



Anti homomorphism and homomorphism of bipolar valued multi Fuzzy HX-subgroups and it's normal

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Abstract

In this article, homomorphism and anti-homomorphism bipolar valued multi normal fuzzy HX subgroup of a HX group is studied, discussed and their properties are introduced. The theorems presented in this article are considered a generalization of what has been published through the concept of fuzzy sets. As the concept of fuzzy sets was introduced by Zadeh in 1965, after that this concept, we expanded into more than one types, intuitionistic fuzzy set, bipolar fuzzy sets, tripolar fuzzy set, and bipolar valued multi fuzzy set are one of this.

Keywords: Fuzzy sets; bipolar valued fuzzy set; bipolar valued multi fuzzy set; bipolar valued multi fuzzy HX subgroup; bipolar valued multi fuzzy normal HX subgroup homomorphism; antihomomorphism.

1 introduction

An algebra (G, V) is called a finite group, and an empty set $\nu \subset Z^G - \{\phi\}$ is called a HX group of G , if ν is a group with respect to the algebraic operation given by $U \cdot V = \{uv; u \in U, v \in V\}$, such that its unit element is represented by E . Zadeh¹ in 1965 introduced the concept of fuzzy set, this concept is useful mathematical structure kind to represent the objects collection whose boundary is vague. For this reason, this concept has become an active field with many scientific papers being completed in many domains, examples of such papers P. Bhattacharya² in 1981 studied some characterizations of fuzzy groups, M. S. Eroglu³ in 1989 discussed the homomorphic image of a fuzzy subgroups, M. O. Massa'deh el al⁴⁻⁶ introduced and studied upper fuzzy order, M -fuzzy subrings and isomorphic fuzzy graphs. Many generalizations of fuzzy sets concept have emerged such as interval valued fuzzy set, intuitionistic fuzzy set, hyper fuzzy set, etc. Many scholars have relied on these concepts and applied them in several fields. M. Akram and W.A. Dudek⁷ in 2011 studied and discussed interval valued fuzzy graphs. In 2011 M.F. Marashdeh and A.R. Salleh⁸ studied intuitionistic fuzzy rings, while⁹⁻¹¹ studied intuitionistic Q -fuzzy ku -ideals, Q -fuzzy k -ideals in Γ -semirings and hyper Q -fuzzy normal HX subgroup. In 1998 Zhang¹² introduced the concept of bipolar valued fuzzy sets, where a bipolar valued fuzzy sets are an extension of fuzzy sets whose range of membership degree is enleryed from $[0, 1]$ to the interval $[-1, 1]$, a degree "0" of the membership means that elements are irrelevant to the corresponding property, a degree of the membership $[-1, 0]$ indicates that elements somewhat satisfy the implicit counter property and a degree of the membership $[0, 1]$ indicates that elements somewhat satisfy the property. The concept of HX group introduced by Li Hongxing¹³ in 1987. Many researchers have combined the concept of bipolar valued fuzzy set with the concept of groups and HX groups and have derived some new results, where Anitha. M. S., Muruganantha. P. K. and Arjunan. K.¹⁴ discussed a bipolar valued fuzzy subgroups, S. Sahaya¹⁵ introduced the concept of homomorphism of bipolar valued Q -fuzzy subgroups, while M. O. Massa'deh el al^{16,17} studied a bipolar value Q -fuzzy HX subgroup and anti bipolar Q -fuzzy normal semigroups. The notion of multi fuzzy set was introduced by S. Sebastian and T. V. Ramakrishnan¹⁸ in 2010.

Some researchers have introduced this concept to a some of topics, including Santhi V. K. and Ainbarasi. K.¹⁹ in 2015 discussed bipolar valued multi fuzzy subhemirings and M. O. Massa'deh and A. O. Fallatah²⁰ in 2023 studied bipolar $M.N.$ -multi Q -fuzzy subgroups. In this article homomorphism and anti homomorphism over bipolar valued multi fuzzy HX subgroup are introduced, studied and discussed some new results.

1.1 Preliminaries

Definition 1.1.¹ Let G be a non empty set. A fuzzy set λ of G is a function $\lambda : G \rightarrow [0, 1]$.

