^L(\alpha), F_\Lambda^U(\alpha)]$ for all $\alpha \in X$. Also, note that $\overline{T_\Lambda(\alpha)} = 1 - T_\Lambda(\alpha) = [1 - T_\Lambda^U(\alpha), 1 - T_\Lambda^L(\alpha)]$,

$$\overline{I_\Lambda(\alpha)} = 1 - I_\Lambda(\alpha) = [1 - I_\Lambda^U(\alpha), 1 - I_\Lambda^L(\alpha)]$$

and $\overline{F_\Lambda(\alpha)} = 1 - F_\Lambda(\alpha) = [1 - F_\Lambda^U(\alpha), 1 - F_\Lambda^L(\alpha)]$ for all $\alpha \in X$, where $(\alpha, \overline{T_\Lambda(\alpha)}, \overline{I_\Lambda(\alpha)}, \overline{F_\Lambda(\alpha)})$ represents the complement of α in Λ .

We define $\overline{\Lambda} = (\overline{T_\Lambda}, \overline{I_\Lambda}, \overline{F_\Lambda})$ as the complement of $\Lambda = (T_\Lambda, I_\Lambda, F_\Lambda)$. For the sake of simplicity, we shall use the symbol $\Lambda = (T_\Lambda, I_\Lambda, F_\Lambda)$ for the *IVNS* is given by $\Lambda = \{(\alpha, T_\Lambda(\alpha), I_\Lambda(\alpha), F_\Lambda(\alpha)) | \alpha \in U\}$.

Definition 2.9. ² A NS Λ in U is called a neutrosophic *INK*-subalgebra, (briefly, *N INK*-subalg) of U if it satisfies the following condition, for all $\alpha, \beta \in U$.

- (i) $T_{\Lambda}(\alpha * \beta) \geq \min\{T_{\Lambda}(\alpha), T_{\Lambda}(\beta)\}$,
- (ii) $I_{\Lambda}(\alpha * \beta) \leq \max\{I_{\Lambda}(\alpha), I_{\Lambda}(\beta)\}$,
- (iii) $F_{\Lambda}(\alpha * \beta) \geq \min\{F_{\Lambda}(\alpha), F_{\Lambda}(\beta)\}$.

3 Interval valued neutrosophic *INK*-ideal in *INK*-algebra

In this section, we introduce the concept of *IVN INK*-ideal and investigate some related properties.

Definition 3.1. An *IVNS* Λ in U is called an interval valued neutrosophic *INK*-subalgebra, (briefly, *IVN INK*-subalg) of U if

- (i) $T_{\Lambda}^+(\alpha * \beta) \geq r \min\{T_{\Lambda}^+(\alpha), T_{\Lambda}^+(\beta)\}$
- (ii) $I_{\Lambda}^+(\alpha * \beta) \leq r \max\{I_{\Lambda}^+(\alpha), I_{\Lambda}^+(\beta)\}$
- (iii) $F_{\Lambda}^+(\alpha * \beta) \geq r \min\{F_{\Lambda}^+(\alpha), F_{\Lambda}^+(\beta)\}$, for all $\alpha, \beta \in U$.

Definition 3.2. An *IVN INK*-algebra U is called an interval valued neutrosophic *INK*-ideal (*IVN INK*-ideal) of U if it satisfies

- (i) $T_{\Lambda}^+(0) \geq T_{\Lambda}^+(\alpha)$
 $I_{\Lambda}^+(0) \leq I_{\Lambda}^+(\alpha)$
 $F_{\Lambda}^+(0) \geq F_{\Lambda}^+(\alpha)$
- (ii) $T_{\Lambda}^+(\alpha) \geq r \min\{T_{\Lambda}^+(\gamma * \alpha) * (\gamma * \beta), T_{\Lambda}^+(\beta)\}$,
 $I_{\Lambda}^+(\alpha) \leq r \max\{I_{\Lambda}^+(\gamma * \alpha) * (\gamma * \beta), I_{\Lambda}^+(\beta)\}$,
 $F_{\Lambda}^+(\alpha) \geq r \min\{F_{\Lambda}^+(\gamma * \alpha) * (\gamma * \beta), F_{\Lambda}^+(\beta)\}$, for all $\alpha, \beta, \gamma \in U$.

Theorem 3.3. If Λ be an *IVN INK*-ideal of U , then the set

- (i) $X_0 = \{\alpha \in U | T_{\Lambda}^+(\alpha) = T_{\Lambda}^+(0)\}$
- (ii) $X_0 = \{\alpha \in U | I_{\Lambda}^+(\alpha) = I_{\Lambda}^+(0)\}$
- (iii) $X_0 = \{\alpha \in U | F_{\Lambda}^+(\alpha) = F_{\Lambda}^+(0)\}$

are *IVN INK*-ideal of U .

Proof. If it clear that $0 \in X_0$. If $\beta \in X_0$ and $(\gamma * \alpha) * (\gamma * \beta) \in X_0$, then

- (i) $T_{\Lambda}^+((\gamma * \alpha) * (\gamma * \beta)) = T_{\Lambda}^+(\beta) = T_{\Lambda}^+(0)$
- (ii) $I_{\Lambda}^+((\gamma * \alpha) * (\gamma * \beta)) = I_{\Lambda}^+(\beta) = I_{\Lambda}^+(0)$
- (iii) $F_{\Lambda}^+((\gamma * \alpha) * (\gamma * \beta)) = F_{\Lambda}^+(\beta) = F_{\Lambda}^+(0)$

Since, Λ be an *IVN INK*-ideal of U , hence

$$\begin{aligned} T_{\Lambda}^{+}(0) &\geq T_{\Lambda}^{+}(\alpha) \geq r \min\{T_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda}^{+}(\beta)\} \\ &= r \min\{T_{\Lambda}^{+}(0), T_{\Lambda}^{+}(0)\} \\ &= T_{\Lambda}^{+}(0) \\ I_{\Lambda}^{+}(0) &\leq I_{\Lambda}^{+}(\alpha) \leq r \max\{I_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda}^{+}(\beta)\} \\ &= r \max\{I_{\Lambda}^{+}(0), I_{\Lambda}^{+}(0)\} \\ &= I_{\Lambda}^{+}(0) \\ F_{\Lambda}^{+}(0) &\geq F_{\Lambda}^{+}(\alpha) \geq r \min\{F_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda}^{+}(\beta)\} \\ &= r \min\{F_{\Lambda}^{+}(0), F_{\Lambda}^{+}(0)\} \\ &= F_{\Lambda}^{+}(0) \end{aligned}$$

It follows that $\alpha \in X_0$, therefore the set X_0 is *IVN INK*-ideal of U . \square

Theorem 3.4. An *IVNS* $\Lambda = [T_{\Lambda}, I_{\Lambda}, F_{\Lambda}]$ in U is a *IVN INK*-ideal of U iff $T_{\Lambda}^L, T_{\Lambda}^U, I_{\Lambda}^L, I_{\Lambda}^U, F_{\Lambda}^L$ and F_{Λ}^U are neutrosophic *INK*-ideals of U .

Proof. Let $T_{\Lambda}^L, T_{\Lambda}^U, I_{\Lambda}^L, I_{\Lambda}^U, F_{\Lambda}^L$ and F_{Λ}^U are neutrosophic *INK*-ideals of U and $\alpha, \beta, \gamma \in U$. Since,

(i) $T_{\Lambda}^L(0) \geq T_{\Lambda}^L(\alpha)$ and $T_{\Lambda}^U(0) \geq T_{\Lambda}^U(\alpha)$. Therefore, $T_{\Lambda}^{+}(0) \geq T_{\Lambda}^{+}(\alpha)$.

$$\begin{aligned} T_{\Lambda}^{+}(\alpha) &= [T_{\Lambda}^L(\alpha), T_{\Lambda}^U(\alpha)] \\ &\geq [\min\{T_{\Lambda}^L(\gamma * \alpha) * (\gamma * \beta), T_{\Lambda}^L(\beta)\}, \min\{T_{\Lambda}^U(\gamma * \alpha) * (\gamma * \beta), T_{\Lambda}^U(\beta)\}] \\ &\geq r \min\{[T_{\Lambda}^L(\gamma * \alpha) * (\gamma * \beta), T_{\Lambda}^U(\gamma * \alpha) * (\gamma * \beta)], [T_{\Lambda}^L(\beta), T_{\Lambda}^U(\beta)]\} \\ &= r \min\{T_{\Lambda}^{+}(\gamma * \alpha) * (\gamma * \beta), T_{\Lambda}^{+}(\beta)\}. \end{aligned}$$

(ii) $I_{\Lambda}^L(0) \leq I_{\Lambda}^L(\alpha)$ and $I_{\Lambda}^U(0) \leq I_{\Lambda}^U(\alpha)$. Therefore, $I_{\Lambda}^{+}(0) \leq I_{\Lambda}^{+}(\alpha)$.

$$\begin{aligned} I_{\Lambda}^{+}(\alpha) &= [I_{\Lambda}^L(\alpha), I_{\Lambda}^U(\alpha)] \\ &\leq [\max\{I_{\Lambda}^L(\gamma * \alpha) * (\gamma * \beta), I_{\Lambda}^L(\beta)\}, \max\{I_{\Lambda}^U(\gamma * \alpha) * (\gamma * \beta), I_{\Lambda}^U(\beta)\}] \\ &\leq r \max\{[I_{\Lambda}^L(\gamma * \alpha) * (\gamma * \beta), I_{\Lambda}^U(\gamma * \alpha) * (\gamma * \beta)], [I_{\Lambda}^L(\beta), I_{\Lambda}^U(\beta)]\} \\ &= r \max\{I_{\Lambda}^{+}(\gamma * \alpha) * (\gamma * \beta), I_{\Lambda}^{+}(\beta)\}. \end{aligned}$$

(iii) $F_{\Lambda}^L(0) \geq F_{\Lambda}^L(\alpha)$ and $F_{\Lambda}^U(0) \geq F_{\Lambda}^U(\alpha)$. Therefore, $F_{\Lambda}^{+}(0) \geq F_{\Lambda}^{+}(\alpha)$.

$$\begin{aligned} F_{\Lambda}^{+}(\alpha) &= [F_{\Lambda}^L(\alpha), F_{\Lambda}^U(\alpha)] \\ &\geq [\min\{F_{\Lambda}^L(\gamma * \alpha) * (\gamma * \beta), F_{\Lambda}^L(\beta)\}, \min\{F_{\Lambda}^U(\gamma * \alpha) * (\gamma * \beta), F_{\Lambda}^U(\beta)\}] \\ &\geq r \min\{[F_{\Lambda}^L(\gamma * \alpha) * (\gamma * \beta), F_{\Lambda}^U(\gamma * \alpha) * (\gamma * \beta)], [F_{\Lambda}^L(\beta), F_{\Lambda}^U(\beta)]\} \\ &= r \min\{F_{\Lambda}^{+}(\gamma * \alpha) * (\gamma * \beta), F_{\Lambda}^{+}(\beta)\}. \end{aligned}$$

Hence Λ is a *IVN INK*-ideal of U .

Conversely, suppose that Λ is a *IVN INK*-ideal of U . For all $\alpha, \beta, \gamma \in U$.

$$\begin{aligned}
 [T_{\Lambda}^L(\alpha), T_{\Lambda}^U(\alpha)] &= T_{\Lambda}^+(\alpha) \\
 &\geq r \min\{T_{\Lambda}^+(\gamma * \alpha) * (\gamma * \beta), T_{\Lambda}^+(\beta)\} \\
 &= r \min\{[T_{\Lambda}^L(\gamma * \alpha) * (\gamma * \beta), T_{\Lambda}^U(\gamma * \alpha) * (\gamma * \beta)], [T_{\Lambda}^L(\beta), T_{\Lambda}^U(\beta)]\} \\
 &= [\min\{T_{\Lambda}^L(\gamma * \alpha) * (\gamma * \beta), T_{\Lambda}^L(\beta)\}, \min\{T_{\Lambda}^U(\gamma * \alpha) * (\gamma * \beta), T_{\Lambda}^U(\beta)\}] \\
 T_{\Lambda}^L(\alpha) &\geq \min\{T_{\Lambda}^L(\gamma * \alpha) * (\gamma * \beta), T_{\Lambda}^L(\beta)\} \\
 T_{\Lambda}^U(\alpha) &\geq \min\{T_{\Lambda}^U(\gamma * \alpha) * (\gamma * \beta), T_{\Lambda}^U(\beta)\}.
 \end{aligned}$$

$$\begin{aligned}
 [I_{\Lambda}^L(\alpha), I_{\Lambda}^U(\alpha)] &= I_{\Lambda}^+(\alpha) \\
 &\leq r \max\{I_{\Lambda}^+(\gamma * \alpha) * (\gamma * \beta), I_{\Lambda}^+(\beta)\} \\
 &= r \max\{[I_{\Lambda}^L(\gamma * \alpha) * (\gamma * \beta), I_{\Lambda}^U(\gamma * \alpha) * (\gamma * \beta)], [I_{\Lambda}^L(\beta), I_{\Lambda}^U(\beta)]\} \\
 &= [\max\{I_{\Lambda}^L(\gamma * \alpha) * (\gamma * \beta), I_{\Lambda}^L(\beta)\}, \max\{I_{\Lambda}^U(\gamma * \alpha) * (\gamma * \beta), I_{\Lambda}^U(\beta)\}] \\
 I_{\Lambda}^L(\alpha) &\leq \max\{I_{\Lambda}^L(\gamma * \alpha) * (\gamma * \beta), I_{\Lambda}^L(\beta)\} \\
 I_{\Lambda}^U(\alpha) &\leq \max\{I_{\Lambda}^U(\gamma * \alpha) * (\gamma * \beta), I_{\Lambda}^U(\beta)\}.
 \end{aligned}$$

$$\begin{aligned}
 [F_{\Lambda}^L(\alpha), F_{\Lambda}^U(\alpha)] &= F_{\Lambda}^+(\alpha) \\
 &\geq r \min\{F_{\Lambda}^+(\gamma * \alpha) * (\gamma * \beta), F_{\Lambda}^+(\beta)\} \\
 &= r \min\{[F_{\Lambda}^L(\gamma * \alpha) * (\gamma * \beta), F_{\Lambda}^U(\gamma * \alpha) * (\gamma * \beta)], [F_{\Lambda}^L(\beta), F_{\Lambda}^U(\beta)]\} \\
 &= [\min\{F_{\Lambda}^L(\gamma * \alpha) * (\gamma * \beta), F_{\Lambda}^L(\beta)\}, \min\{F_{\Lambda}^U(\gamma * \alpha) * (\gamma * \beta), F_{\Lambda}^U(\beta)\}] \\
 F_{\Lambda}^L(\alpha) &\geq \min\{F_{\Lambda}^L(\gamma * \alpha) * (\gamma * \beta), F_{\Lambda}^L(\beta)\} \\
 F_{\Lambda}^U(\alpha) &\geq \min\{F_{\Lambda}^U(\gamma * \alpha) * (\gamma * \beta), F_{\Lambda}^U(\beta)\}.
 \end{aligned}$$

Hence $T_{\Lambda}^L, T_{\Lambda}^U, I_{\Lambda}^L, I_{\Lambda}^U, F_{\Lambda}^L$ and F_{Λ}^U are *N INK*-ideals of U . □

Example 3.5. Let $U = \{0, \alpha, \beta, \gamma\}$ be a set with the following table:

*	0	α	β	γ
0	0	α	β	γ
α	α	0	γ	β
β	β	γ	0	α
γ	γ	β	α	0

Then U is a *INK*-ideal. We define an *IVN INK* ideal $\Lambda = \{T_{\Lambda}, I_{\Lambda}, F_{\Lambda}\}$ as following: $T_{\Lambda}^+(\alpha) = F_{\Lambda}^+(\alpha) = [0.9, 0.7]$

$$T_{\Lambda}^+(\alpha) = F_{\Lambda}^+(\alpha) = [0.6, 0.7]$$

$$T_{\Lambda}^+(\beta) = T_{\Lambda}^+(\gamma) = F_{\Lambda}^+(\beta) = F_{\Lambda}^+(\gamma) = [0.5, 0.6]$$

$$I_{\Lambda}^+(\alpha) = [0.2, 0.3]$$

$$I_{\Lambda}^+(\beta) = [0.4, 0.7]$$

$$I_{\Lambda}^+(\gamma) = [0.5, 0.7]$$

Hence Λ is an *IVN INK* -ideal of U .

Theorem 3.6. Every *INK*-ideal of U can be realized as an *IV* level set of *IVN*-ideal of U .

Proof. Let V be an *INK*-ideal of U and let Λ be an *IVN* set on U defined by

$$T_{\Lambda}^{+}(\alpha) = \begin{cases} [\alpha_1, \alpha_2], & \alpha \in V \\ [1, 1], & \alpha \notin V \end{cases}$$

where $\alpha_1, \alpha_2 \in [0, 1]$ and $\alpha_1 < \alpha_2$.

$$I_{\Lambda}^{+}(\alpha) = \begin{cases} [\alpha_1, \alpha_2], & \alpha \in V \\ [1, 1], & \alpha \notin V \end{cases}$$

where $\alpha_1, \alpha_2 \in [0, 1]$ and $\alpha_1 > \alpha_2$.

$$F_{\Lambda}^{+}(\alpha) = \begin{cases} [\alpha_1, \alpha_2], & \alpha \in V \\ [1, 1], & \alpha \notin V \end{cases}$$

where $\alpha_1, \alpha_2 \in [0, 1]$ and $\alpha_1 < \alpha_2$.