Definition 1.2.¹² A bipolar valued fuzzy set δ in ν is defined as an object of the form

$$\delta_\lambda = \{A; \delta_\lambda^+(A), \delta_\lambda^-(A); A \in \nu\}$$

such that $\delta_\lambda^- : \nu \rightarrow [-1, 0]$, $\delta_\lambda^+ : \nu \rightarrow [0, 1]$. The negative degree of membership $\delta_\lambda^-(A)$ represent the satisfaction degree of an element A to some implicit counter property corresponding to a bipolar valued fuzzy set δ and the positive degree of membership $\delta_\lambda^+(A)$ represent the degree of satisfaction of an element A to the property corresponding to a bipolar valued fuzzy set A .

Example 1.3. Let $\nu = \{x, y, z\}$, then $\delta_\lambda = \{ \langle x, 0.8, -0.2 \rangle, \langle y, 0.1, -0.9 \rangle, \langle z, 0.4, -0.5 \rangle \}$ is a bipolar valued fuzzy subset of ν .

Definition 1.4.¹³ If G is a finite group in $2^G - \{\phi\}$ a non empty set $\nu \subset 2^G - \{\phi\}$ is called a HX group of G , if ν is a group with respect to algebraic operation defined by $AB = \{ab; a \in A, b \in B\}$ which its unit element is denoted by E , the inverse element of A is denoted by A^{-1} .

Example 1.5. Consider $G = \{-1, 1, i, -i, \cdot\}$ be a group and $\nu = \{i, -i, \{1, -1\}\}$ then (ν, \cdot) is an HX group. Hence $E = \{-1, 1\}$.

Definition 1.6.¹⁸ A bipolar valued multi fuzzy set δ_λ in ν is defined as an object of the form $\delta = \{ \langle A, \delta_{\lambda_1}^-(A), \delta_{\lambda_2}^-(A), \dots, \delta_{\lambda_k}^-(A), \delta_{\lambda_1}^+(A), \delta_{\lambda_2}^+(A), \dots, \delta_{\lambda_k}^+(A) \rangle, A \in \nu \}$ such that $\delta_{\lambda_j}^- : \nu \rightarrow [-1, 0]$ and $\delta_{\lambda_j}^+ : \nu \rightarrow [0, 1]$ for all $j = 1, 2, \dots, k$. The negative degree of membership $\delta_{\lambda_j}^-(A)$ represent to the degree of satisfaction of an element A to some implicit conuter property corresponding to a bipolar valued multi fuzzy set δ_λ and the positive degrees of membership $\delta_{\lambda_j}^+(A)$ represent the degree of satisfaction of an element A to the corresponding of property to a bipolar valued multi fuzzy set δ_λ it is denoted as $\delta_\lambda = \langle \delta_{\lambda_1}^-, \delta_{\lambda_2}^-, \dots, \delta_{\lambda_k}^-, \delta_{\lambda_1}^+, \delta_{\lambda_2}^+, \dots, \delta_{\lambda_k}^+ \rangle$.

Example 1.7. Let $\nu = \{X, Y, Z\}$, then $\delta_\lambda = \{ \langle X, 0.3, 0.7, 0.9, -0.3, -0.5, -0.8 \rangle, \langle Y, 0.4, 0.8, 0.2, -0.9, -0.4, -0.1 \rangle, \langle Z, 0.6, 0.4, 0.5, -0.7, -0.6, -0.2 \rangle \}$ is a bipolar valued multi fuzzy subset of ν .

1.2 Bipolar valued multi fuzzy HX-subgroups

Definition 1.8. Let $\nu \subset 2^G - \{\phi\}$ be an HX-group of G . A bipolar valued multi fuzzy subset $\delta_\lambda = \langle \delta_{\lambda_1}^-, \delta_{\lambda_2}^-, \dots, \delta_{\lambda_k}^-, \delta_{\lambda_1}^+, \delta_{\lambda_2}^+, \dots, \delta_{\lambda_k}^+ \rangle$ of ν is said to be a bipolar valued multi fuzzy HX-subgroup of ν if satisfied the following axioms for all j :

1. $\delta_{\lambda_j}^+(AB) \geq \min\{\delta_{\lambda_j}^+(A), \delta_{\lambda_j}^+(B)\}$.
2. $\delta_{\lambda_j}^-(AB) \leq \max\{\delta_{\lambda_j}^-(A), \delta_{\lambda_j}^-(B)\}$.
3. $\delta_{\lambda_j}^+(A^{-1}) = \delta_{\lambda_j}^+(A)$, $\delta_{\lambda_j}^-(A^{-1}) = \delta_{\lambda_j}^-(A)$.

for all $A, B \in \nu$ where $\{\delta_{\lambda_j}^+(A) = \max\{\lambda_j^+(a); \forall a \in A \subseteq G\}$ and $\{\delta_{\lambda_j}^-(A) = \min\{\lambda_j^-(a); \forall a \in A \subseteq G\}$.