It is clear that $U^{+}(\Lambda; [\alpha_1, \alpha_2]) = V$. We show that Λ be an *IVN INK*-ideal of U . Let $\alpha, \beta, \gamma \in U$. If $(\gamma * \alpha) * (\gamma * \beta) \in V$ and $\beta \in V$, then $\alpha \in V$, this shows that,

- (i) $T_{\Lambda}^{+}(\alpha) = T_{\Lambda}^{+}(\beta) = T_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)) = [\alpha_1, \alpha_2]$. Thus $T_{\Lambda}^{+}(\alpha) \geq r \min\{T_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda}^{+}(\beta)\}$. If $(\gamma * \alpha) * (\gamma * \beta) \notin V$ and $\beta \notin V$, then $T_{\Lambda}^{+}(\beta) = T_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)) = [1, 1]$. Thus $T_{\Lambda}^{+}(\alpha) \geq [1, 1] = r \min\{T_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda}^{+}(\beta)\}$. If $((\gamma * \alpha) * (\gamma * \beta)) \in V$ and $\beta \notin V$ then $T_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)) = [\alpha_1, \alpha_2]$, $T_{\Lambda}^{+}(\beta) = [1, 1]$. Thus, $r \min\{T_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda}^{+}(\beta)\} = [1, 1]$. Hence $T_{\Lambda}^{+}(\alpha) \geq r \min\{T_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda}^{+}(\beta)\}$.
- (ii) $I_{\Lambda}^{+}(\alpha) = I_{\Lambda}^{+}(\beta) = I_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)) = [\alpha_1, \alpha_2]$. Thus $I_{\Lambda}^{+}(\alpha) \leq r \max\{I_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda}^{+}(\beta)\}$. If $(\gamma * \alpha) * (\gamma * \beta) \notin V$ and $\beta \notin V$, then $I_{\Lambda}^{+}(\beta) = I_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)) = [1, 1]$. Thus $I_{\Lambda}^{+}(\alpha) \leq [1, 1] = r \max\{I_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda}^{+}(\beta)\}$. If $((\gamma * \alpha) * (\gamma * \beta)) \in V$ and $\beta \notin V$ then $I_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)) = [\alpha_1, \alpha_2]$, $I_{\Lambda}^{+}(\beta) = [1, 1]$. Thus, $r \max\{I_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda}^{+}(\beta)\} = [1, 1]$. Hence $I_{\Lambda}^{+}(\alpha) \leq r \max\{I_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda}^{+}(\beta)\}$.
- (iii) $F_{\Lambda}^{+}(\alpha) = F_{\Lambda}^{+}(\beta) = F_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)) = [\alpha_1, \alpha_2]$. Thus $F_{\Lambda}^{+}(\alpha) \geq r \min\{F_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda}^{+}(\beta)\}$. If $(\gamma * \alpha) * (\gamma * \beta) \notin V$ and $\beta \notin V$, then $F_{\Lambda}^{+}(\beta) = F_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)) = [1, 1]$. Thus $F_{\Lambda}^{+}(\alpha) \geq [1, 1] = r \min\{F_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda}^{+}(\beta)\}$. If $((\gamma * \alpha) * (\gamma * \beta)) \in V$ and $\beta \notin V$ then $F_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)) = [\alpha_1, \alpha_2]$, $F_{\Lambda}^{+}(\beta) = [1, 1]$. Thus, $r \min\{F_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda}^{+}(\beta)\} = [1, 1]$. Hence $F_{\Lambda}^{+}(\alpha) \geq r \min\{F_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda}^{+}(\beta)\}$.

Similarly for the case $(\gamma * \alpha) * (\gamma * \beta) \notin V$ and $\beta \in V$. We get,

- (i) $T_{\Lambda}^{+}(\alpha) \geq r \min\{T_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda}^{+}(\beta)\}$.
On the other hand, since $0 \in V$, thus $T_{\Lambda}^{+}(0) = [\alpha_1, \alpha_1]$, that is $T_{\Lambda}^{+}(0) \geq T_{\Lambda}^{+}(\alpha)$.
- (ii) $I_{\Lambda}^{+}(\alpha) \leq r \max\{I_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda}^{+}(\beta)\}$.
On the other hand, since $0 \in V$, thus $I_{\Lambda}^{+}(0) = [\alpha_1, \alpha_1]$, that is $I_{\Lambda}^{+}(0) \leq I_{\Lambda}^{+}(\alpha)$.
- (iii) $F_{\Lambda}^{+}(\alpha) \geq r \min\{F_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda}^{+}(\beta)\}$.
On the other hand, since $0 \in V$, thus $F_{\Lambda}^{+}(0) = [\alpha_1, \alpha_1]$, that is $F_{\Lambda}^{+}(0) \geq I_{\Lambda}^{+}(\alpha)$.

Hence Λ is *IVN INK*-ideal of U . □

Theorem 3.7. Let V be a nonempty subset of U and let Λ be an IVN set on U defined by

$$T_{\Lambda}^{+}(\alpha) = \begin{cases} [\alpha_1, \alpha_2], & \alpha \in V \\ [1, 1], & \alpha \notin V \end{cases}$$

where $\alpha_1, \alpha_2 \in [0, 1]$ and $\alpha_1 < \alpha_2$.

$$I_{\Lambda}^{+}(\alpha) = \begin{cases} [\alpha_1, \alpha_2], & \alpha \in V \\ [1, 1], & \alpha \notin V \end{cases}$$

where $\alpha_1, \alpha_2 \in [0, 1]$ and $\alpha_1 > \alpha_2$.

$$F_{\Lambda}^{+}(\alpha) = \begin{cases} [\alpha_1, \alpha_2], & \alpha \in V \\ [1, 1], & \alpha \notin V \end{cases}$$

where $\alpha_1, \alpha_2 \in [0, 1]$ and $\alpha_1 < \alpha_2$.

If U is an IVN INK -ideal of U . Then V is an IVN INK -ideal of U .

Proof. Assume that Λ is IVN INK ideal of U . Then,

- (i) $T_{\Lambda}^{+}(0) \geq T_{\Lambda}^{+}(\alpha)$, $T_{\Lambda}^{+}(\alpha) \geq r \min\{T_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda}^{+}(\beta)\} \forall \alpha, \beta, \gamma \in U$.
- (ii) $I_{\Lambda}^{+}(0) \leq I_{\Lambda}^{+}(\alpha)$, $I_{\Lambda}^{+}(\alpha) \leq r \max\{I_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda}^{+}(\beta)\} \forall \alpha, \beta, \gamma \in U$.
- (iii) $F_{\Lambda}^{+}(0) \geq F_{\Lambda}^{+}(\alpha)$, $F_{\Lambda}^{+}(\alpha) \geq r \min\{F_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda}^{+}(\beta)\} \forall \alpha, \beta, \gamma \in U$.

Since V be a non - empty subset of U , thus there exists $\alpha \in V$ such that,

- (i) $T_{\Lambda}^{+}(\alpha) = [\alpha_1, \alpha_2]$, so $T_{\Lambda}^{+}(0) \geq T_{\Lambda}^{+}(\alpha) = [\alpha_1, \alpha_2]$
- (ii) $I_{\Lambda}^{+}(\alpha) = [\alpha_1, \alpha_2]$, so $I_{\Lambda}^{+}(0) \leq I_{\Lambda}^{+}(\alpha) = [\alpha_1, \alpha_2]$.
- (iii) $F_{\Lambda}^{+}(\alpha) = [\alpha_1, \alpha_2]$, so $F_{\Lambda}^{+}(0) \geq F_{\Lambda}^{+}(\alpha) = [\alpha_1, \alpha_2]$.

This shows that $0 \in V$. If $(\gamma * \alpha) * (\gamma * \beta) \in V$ and $\beta \in V$, then

- (i) $T_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)) = T_{\Lambda}^{+}(\alpha) = [\alpha_1, \alpha_2]$. Therefore,

$$\begin{aligned} T_{\Lambda}^{+}(\alpha) &\geq r \min\{T_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda}^{+}(\beta)\} \\ &= r \min\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} \\ &= [\alpha_1, \alpha_2]. \end{aligned}$$

So, $T_{\Lambda}^{+}(\alpha) = [\alpha_1, \alpha_2]$, that is $\alpha \in V$

- (ii) $I_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)) = I_{\Lambda}^{+}(\alpha) = [\alpha_1, \alpha_2]$. Therefore,

$$\begin{aligned} I_{\Lambda}^{+}(\alpha) &\leq r \max\{I_{\Lambda}^{+}((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda}^{+}(\beta)\} \\ &= r \max\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} \\ &= [\alpha_1, \alpha_2]. \end{aligned}$$

So, $I_{\Lambda}^{+}(\alpha) = [\alpha_1, \alpha_2]$, that is $\alpha \in V$

(iii) $F_{\Lambda}^+((\gamma * \alpha) * (\gamma * \beta)) = F_{\Lambda}^+(\alpha) = [\alpha_1, \alpha_2]$. Therefore,

$$\begin{aligned} F_{\Lambda}^+(\alpha) &\geq r \min\{F_{\Lambda}^+((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda}^+(\beta)\} \\ &= r \min\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} \\ &= [\alpha_1, \alpha_2]. \end{aligned}$$

So, $F_{\Lambda}^+(\alpha) = [\alpha_1, \alpha_2]$, that is $\alpha \in V$

Hence V is IVN INK -ideal of U . □

Definition 3.8. Let $\Lambda_1^+ = \{(\alpha, T_{\Lambda_1}^+(\alpha), I_{\Lambda_1}^+(\alpha), F_{\Lambda_1}^+(\alpha)), \alpha \in U\}$ and $\Lambda_2^+ = \{(\alpha, T_{\Lambda_2}^+(\alpha), I_{\Lambda_2}^+(\alpha), F_{\Lambda_2}^+(\alpha)), \alpha \in U\}$ are two $IVNS$'s on a INK -algebra U , define the $IVNS$ $(\Lambda_1 \cap \Lambda_2)^+ = (T_{\Lambda_1 \cap \Lambda_2}^+, I_{\Lambda_1 \cap \Lambda_2}^+, F_{\Lambda_1 \cap \Lambda_2}^+)$ on U by,

(i) $T_{\Lambda_1 \cap \Lambda_2}^+(\alpha) = r \min\{T_{\Lambda_1}^+(\alpha), T_{\Lambda_2}^+(\alpha)\}, \alpha \in U$.