Example 1.9. Let $G = \{e, a, b, ab\}$ be a Klein's four group, where $e = a^2 = b^2$, $ab = ba$ and e is the identity element of G .

Define a bipolar multi fuzzy subset $\lambda = (\lambda_j^-, \lambda_j^+)$ on a G as:

$$\begin{aligned} \lambda_1^+(x) &= 0.7, \lambda_2^+(x) = 0.8, \lambda_3^+(x) = 0.9 \text{ if } x = e \\ \lambda_1^-(x) &= -0.7, \lambda_2^-(x) = -0.8, \lambda_3^-(x) = -0.9 \text{ if } x = e \\ \lambda_1^+(x) &= 0.4, \lambda_2^+(x) = 0.5, \lambda_3^+(x) = 0.6 \text{ if } x = a \\ \lambda_1^-(x) &= -0.4, \lambda_2^-(x) = -0.5, \lambda_3^-(x) = -0.6 \text{ if } x = a \\ \lambda_1^+(x) &= 0.3, \lambda_2^+(x) = 0.4, \lambda_3^+(x) = 0.5 \text{ if } x = b, ab \\ \lambda_1^-(x) &= -0.3, \lambda_2^-(x) = -0.4, \lambda_3^-(x) = -0.5 \text{ if } x = b, ab. \end{aligned}$$

Let $\nu = \{E, A\} = \{\{e, a\}, \{b, ab\}\}$ be a HX-group on G and define $\delta_{\lambda_i}^- : \nu \rightarrow [-1, 0]$, $\delta_{\lambda_i}^+ : \nu \rightarrow [0, 1]$ such that $\delta_{\lambda_i}^+(A) = \max\{\lambda_i^+(a); \forall a \in A \subseteq G\}$ and $\delta_{\lambda_i}^-(A) = \min\{\lambda_i^-(a); \forall a \in A \subseteq G\}$.

Then

$$\begin{aligned} \delta &= \{ \langle e, 0.7, 0.8, 0.9, -0.7, -0.8, -0.9 \rangle \\ &\quad \langle a, 0.4, 0.5, 0.6, -0.4, -0.5, -0.6 \rangle \\ &\quad \langle b, 0.3, 0.4, 0.5, -0.3, -0.4, -0.5 \rangle \\ &\quad \langle ab, 0.3, 0.4, 0.5, -0.3, -0.4, -0.5 \rangle \} \end{aligned}$$

is a bipolar multi fuzzy HX-subgroup of ν .

Definition 1.10. if ν is an HX-group. A bipolar valued multi fuzzy HX-subgroup δ_λ of ν is called a bipolar valued multi fuzzy normal HX-subgroup of ν if $\delta_{\lambda_j}^-(AB) = \delta_{\lambda_j}^-(BA)$ and $\delta_{\lambda_j}^+(AB) = \delta_{\lambda_j}^+(BA)$ for all $A, B \in \nu$ and for all j .

Definition 1.11. Let ν_1 and ν_2 any two HX-groups then:

- 1)- the function $\phi : \nu_1 \rightarrow \nu_2$ is said to be homomorphism if $\phi(AB) = \phi(A)\phi(B)$.
- 2)- The function $\phi : \nu_1 \rightarrow \nu_2$, (ν_1 and ν_2 not necessarily commutitive) is said to be an antihomomorphism if $\phi(AB) = \phi(B)\phi(A)$ for all $A, B \in \nu_1$.

Definition 1.12. Let U_1 and U_2 be two sets, $\psi : U_1 \rightarrow U_2$ be any function and let δ_λ be a bipolar valued multi fuzzy subset in U_r , φ_λ be a bipolar valued subset in $psi(U_1) = U_2$ defined by $\varphi_{\lambda_j}^+ = \sup_{a \in \psi^{-1}(b)} \delta_{\lambda_j}^+(a)$ and $\varphi_{\lambda_j}^- = \inf_{a \in \psi^{-1}(b)} \delta_{\lambda_j}^-(a)$, for all $a \in U_1$ and $b \in U_2$ and for all i , δ_λ is called a preimage of φ under ψ and is denoted by $\psi^{-1}(\varphi)$.