Then, $T_{\Lambda_1 \cap \Lambda_2}^+$ is called the intersection of $T_{\Lambda_1}^+$ and $T_{\Lambda_2}^+$.

(ii) $I_{\Lambda_1 \cap \Lambda_2}^+(\alpha) = r \max\{I_{\Lambda_1}^+(\alpha), I_{\Lambda_2}^+(\alpha)\}, \alpha \in U$.

Then, $I_{\Lambda_1 \cap \Lambda_2}^+$ is called the intersection of $I_{\Lambda_1}^+$ and $I_{\Lambda_2}^+$.

(iii) $F_{\Lambda_1 \cap \Lambda_2}^+(\alpha) = r \min\{F_{\Lambda_1}^+(\alpha), F_{\Lambda_2}^+(\alpha)\}, \alpha \in U$.

Then, $F_{\Lambda_1 \cap \Lambda_2}^+$ is called the intersection of $F_{\Lambda_1}^+$ and $F_{\Lambda_2}^+$.

Theorem 3.9. Let Λ_1 & Λ_2 be two IVN INK -ideals of a INK -algebra U . Then $\Lambda_1 \cap \Lambda_2$ is a IVN INK -ideal of U .

Proof. Assume that Λ_1 and Λ_2 be two IVN INK -ideals of INK -algebra U . Then for all $\alpha, \beta, \gamma \in U$, we have

(i) $T_{\Lambda_1}^+(0) \geq T_{\Lambda_1}^+(\alpha), T_{\Lambda_1}^+(\alpha) \geq r \min\{T_{\Lambda_1}^+((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda_1}^+(\beta)\},$

$T_{\Lambda_2}^+(0) \geq T_{\Lambda_2}^+(\alpha), T_{\Lambda_2}^+(\alpha) \geq r \min\{T_{\Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda_2}^+(\beta)\}.$

Hence,

$$\begin{aligned} T_{\Lambda_1 \cap \Lambda_2}^+(0) &= r \min\{T_{\Lambda_1}^+(0), T_{\Lambda_2}^+(0)\} \\ &\geq r \min\{T_{\Lambda_1}^+(\alpha), T_{\Lambda_2}^+(\alpha)\} \\ &= T_{\Lambda_1 \cap \Lambda_2}^+(\alpha) \end{aligned}$$

$$\begin{aligned} T_{\Lambda_1 \cap \Lambda_2}^+(\alpha) &= r \min\{T_{\Lambda_1}^+(\alpha), T_{\Lambda_2}^+(\alpha)\} \\ &\geq r \min\{r \min\{T_{\Lambda_1}^+((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda_1}^+(\beta)\}, r \min\{T_{\Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda_2}^+(\beta)\}\} \\ &= r \min\{r \min\{T_{\Lambda_1}^+((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta))\}, r \min\{T_{\Lambda_1}^+(\beta), T_{\Lambda_2}^+(\beta)\}\} \\ &= r \min\{T_{\Lambda_1 \cap \Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda_1 \cap \Lambda_2}^+(\beta)\}. \end{aligned}$$

(ii) $I_{\Lambda_1}^+(0) \leq I_{\Lambda_1}^+(\alpha), I_{\Lambda_1}^+(\alpha) \leq r \max\{I_{\Lambda_1}^+((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda_1}^+(\beta)\},$

$I_{\Lambda_2}^+(0) \leq I_{\Lambda_2}^+(\alpha), I_{\Lambda_2}^+(\alpha) \leq r \max\{I_{\Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda_2}^+(\beta)\}.$

Hence,

$$\begin{aligned} I_{\Lambda_1 \cap \Lambda_2}^+(0) &= r \max\{I_{\Lambda_1}^+(0), I_{\Lambda_2}^+(0)\} \\ &\leq r \max\{I_{\Lambda_1}^+(\alpha), I_{\Lambda_2}^+(\alpha)\} \\ &= I_{\Lambda_1 \cap \Lambda_2}^+(\alpha) \end{aligned}$$

$$\begin{aligned} I_{\Lambda_1 \cap \Lambda_2}^+(\alpha) &= r \min\{I_{\Lambda_1}^+(\alpha), I_{\Lambda_2}^+(\alpha)\} \\ &\leq r \min\{r \max\{I_{\Lambda_1}^+((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda_1}^+(\beta)\}, r \max\{I_{\Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda_2}^+(\beta)\}\} \\ &= r \max\{r \min\{I_{\Lambda_1}^+((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta))\}, r \min\{T_{\Lambda_1}^+(\beta), I_{\Lambda_2}^+(\beta)\}\} \\ &= r \max\{I_{\Lambda_1 \cap \Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda_1 \cap \Lambda_2}^+(\beta)\}. \end{aligned}$$

- (iii) $F_{\Lambda_1}^+(0) \geq F_{\Lambda_1}^+(\alpha), F_{\Lambda_1}^+(\alpha) \geq r \min\{F_{\Lambda_1}^+((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda_1}^+(\beta)\},$
 $F_{\Lambda_2}^+(0) \geq F_{\Lambda_2}^+(\alpha), F_{\Lambda_2}^+(\alpha) \geq r \min\{F_{\Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda_2}^+(\beta)\}.$

Hence,

$$\begin{aligned} F_{\Lambda_1 \cap \Lambda_2}^+(0) &= r \min\{F_{\Lambda_1}^+(0), F_{\Lambda_2}^+(0)\} \\ &\leq r \min\{F_{\Lambda_1}^+(\alpha), F_{\Lambda_2}^+(\alpha)\} \\ &= F_{\Lambda_1 \cap \Lambda_2}^+(\alpha) \\ F_{\Lambda_1 \cap \Lambda_2}^+(\alpha) &= r \min\{F_{\Lambda_1}^+(\alpha), F_{\Lambda_2}^+(\alpha)\} \\ &\geq r \min\{r \min\{F_{\Lambda_1}^+((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda_1}^+(\beta)\}, r \min\{F_{\Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda_2}^+(\beta)\}\} \\ &= r \min\{r \min\{F_{\Lambda_1}^+((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta))\}, r \min\{F_{\Lambda_1}^+(\beta), F_{\Lambda_2}^+(\beta)\}\} \\ &= r \min\{F_{\Lambda_1 \cap \Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda_1 \cap \Lambda_2}^+(\beta)\}. \end{aligned}$$

Hence the intersection of Λ_1 and Λ_2 is also *IVN INK*-ideal of U .

□

Definition 3.10. Let $\Lambda_1^+ = \{(\alpha, T_{\Lambda_1}^+(\alpha), I_{\Lambda_1}^+(\alpha), F_{\Lambda_1}^+(\alpha)), \alpha \in U\}$ and $\Lambda_2^+ = \{(\alpha, T_{\Lambda_2}^+(\alpha), I_{\Lambda_2}^+(\alpha), F_{\Lambda_2}^+(\alpha)), \alpha \in U\}$ are two *IVNS*'s on a *INK*-algebra U , define the *IVNS* $(\Lambda_1 \cup \Lambda_2)^+ = (T_{\Lambda_1 \cup \Lambda_2}^+, I_{\Lambda_1 \cup \Lambda_2}^+, F_{\Lambda_1 \cup \Lambda_2}^+)$ on U by,

- (i) $T_{\Lambda_1 \cup \Lambda_2}^+(\alpha) = r \max\{T_{\Lambda_1}^+(\alpha), T_{\Lambda_2}^+(\alpha)\}, \alpha \in U.$

Then, $T_{\Lambda_1 \cup \Lambda_2}^+$ is called the union of $T_{\Lambda_1}^+$ and $T_{\Lambda_2}^+$.

- (ii) $I_{\Lambda_1 \cup \Lambda_2}^+(\alpha) = r \min\{I_{\Lambda_1}^+(\alpha), I_{\Lambda_2}^+(\alpha)\}, \alpha \in U.$

Then, $I_{\Lambda_1 \cup \Lambda_2}^+$ is called the union of $I_{\Lambda_1}^+$ and $I_{\Lambda_2}^+$.

- (iii) $F_{\Lambda_1 \cup \Lambda_2}^+(\alpha) = r \max\{F_{\Lambda_1}^+(\alpha), F_{\Lambda_2}^+(\alpha)\}, \alpha \in U.$

Then, $F_{\Lambda_1 \cup \Lambda_2}^+$ is called the union of $F_{\Lambda_1}^+$ and $F_{\Lambda_2}^+$.

Theorem 3.11. Let Λ_1 & Λ_2 be two *IVN INK*-ideals of a *INK*-algebra U . Then $\Lambda_1 \cup \Lambda_2$ is a *IVN INK*-ideal of U .