Definition 1.13. If $\delta_\lambda = \langle \delta_{\lambda_1}^-, \delta_{\lambda_2}^-, \dots, \delta_{\lambda_k}^-, \delta_{\lambda_1}^+, \delta_{\lambda_2}^+, \dots, \delta_{\lambda_k}^+ \rangle$ is a bipolar valued multi fuzzy subset of U . for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_j), \alpha_j \in [0, 1]$ and $\beta = (\beta_1, \beta_2, \dots, \beta_j), \beta_j \in [-1, 0]$ then $\langle \alpha\beta \rangle = \langle (\alpha_1, \alpha_2, \dots, \alpha_j)(\beta_1, \beta_2, \dots, \beta_j) \rangle$ have subset of δ_λ is the set $\delta_{\lambda \langle (\alpha_1, \alpha_2, \dots, \alpha_j), (\beta_1, \beta_2, \dots, \beta_j) \rangle} = \{u \in U; \delta_\lambda^-(u) \leq \beta_j \text{ and } \delta_\lambda^+(u) \geq \alpha_j\}$ for all j .

1.3 Theorems and properties

Theorem 1.14. If ν_1 and ν_2 are two HX-groups, the homomorphic image of a bipolar valued multi fuzzy HX subgroup of ν_1 is a bipolar valued multi fuzzy HX-subgroup of ν_2 .

Theorem 1.15. If ν_1 and ν_2 are any two HX-groups. The antihomomorphic image of a bipolar valued multi fuzzy HX subgroup of ν_1 is a bipolar valued multi fuzzy HX-subgroup of ν_2 .

Theorem 1.16. If ν_1 and ν_2 are any two HX-groups, the homomorphic preimage of a bipolar valued multi fuzzy HX subgroup of ν_2 is a bipolar valued multi fuzzy HX-subgroup of ν_1 .

Theorem 1.17. If ν_1 and ν_2 are any two HX-groups. The antihomomorphic preimage of a bipolar valued multi fuzzy HX subgroup of ν_2 is a bipolar valued multi fuzzy HX-subgroup of ν_1 .

Theorem 1.18. If ν_1 and ν_2 are any two HX-group, the homomorphic image of a bipolar valued multi fuzzy normal HX subgroup of ν_1 is a bipolar valued multi fuzzy normal HX-subgroup of ν_2 .

Proof. Let $U = \psi(\delta_\lambda)$, such that $\delta_\lambda = \langle \delta_{\lambda_1}^-, \delta_{\lambda_2}^-, \dots, \delta_{\lambda_k}^-, \delta_{\lambda_1}^+, \delta_{\lambda_2}^+, \dots, \delta_{\lambda_k}^+ \rangle$ be a bipolar valued multi fuzzy normal HX-subgroup of ν_1 , we have to prove that $\psi(\delta_\lambda) = U$ be a bipolar valued multi fuzzy normal HX-subgroup of ν_2 . Now, for $\psi(g_1), \psi(g_2) \in \nu_2$ for all j , clearly U is a bipolar valued multi fuzzy

HX subgroup of ν_2 , that is, $U_j^+(\psi(g_1)\psi(g_2)) = U_j^+(\psi(g_1g_2)) \geq \delta_{\lambda_j}^+(g_1g_2) = \delta_{\lambda_j}^+(g_2g_1) \leq U_j^+(\psi(g_2g_1)) = U_j^+(\psi(g_2)\psi(g_1))$ which implies that $U_j^+(\psi(g_1)\psi(g_2)) = U_j^+(\psi(g_2)\psi(g_1))$ and $U_j^-(\psi(g_1)\psi(g_2)) = U_j^-(\psi(g_1g_2)) \geq \delta_{\lambda_j}^-(g_1g_2) = \delta_{\lambda_j}^-(g_2g_1) \leq U_j^-(\psi(g_2g_1)) = U_j^-(\psi(g_2)\psi(g_1))$ which implies that $U_j^-(\psi(g_1)\psi(g_2)) = U_j^-(\psi(g_2)\psi(g_1))$. Therefore U is a bipolar valued multi fuzzy normal HX-subgroup of ν_2 . \square

Theorem 1.19. *If ν_1 and ν_2 are any two HX-group, the homomorphic preimage of a bipolar valued multi fuzzy normal HX subgroup of ν_2 is a bipolar valued multi fuzzy normal HX-subgroup of ν_1 .*