Proof. Assume that Λ_1 and Λ_2 be two *IVN INK*-ideals of *INK*-algebra U . Then for all $\alpha, \beta, \gamma \in U$, we have

- (i) $T_{\Lambda_1}^+(0) \geq T_{\Lambda_1}^+(\alpha), T_{\Lambda_1}^+(\alpha) \geq r \min\{T_{\Lambda_1}^+((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda_1}^+(\beta)\},$

- $T_{\Lambda_2}^+(0) \geq T_{\Lambda_2}^+(\alpha), T_{\Lambda_2}^+(\alpha) \geq r \min\{T_{\Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda_2}^+(\beta)\}.$

Hence,

$$\begin{aligned} T_{\Lambda_1 \cup \Lambda_2}^+(0) &= r \min\{T_{\Lambda_1}^+(0), T_{\Lambda_2}^+(0)\} \\ &\geq r \min\{T_{\Lambda_1}^+(\alpha), T_{\Lambda_2}^+(\alpha)\} \\ &= T_{\Lambda_1 \cup \Lambda_2}^+(\alpha) \\ T_{\Lambda_1 \cup \Lambda_2}^+(\alpha) &= r \max\{T_{\Lambda_1}^+(\alpha), T_{\Lambda_2}^+(\alpha)\} \\ &\geq r \max\{r \min\{T_{\Lambda_1}^+((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda_1}^+(\beta)\}, r \min\{T_{\Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda_2}^+(\beta)\}\} \\ &= r \min\{r \max\{T_{\Lambda_1}^+((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta))\}, r \max\{T_{\Lambda_1}^+(\beta), T_{\Lambda_2}^+(\beta)\}\} \\ &= r \min\{T_{\Lambda_1 \cup \Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda_1 \cup \Lambda_2}^+(\beta)\}. \end{aligned}$$

- (ii) $I_{\Lambda_1}^+(0) \leq I_{\Lambda_1}^+(\alpha), I_{\Lambda_1}^+(\alpha) \leq r \max\{I_{\Lambda_1}^+((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda_1}^+(\beta)\},$
 $I_{\Lambda_2}^+(0) \leq I_{\Lambda_2}^+(\alpha), I_{\Lambda_2}^+(\alpha) \leq r \max\{I_{\Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda_2}^+(\beta)\}.$

Hence,

$$\begin{aligned} I_{\Lambda_1 \cup \Lambda_2}^+(0) &= r \min\{I_{\Lambda_1}^+(0), I_{\Lambda_2}^+(0)\} \\ &\leq r \min\{I_{\Lambda_1}^+(\alpha), I_{\Lambda_2}^+(\alpha)\} \\ &= I_{\Lambda_1 \cup \Lambda_2}^+(\alpha) \\ I_{\Lambda_1 \cup \Lambda_2}^+(\alpha) &= r \min\{I_{\Lambda_1}^+(\alpha), I_{\Lambda_2}^+(\alpha)\} \\ &\leq r \min\{r \max\{I_{\Lambda_1}^+((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda_1}^+(\beta)\}, r \max\{I_{\Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda_2}^+(\beta)\}\} \\ &= r \max\{r \min\{I_{\Lambda_1}^+((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta))\}, r \min\{I_{\Lambda_1}^+(\beta), I_{\Lambda_2}^+(\beta)\}\} \\ &= r \max\{I_{\Lambda_1 \cup \Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda_1 \cup \Lambda_2}^+(\beta)\}. \end{aligned}$$

- (iii) $F_{\Lambda_1}^+(0) \geq F_{\Lambda_1}^+(\alpha), F_{\Lambda_1}^+(\alpha) \geq r \min\{F_{\Lambda_1}^+((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda_1}^+(\beta)\},$
 $F_{\Lambda_2}^+(0) \geq F_{\Lambda_2}^+(\alpha), F_{\Lambda_2}^+(\alpha) \geq r \min\{F_{\Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda_2}^+(\beta)\}.$

Hence,

$$\begin{aligned} F_{\Lambda_1 \cup \Lambda_2}^+(0) &= r \min\{F_{\Lambda_1}^+(0), F_{\Lambda_2}^+(0)\} \\ &\geq r \min\{F_{\Lambda_1}^+(\alpha), F_{\Lambda_2}^+(\alpha)\} \\ &= F_{\Lambda_1 \cup \Lambda_2}^+(\alpha) \\ F_{\Lambda_1 \cup \Lambda_2}^+(\alpha) &= r \min\{F_{\Lambda_1}^+(\alpha), F_{\Lambda_2}^+(\alpha)\} \\ &\geq r \min\{r \max\{F_{\Lambda_1}^+((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda_1}^+(\beta)\}, r \max\{F_{\Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda_2}^+(\beta)\}\} \\ &= r \max\{r \min\{F_{\Lambda_1}^+((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta))\}, r \min\{F_{\Lambda_1}^+(\beta), F_{\Lambda_2}^+(\beta)\}\} \\ &= r \max\{F_{\Lambda_1 \cup \Lambda_2}^+((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda_1 \cup \Lambda_2}^+(\beta)\}. \end{aligned}$$

Hence the union of Λ_1 and Λ_2 is also *IVN INK*-ideal of U .

□

Definition 3.12. Let Λ_1 and Λ_2 be two *IVNFS*'s of U , then the direct product of Λ_1 and Λ_2 are

- (i) $\Lambda_1 \times \Lambda_2 = \{(\alpha, \beta), [T_{\Lambda_1 \times \Lambda_2}^L(\alpha, \beta), T_{\Lambda_1 \times \Lambda_2}^U(\alpha, \beta)]\}$ is defined by, $\bar{T}_{\Lambda_1 \times \Lambda_2}^+(\alpha, \beta) = r \min\{\bar{T}_{\Lambda_1}^+(\alpha), \bar{T}_{\Lambda_2}^+(\beta)\}.$
- (ii) $\Lambda_1 \times \Lambda_2 = \{(\alpha, \beta), [I_{\Lambda_1 \times \Lambda_2}^L(\alpha, \beta), I_{\Lambda_1 \times \Lambda_2}^U(\alpha, \beta)]\}$ is defined by, $\bar{I}_{\Lambda_1 \times \Lambda_2}^+(\alpha, \beta) = r \max\{\bar{I}_{\Lambda_1}^+(\alpha), \bar{I}_{\Lambda_2}^+(\beta)\}.$
- (iii) $\Lambda_1 \times \Lambda_2 = \{(\alpha, \beta), [F_{\Lambda_1 \times \Lambda_2}^L(\alpha, \beta), F_{\Lambda_1 \times \Lambda_2}^U(\alpha, \beta)]\}$ is defined by, $\bar{F}_{\Lambda_1 \times \Lambda_2}^+(\alpha, \beta) = r \min\{\bar{F}_{\Lambda_1}^+(\alpha), \bar{F}_{\Lambda_2}^+(\beta)\}.$

Theorem 3.13. Let Λ_1 and Λ_2 be any two *IVN INK*-ideals of U , then the direct product $\Lambda_1 \times \Lambda_2$ is *IVN INK*-ideal of $U \times U$.

Proof. Assume that Λ_1 and Λ_2 be any two *IVN INK*-ideals of U . Then $\forall \alpha, \beta, \gamma \in U$, we have

- (i) $\bar{T}_{\Lambda_1}^+(0) \geq \bar{T}_{\Lambda_1}^+(\alpha),$
 $\bar{T}_{\Lambda_1}^+(\alpha) \geq r \min\{\bar{T}_{\Lambda_1}^+(\gamma * \alpha) * (\gamma * \beta), \bar{T}_{\Lambda_1}^+(\beta)\}$
 $\bar{T}_{\Lambda_2}^+(0) \geq \bar{T}_{\Lambda_2}^+(\alpha),$
 $\bar{T}_{\Lambda_2}^+(\alpha) \geq r \min\{\bar{T}_{\Lambda_2}^+(\gamma * \alpha) * (\gamma * \beta), \bar{T}_{\Lambda_2}^+(\beta)\}.$
 Hence for all $(\alpha, \beta) \in U \times U$.

$$\begin{aligned} \bar{T}_{\Lambda_1 \times \Lambda_2}^+(0, 0) &= r \min\{\bar{T}_{\Lambda_1}^+(0), \bar{T}_{\Lambda_2}^+(0)\} \\ &\geq r \min\{\bar{T}_{\Lambda_1}^+(\alpha), \bar{T}_{\Lambda_2}^+(\beta)\} \\ &= \bar{T}_{\Lambda_1 \times \Lambda_2}^+(\alpha, \beta). \end{aligned}$$

for all $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ and $(\alpha_3, \beta_3) \in U \times U$, we have

$$\begin{aligned} &\bar{T}_{\Lambda_1 \times \Lambda_2}^+(\alpha_1, \beta_1) \\ &= r \min\{\bar{T}_{\Lambda_1}^+(\alpha_1), \bar{T}_{\Lambda_2}^+(\beta_1)\} \\ &\geq r \min\{r \min\{\bar{T}_{\Lambda_1}^+(\alpha_3 * \alpha_1) * (\alpha_3 * \alpha_2), \bar{T}_{\Lambda_1}^+(\alpha_2)\}, r \min\{\bar{T}_{\Lambda_2}^+(\beta_3 * \beta_1) * (\beta_3 * \beta_2), \{\bar{T}_{\Lambda_2}^+(\beta_2)\}\}\} \\ &= r \min\{r \min\{\bar{T}_{\Lambda_1}^+(\alpha_3 * \alpha_1) * (\alpha_3 * \alpha_2), \bar{T}_{\Lambda_2}^+(\beta_3 * \beta_1) * (\beta_3 * \beta_2)\}, r \min\{\bar{T}_{\Lambda_1}^+(\alpha_2), \bar{T}_{\Lambda_2}^+(\beta_2)\}\} \\ &= r \min\{\bar{T}_{\Lambda_1 \times \Lambda_2}^+((\alpha_3 * \alpha_1) * (\alpha_3 * \alpha_2), (\beta_3 * \beta_1) * (\beta_3 * \beta_2)), \bar{T}_{\Lambda_1 \times \Lambda_2}^+(\alpha_2, \beta_2)\} \\ &= r \min\{\bar{T}_{\Lambda_1 \times \Lambda_2}^+((\alpha_3 * \alpha_1) * (\beta_3 * \beta_1), (\alpha_3 * \alpha_2) * (\beta_3 * \beta_2)), \bar{T}_{\Lambda_1 \times \Lambda_2}^+(\alpha_2, \beta_2)\} \\ &\bar{T}_{\Lambda_1 \times \Lambda_2}^+(\alpha_1, \beta_1) \\ &\geq r \min\{\bar{T}_{\Lambda_1 \times \Lambda_2}^+(((\alpha_3, \beta_3) * (\alpha_1, \beta_1)), ((\alpha_3, \beta_3) * (\alpha_2, \beta_2))), \bar{T}_{\Lambda_1 \times \Lambda_2}^+(\alpha_2, \beta_2)\}. \end{aligned}$$

(ii) $\bar{I}_{\Lambda_1}^+(0) \leq \bar{I}_{\Lambda_1}^+(\alpha)$,
 $\bar{I}_{\Lambda_1}^+(\alpha) \leq r \max\{\bar{I}_{\Lambda_1}^+(\gamma * \alpha) * (\gamma * \beta), \bar{I}_{\Lambda_1}^+(\beta)\}$
 $\bar{I}_{\Lambda_2}^+(0) \leq \bar{I}_{\Lambda_2}^+(\alpha)$,
 $\bar{I}_{\Lambda_2}^+(\alpha) \leq r \max\{\bar{I}_{\Lambda_2}^+(\gamma * \alpha) * (\gamma * \beta), \bar{I}_{\Lambda_2}^+(\beta)\}$.
Hence for all $(\alpha, \beta) \in U \times U$.

$$\begin{aligned} \bar{I}_{\Lambda_1 \times \Lambda_2}^+(0, 0) &= r \max\{\bar{I}_{\Lambda_1}^+(0), \bar{I}_{\Lambda_2}^+(0)\} \\ &\leq r \max\{\bar{I}_{\Lambda_1}^+(\alpha), \bar{I}_{\Lambda_2}^+(\beta)\} \\ &= \bar{I}_{\Lambda_1 \times \Lambda_2}^+(\alpha, \beta). \end{aligned}$$

for all $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ and $(\alpha_3, \beta_3) \in U \times U$, we have

$$\begin{aligned} &\bar{I}_{\Lambda_1 \times \Lambda_2}^+(\alpha_1, \beta_1) \\ &= r \min\{\bar{I}_{\Lambda_1}^+(\alpha_1), \bar{I}_{\Lambda_2}^+(\beta_1)\} \\ &\leq r \min\{r \min\{\bar{I}_{\Lambda_1}^+(\alpha_3 * \alpha_1) * (\alpha_3 * \alpha_2), \bar{I}_{\Lambda_1}^+(\alpha_2)\}, r \min\{\bar{I}_{\Lambda_2}^+(\beta_3 * \beta_1) * (\beta_3 * \beta_2), \{\bar{I}_{\Lambda_2}^+(\beta_2)\}\}\} \\ &= r \min\{r \min\{\bar{I}_{\Lambda_1}^+(\alpha_3 * \alpha_1) * (\alpha_3 * \alpha_2), \bar{I}_{\Lambda_2}^+(\beta_3 * \beta_1) * (\beta_3 * \beta_2)\}, r \min\{\bar{I}_{\Lambda_1}^+(\alpha_2), \bar{I}_{\Lambda_2}^+(\beta_2)\}\} \\ &= r \min\{\bar{I}_{\Lambda_1 \times \Lambda_2}^+((\alpha_3 * \alpha_1) * (\alpha_3 * \alpha_2), (\beta_3 * \beta_1) * (\beta_3 * \beta_2)), \bar{I}_{\Lambda_1 \times \Lambda_2}^+(\alpha_2, \beta_2)\} \\ &= r \min\{\bar{I}_{\Lambda_1 \times \Lambda_2}^+((\alpha_3 * \alpha_1) * (\beta_3 * \beta_1), (\alpha_3 * \alpha_2) * (\beta_3 * \beta_2)), \bar{I}_{\Lambda_1 \times \Lambda_2}^+(\alpha_2, \beta_2)\} \\ &\bar{I}_{\Lambda_1 \times \Lambda_2}^+(\alpha_1, \beta_1) \\ &\leq r \min\{\bar{I}_{\Lambda_1 \times \Lambda_2}^+(((\alpha_3, \beta_3) * (\alpha_1, \beta_1)), ((\alpha_3, \beta_3) * (\alpha_2, \beta_2))), \bar{I}_{\Lambda_1 \times \Lambda_2}^+(\alpha_2, \beta_2)\}. \end{aligned}$$

(iii) $\bar{F}_{\Lambda_1}^+(0) \geq \bar{F}_{\Lambda_1}^+(\alpha)$,
 $\bar{F}_{\Lambda_1}^+(\alpha) \geq r \min\{\bar{F}_{\Lambda_1}^+(\gamma * \alpha) * (\gamma * \beta), \bar{F}_{\Lambda_1}^+(\beta)\}$
 $\bar{F}_{\Lambda_2}^+(0) \geq \bar{F}_{\Lambda_2}^+(\alpha)$,
 $\bar{F}_{\Lambda_2}^+(\alpha) \geq r \min\{\bar{F}_{\Lambda_2}^+(\gamma * \alpha) * (\gamma * \beta), \bar{F}_{\Lambda_2}^+(\beta)\}$.
Hence for all $(\alpha, \beta) \in U \times U$.

$$\begin{aligned} \bar{F}_{\Lambda_1 \times \Lambda_2}^+(0, 0) &= r \min\{\bar{F}_{\Lambda_1}^+(0), \bar{F}_{\Lambda_2}^+(0)\} \\ &\geq r \min\{\bar{F}_{\Lambda_1}^+(\alpha), \bar{F}_{\Lambda_2}^+(\beta)\} \\ &= \bar{F}_{\Lambda_1 \times \Lambda_2}^+(\alpha, \beta). \end{aligned}$$

for all $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ and $(\alpha_3, \beta_3) \in U \times U$,

we have

$$\begin{aligned} & \bar{F}_{\Lambda_1 \times \Lambda_2}^+(\alpha_1, \beta_1) \\ &= r \min\{\bar{F}_{\Lambda_1}^+(\alpha_1), \bar{F}_{\Lambda_2}^+(\beta_1)\} \\ &\geq r \min\{r \min\{\bar{F}_{\Lambda_1}^+(\alpha_3 * \alpha_1) * (\alpha_3 * \alpha_2), \bar{F}_{\Lambda_1}^+(\alpha_2)\}, r \min\{\bar{F}_{\Lambda_2}^+(\beta_3 * \beta_1) * (\beta_3 * \beta_2), \{\bar{F}_{\Lambda_2}^+(\beta_2)\}\}\} \\ &= r \min\{r \min\{\bar{F}_{\Lambda_1}^+(\alpha_3 * \alpha_1) * (\alpha_3 * \alpha_2), \bar{F}_{\Lambda_2}^+(\beta_3 * \beta_1) * (\beta_3 * \beta_2)\}, r \min\{\bar{F}_{\Lambda_1}^+(\alpha_2), \bar{F}_{\Lambda_2}^+(\beta_2)\}\} \\ &= r \min\{\bar{F}_{\Lambda_1 \times \Lambda_2}^+((\alpha_3 * \alpha_1) * (\alpha_3 * \alpha_2), (\beta_3 * \beta_1) * (\beta_3 * \beta_2)), \bar{F}_{\Lambda_1 \times \Lambda_2}^+(\alpha_2, \beta_2)\} \\ &= r \min\{\bar{F}_{\Lambda_1 \times \Lambda_2}^+((\alpha_3 * \alpha_1) * (\beta_3 * \beta_1), (\alpha_3 * \alpha_2) * (\beta_3 * \beta_2)), \bar{F}_{\Lambda_1 \times \Lambda_2}^+(\alpha_2, \beta_2)\} \\ &\bar{F}_{\Lambda_1 \times \Lambda_2}^+(\alpha_1, \beta_1) \\ &\geq r \min\{\bar{F}_{\Lambda_1 \times \Lambda_2}^+(((\alpha_3, \beta_3) * (\alpha_1, \beta_1)), ((\alpha_3, \beta_3) * (\alpha_2, \beta_2))), \bar{F}_{\Lambda_1 \times \Lambda_2}^+(\alpha_2, \beta_2)\}. \end{aligned}$$

Therefore, the direct product $\Lambda_1 \times \Lambda_2$ is *IVN INK*-ideal of $U \times U$.

□

Theorem 3.14. Let U be a *INK*-algebra and Λ be an *IVN* subset in U . Then Λ is an *IVN INK*-ideal of U , iff

- (i) $\cup^+(\Lambda[\vartheta_1, \vartheta_2]) = \{\alpha \in U / T_{\Lambda}^+(\alpha) \geq [\vartheta_1, \vartheta_2]\}$
- (ii) $\cup^+(\Lambda[\vartheta_1, \vartheta_2]) = \{\alpha \in U / I_{\Lambda}^+(\alpha) \leq [\vartheta_1, \vartheta_2]\}$
- (iii) $\cup^+(\Lambda[\vartheta_1, \vartheta_2]) = \{\alpha \in U / F_{\Lambda}^+(\alpha) \geq [\vartheta_1, \vartheta_2]\}$

are *N INK*-ideal of $\Lambda \forall [\vartheta_1, \vartheta_2] \in D[0, 1]$. We call $\cup^+(\Lambda : [\vartheta_1, \vartheta_2])$ the *IV level N INK*-ideal of Λ .

Proof. Assume that Λ is an *IVN INK*-ideal of U . Let $[\vartheta_1, \vartheta_2] \in D[0, 1]$, such that $(\gamma * \alpha) * (\gamma * \beta), \beta \in \cup^+(\Lambda : [\vartheta_1, \vartheta_2])$

$$\begin{aligned} T_{\Lambda}^+(\alpha) &\geq r \min\{T_{\Lambda}^+(\gamma * \alpha) * (\gamma * \beta), T_{\Lambda}^+(\beta)\} \\ &\geq r \min\{[\vartheta_1, \vartheta_2], [\vartheta_1, \vartheta_2]\} \\ &= [\vartheta_1, \vartheta_2]. \end{aligned}$$

$$\begin{aligned} I_{\Lambda}^+(\alpha) &\leq r \max\{I_{\Lambda}^+(\gamma * \alpha) * (\gamma * \beta), I_{\Lambda}^+(\beta)\} \\ &\leq r \max\{[\vartheta_1, \vartheta_2], [\vartheta_1, \vartheta_2]\} \\ &= [\vartheta_1, \vartheta_2]. \end{aligned}$$

$$\begin{aligned} F_{\Lambda}^+(\alpha) &\geq r \min\{F_{\Lambda}^+(\gamma * \alpha) * (\gamma * \beta), F_{\Lambda}^+(\beta)\} \\ &\geq r \min\{[\vartheta_1, \vartheta_2], [\vartheta_1, \vartheta_2]\} \\ &= [\vartheta_1, \vartheta_2]. \end{aligned}$$

Therefore $\alpha \in \cup^+(\Lambda : [\vartheta_1, \vartheta_2])$ and then $\cup^+(\Lambda : [\vartheta_1, \vartheta_2])$ is *IV level N INK*-ideal of Λ .

Conversely, assume that $\cup^+(\Lambda : [\vartheta_1, \vartheta_2]) \neq \phi$ is a *N INK*-ideal of $U, \forall [\vartheta_1, \vartheta_2] \in D[0, 1]$. On the contrary, suppose that there exist $\alpha_0, \beta_0, \gamma_0 \in U$ such that

(i)

$$\begin{aligned}
 T_{\Lambda}^+(\alpha_0) &\geq r \min\{T_{\Lambda}^+(\gamma_0 * \alpha_0) * (\gamma_0 * \beta_0), T_{\Lambda}^+(\beta_0)\} \\
 T_{\Lambda}^+(\alpha_0) &= [\vartheta_1, \vartheta_2], \\
 T_{\Lambda}^+(\gamma_0 * \alpha_0) * (\gamma_0 * \beta_0) &= [\eta_1, \eta_2], \\
 T_{\Lambda}^+(\beta_0) &= [\vartheta_3, \vartheta_4] \\
 \text{If, } [\vartheta_1, \vartheta_2] &< r \min\{[\eta_1, \eta_2], [\eta_3, \eta_4]\} \\
 &= \min[\eta_1, \eta_2], \min[\eta_3, \eta_4].
 \end{aligned}$$

So $\vartheta_1 < \min[\eta_1, \eta_2]$ and $\vartheta_2 < \min[\eta_3, \eta_4]$.

$$\begin{aligned}
 \text{Consider } [\Lambda_3, \Lambda_4] &= 1/2\{T_{\Lambda}^+(\alpha_0) + T_{\Lambda}^+(\gamma_0 * \alpha_0) * (\gamma_0 * \beta_0), T_{\Lambda}^+(\beta_0)\}. \\
 \text{Then we get } [\Lambda_1, \Lambda_2] &= 1/2\{[\eta_1, \eta_2] + r \min\{[\eta_1, \eta_2], [\eta_3, \eta_4]\}\} \\
 &= 1/2\{(\vartheta_1 + \min\{[\eta_1, \eta_3]\}), (\vartheta_2 + \min\{[\eta_3, \eta_4]\})\} \\
 \min\{\eta_1, \eta_3\} &> \Lambda_1 = 1/2([\eta_1, \eta_3]) > \eta_1 \\
 \max\{\eta_2, \eta_4\} &> \Lambda_2 = 1/2([\eta_2, \eta_4]) > \eta_2.
 \end{aligned}$$

Hence $\{\min[\eta_1, \eta_3], \min[\eta_2, \eta_4]\} > [\Lambda_1, \Lambda_2] > [\vartheta_1, \vartheta_2]$. $(\gamma_0 * \alpha_0) * (\gamma_0 * \beta_0) \in \cup^+(\Lambda : [\Lambda_1, \Lambda_2])$.

(ii)

$$\begin{aligned}
 I_{\Lambda}^+(\alpha_0) &\leq r \max\{I_{\Lambda}^+(\gamma_0 * \alpha_0) * (\gamma_0 * \beta_0), I_{\Lambda}^+(\beta_0)\} \\
 I_{\Lambda}^+(\alpha_0) &= [\vartheta_1, \vartheta_2], \\
 I_{\Lambda}^+(\gamma_0 * \alpha_0) * (\gamma_0 * \beta_0) &= [\eta_1, \eta_2], \\
 I_{\Lambda}^+(\beta_0) &= [\vartheta_3, \vartheta_4] \\
 \text{If, } [\vartheta_1, \vartheta_2] &> r \max\{[\eta_1, \eta_2], [\eta_3, \eta_4]\} \\
 &= \max[\eta_1, \eta_2], \min[\eta_3, \eta_4].
 \end{aligned}$$

So $\vartheta_1 > \max[\eta_1, \eta_2]$ and $\vartheta_2 > \max[\eta_3, \eta_4]$.

$$\begin{aligned}
 \text{Consider } [\Lambda_3, \Lambda_4] &= 1/2\{I_{\Lambda}^+(\alpha_0) + I_{\Lambda}^+(\gamma_0 * \alpha_0) * (\gamma_0 * \beta_0), I_{\Lambda}^+(\beta_0)\}. \\
 \text{Then we get } [\Lambda_1, \Lambda_2] &= 1/2\{[\eta_1, \eta_2] + r \max\{[\eta_1, \eta_2], [\eta_3, \eta_4]\}\} \\
 &= 1/2\{(\vartheta_1 + \max\{[\eta_1, \eta_3]\}), (\vartheta_2 + \max\{[\eta_3, \eta_4]\})\} \\
 \max\{\eta_1, \eta_3\} &< \Lambda_1 = 1/2([\eta_1, \eta_3]) < \eta_1 \\
 \max\{\eta_2, \eta_4\} &< \Lambda_2 = 1/2([\eta_2, \eta_4]) < \eta_2.
 \end{aligned}$$

Hence, $\{\max[\eta_1, \eta_3], \max[\eta_2, \eta_4]\} < [\Lambda_1, \Lambda_2] < [\vartheta_1, \vartheta_2]$. $(\gamma_0 * \alpha_0) * (\gamma_0 * \beta_0) \in \cup^+(\Lambda : [\Lambda_1, \Lambda_2])$.

(iii)

$$\begin{aligned}
 F_{\Lambda}^+(\alpha_0) &\geq r \min\{F_{\Lambda}^+(\gamma_0 * \alpha_0) * (\gamma_0 * \beta_0), F_{\Lambda}^+(\beta_0)\} \\
 F_{\Lambda}^+(\alpha_0) &= [\vartheta_1, \vartheta_2], \\
 F_{\Lambda}^+(\gamma_0 * \alpha_0) * (\gamma_0 * \beta_0) &= [\eta_1, \eta_2], \\
 F_{\Lambda}^+(\beta_0) &= [\vartheta_3, \vartheta_4] \\
 \text{If } [\vartheta_1, \vartheta_2] &< r \min\{[\eta_1, \eta_2], [\eta_3, \eta_4]\} \\
 &= \min[\eta_1, \eta_2], \min[\eta_3, \eta_4].
 \end{aligned}$$

So $\vartheta_1 < \min[\eta_1, \eta_2]$ and $\vartheta_2 < \min[\eta_3, \eta_4]$.

$$\begin{aligned}
 \text{Consider } [\Lambda_3, \Lambda_4] &= 1/2\{F_{\Lambda}^+(\alpha_0) + F_{\Lambda}^+(\gamma_0 * \alpha_0) * (\gamma_0 * \beta_0), F_{\Lambda}^+(\beta_0)\}. \\
 \text{Then we get } [\Lambda_1, \Lambda_2] &= 1/2\{[\eta_1, \eta_2] + r \min\{[\eta_1, \eta_2], [\eta_3, \eta_4]\}\} \\
 &= 1/2\{(\vartheta_1 + \min\{[\eta_1, \eta_3]\}), (\vartheta_2 + \min\{[\eta_2, \eta_4]\})\} \\
 \min\{\eta_1, \eta_3\} &> \Lambda_1 = 1/2([\eta_1, \eta_3]) > \eta_1 \\
 \min\{\eta_2, \eta_4\} &> \Lambda_2 = 1/2([\eta_2, \eta_4]) > \eta_2.
 \end{aligned}$$

Hence, $\{\min[\eta_1, \eta_3], \min\{\eta_2, \eta_4\}\} > [\Lambda_1, \Lambda_2] > [\vartheta_1, \vartheta_2]. (\gamma_0 * \alpha_0) * (\gamma_0 * \beta_0) \in \cup^+(\Lambda : [\Lambda_1, \Lambda_2])$.
Then,

- (i) $T_{\Lambda}^+(\alpha) \geq r \min\{T_{\Lambda}^+((\gamma * \alpha) * (\gamma * \beta)), T_{\Lambda}^+(\beta)\}$
- (ii) $I_{\Lambda}^+(\alpha) \leq r \max\{I_{\Lambda}^+((\gamma * \alpha) * (\gamma * \beta)), I_{\Lambda}^+(\beta)\}$
- (iii) $F_{\Lambda}^+(\alpha) \geq r \min\{F_{\Lambda}^+((\gamma * \alpha) * (\gamma * \beta)), F_{\Lambda}^+(\beta)\}$, for all $\alpha, \beta, \gamma \in U$.

□

4 Homomorphism of INK-ideal in INK-algebra

Definition 4.1. Let $g : (U, *, 0) \rightarrow (V, *, 0)$ be a mapping from set U into set V . Let Λ be an IVNS in V . Then, the inverse image of Λ , denoted by $g^{-1}(\Lambda)$ is an IVNS in U with the membership function given,

- (i) $T_{g\Lambda}^+(\alpha) = T_{\Lambda}^+(g(\alpha))$
- (ii) $I_{g\Lambda}^+(\alpha) = I_{\Lambda}^+(g(\alpha))$
- (iii) $F_{g\Lambda}^+(\alpha) = F_{\Lambda}^+(g(\alpha))$, for all $\alpha \in U$.

Theorem 4.2. An into homomorphic preimage of a N INK-ideal is also N INK-ideal.

Proof. (i) Let $f : U \rightarrow U'$ be an into homomorphism of INK-algebra, V a N INK-ideal of U' and T the preimage of V under f ; then $V(f(\alpha)) = T(\alpha)$ for all $\alpha \in U$. Then,

$$\begin{aligned} T(0) &= V(f(0)) \\ T(0) &\geq V(f(\alpha)) \\ T(0) &= V(f(\alpha)) \end{aligned}$$

Let $\alpha, \beta, \gamma \in U$, then

$$\begin{aligned} T(\alpha) &= V(f(\alpha)) \\ T(\alpha) &\geq \min\{V(f(\gamma) * f(\alpha)) * (f(\gamma) * f(\beta)), V(f(\beta))\} \\ T(\alpha) &= \min\{V(f(\gamma * \alpha) * (\gamma * \beta)), V(f(\beta))\} \\ T(\alpha) &= \min\{T((\gamma * \alpha) * (\gamma * \beta)), T(\beta)\}. \end{aligned}$$

Hence, $T(\alpha) = V(f(\alpha)) = V \circ f(\alpha)$ is a N INK ideal of U .

(ii) Let $f : U \rightarrow U'$ be an into homomorphism of INK-algebra, V a N INK-ideal of U' and I the preimage of V under f ; then $V(f(\alpha)) = I(\alpha)$ for all $\alpha \in U$. Then,

$$\begin{aligned} I(0) &= V(f(0)) \\ I(0) &\leq V(f(\alpha)) \\ I(0) &= V(f(\alpha)) \end{aligned}$$

Let $\alpha, \beta, \gamma \in U$, then

$$\begin{aligned} I(\alpha) &= V(f(\alpha)) \\ I(\alpha) &\leq \max\{V(f(\gamma) * f(\alpha)) * (f(\gamma) * f(\beta)), V(f(\beta))\} \\ I(\alpha) &= \max\{V(f(\gamma * \alpha) * (\gamma * \beta)), V(f(\beta))\} \\ I(\alpha) &= \max\{I((\gamma * \alpha) * (\gamma * \beta)), I(\beta)\}. \end{aligned}$$

Hence, $I(\alpha) = V(f(\alpha)) = V \circ f(\alpha)$ is a N INK ideal of U .

(iii) Let $f : U \rightarrow U'$ be an into homomorphism of *INK*-algebra, V a *N INK*-ideal of U' and F the preimage of V under f ; then $V(f(\alpha)) = F(\alpha)$ for all $\alpha \in U$. Then,

$$\begin{aligned} F(0) &= V(f(0)) \\ F(0) &\geq V(f(\alpha)) \\ F(0) &= V(f(\alpha)) \end{aligned}$$

Let $\alpha, \beta, \gamma \in U$, then

$$\begin{aligned} F(\alpha) &= V(f(\alpha)) \\ F(\alpha) &\geq \min\{V(f(\gamma) * f(\alpha)) * (f(\gamma) * f(\beta)), V(f(\beta))\} \\ F(\alpha) &= \min\{V(f(\gamma * \alpha) * (\gamma * \beta)), V(f(\beta))\} \\ F(\alpha) &= \min\{F((\gamma * \alpha) * (\gamma * \beta)), F(\beta)\}. \end{aligned}$$

Hence, $F(\alpha) = V(f(\alpha)) = V \circ f(\alpha)$ is a *N INK* ideal of U .

□

Proposition 4.3. Let $g : U \rightarrow V$. Let $n = [n^L, n^U]$ and $m = [m^L, m^U]$ be *IVN*-subalg in U and V . Then

- (i) $[g^{-1}(m) = [g^{-1}(m^L), g^{-1}(m^U)]$
- (ii) $g(n) = [g(n^L), f(n^U)]$.

Theorem 4.4. Let $g : U \rightarrow V$ be a homomorphism from a *INK*-algebra U into a *INK*-algebra V . If Λ is an *IVN INK*-ideal of β , then the inverse image $f^{-1}(\Lambda)$ of Λ is an *IVN INK*-ideal of U .

Proof. Since, $\Lambda^+ = [T_\Lambda^+, I_\Lambda^+, F_\Lambda^+]$ is an *IVN INK*-ideal of β , it follows from Theorem 3.4 that $T_\Lambda^L, T_\Lambda^U, I_\Lambda^L, I_\Lambda^U, F_\Lambda^L$ and F_Λ^U are *N INK*-ideals of β . Using Theorem 4.2 we know that,

- (i) $f^{-1}(T_\Lambda^+) = [f^{-1}(T_\Lambda^L), f^{-1}(T_\Lambda^U)]$
- (ii) $f^{-1}(I_\Lambda^+) = [f^{-1}(I_\Lambda^L), f^{-1}(I_\Lambda^U)]$
- (iii) $f^{-1}(F_\Lambda^+) = [f^{-1}(F_\Lambda^L), f^{-1}(F_\Lambda^U)]$

are *N INK*-ideals of U . Hence, by Proposition 4.3,

- (i) $f^{-1}(T_\Lambda) = [f^{-1}(T_\Lambda^L), f^{-1}(T_\Lambda^U)]$
- (ii) $f^{-1}(I_\Lambda) = [f^{-1}(I_\Lambda^L), f^{-1}(I_\Lambda^U)]$
- (iii) $f^{-1}(F_\Lambda) = [f^{-1}(F_\Lambda^L), f^{-1}(F_\Lambda^U)]$
are an *N INK*-ideal of U .

□

Theorem 4.5. Let $g : U \rightarrow V$ be a homomorphism. If Λ is an *IVN INK*-ideal of U . Then $f(\Lambda)$ of Λ is an *IVN INK*-ideal of β .

Proof. Assume that, Λ is an *IVN INK*-ideal of U . Note that,

- (i) $T_\Lambda = [T_\Lambda^L, T_\Lambda^U]$
- (ii) $I_\Lambda = [I_\Lambda^L, I_\Lambda^U]$

$$(iii) F_{\Lambda} = [F_{\Lambda}^L, F_{\Lambda}^U]$$

are an *IVN INK*-ideal of U . Let $g : U \rightarrow V$ be a homomorphism between *INK*-algebra U and β . For every *N INK*-ideal T, I, F in U . $f(T), f(I), f(F)$ are a *N INK*-ideal of β . Then, the image $f(T_{\Lambda}^L), f(T_{\Lambda}^U), f(I_{\Lambda}^L), f(I_{\Lambda}^U), f(F_{\Lambda}^L)$ and $f(F_{\Lambda}^U)$ are *N INK*-ideals in β . Combining Theorem 3.4 and Proposition 4.3 we conclude that,

$$(i) f(T_{\Lambda}) = [f^{-1}(T_{\Lambda}^L), f^{-1}(T_{\Lambda}^U)]$$

$$(ii) f(I_{\Lambda}) = [f^{-1}(I_{\Lambda}^L), f^{-1}(I_{\Lambda}^U)]$$

$$(iii) f(F_{\Lambda}) = [f^{-1}(F_{\Lambda}^L), f^{-1}(F_{\Lambda}^U)]$$

are an *IVN INK*-ideal of β . □

5 Conclusions

In this paper, *IVN INK*-ideals is defined and studied some properties of it. Also, homomorphism of *IVN INK*-ideal of *INK*-algebra is defined and presented some of its properties.

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