Proof. Suppose that $U = \psi(\delta_{\lambda})$, such that $U = \langle U_1^-, U_2^-, \dots, U_k^-, U_1^+, U_2^+, \dots, U_k^+ \rangle$ is a bipolar valued multi fuzzy normal HX-subgroup of ν_2 . We have to prove that δ_{λ} is a bipolar valued multi fuzzy normal HX-subgroup of ν_1 . Let $g_1, g_2 \in \nu_1$, since δ_{λ} is a bipolar valued multi fuzzy HX-subgroup of ν_1 , for all j , $\delta_{\lambda_j}^-(g_1g_2) = U_j^-(\psi(g_1g_2)) = U_j^-(\psi(g_1)\psi(g_2)) = U_j^-(\psi(g_2)\psi(g_1)) = U_j^-(\psi(g_2g_1)) = \delta_{\lambda_j}^-(g_2g_1)$. Thus $\delta_{\lambda_j}^-(g_1g_2) = \delta_{\lambda_j}^-(g_2g_1)$. And $\delta_{\lambda_j}^+(g_1g_2) = U_j^+(\psi(g_1g_2)) = U_j^+(\psi(g_1)\psi(g_2)) = U_j^+(\psi(g_2)\psi(g_1)) = U_j^+(\psi(g_2g_1)) = \delta_{\lambda_j}^+(g_2g_1)$ this implies that $\delta_{\lambda_j}^+(g_1g_2) = \delta_{\lambda_j}^+(g_2g_1)$. Therefore δ is a bipolar valued multi fuzzy normal HX-subgroup of ν_1 . \square

Theorem 1.20. *If ν_1 and ν_2 are two HX-group. The antihomomorphic image of a bipolar valued multi fuzzy normal HX subgroup of ν_1 is a bipolar valued multi fuzzy normal HX-subgroup of ν_2 .*

Proof. Let $U = \psi(\delta_{\lambda})$ such that $\delta_{\lambda} = \langle \delta_{\lambda_1}^-, \delta_{\lambda_2}^-, \dots, \delta_{\lambda_k}^-, \delta_{\lambda_1}^+, \delta_{\lambda_2}^+, \dots, \delta_{\lambda_k}^+ \rangle$ is a bipolar valued multi fuzzy normal HX-subgroup of ν_1 . We have to show that U is a bipolar valued multi fuzzy normal HX-subgroup of ν_2 , since it clear that U is a bipolar valued multi fuzzy HX-subgroup of ν_2 . For $\psi(g_1), \psi(g_2) \in \nu_2$ for all j , $U_j^-(\psi(g_1)\psi(g_2)) = U_j^-(\psi(g_2g_1)) \leq \delta_{\lambda_j}^-(g_2g_1) = \delta_{\lambda_j}^-(g_1g_2) \geq U_j^-(\psi(g_2g_1)) = U_j^-(\psi(g_2)\psi(g_1))$. Hence $U_j^-(\psi(g_1)\psi(g_2)) = U_j^-(\psi(g_2)\psi(g_1))$. Also $U_j^+(\psi(g_1)\psi(g_2)) = U_j^+(\psi(g_2g_1)) \geq \delta_{\lambda_j}^+(g_2g_1) = \delta_{\lambda_j}^+(g_1g_2) \leq U_j^+(\psi(g_2g_1)) = U_j^+(\psi(g_2))\psi(g_1)$. Thus $U_j^+(\psi(g_1)\psi(g_2)) = U_j^+(\psi(g_2)\psi(g_1))$. Therefore U is a bipolar valued multi fuzzy normal HX-subgroup of ν_2 . \square

Theorem 1.21. *If ν_1 and ν_2 are two HX-group. The antihomomorphic preimage of a bipolar valued multi fuzzy normal HX subgroup of ν_2 is a bipolar valued multi fuzzy normal HX-subgroup of ν_1 .*

Proof. Suppose that $U = \psi(\delta_{\lambda})$ such that $U = \langle U_1^-, U_2^-, \dots, U_k^-, U_1^+, U_2^+, \dots, U_k^+ \rangle$ be a bipolar valued multi fuzzy normal HX-subgroup of ν_1 . We need to prove that δ_{λ} be a bipolar valued multi fuzzy normal HX-subgroup of ν_1 . If $v_1, v_2 \in \nu_1$. Since δ_{λ} is a bipolar valued multi fuzzy HX-subgroup of ν_1 . Now for all J , $\delta_{\lambda_j}^-(v_1v_2) = U_j^-(\psi(v_1v_2)) = U_j^-(\psi(v_2)\psi(v_1)) = U_j^-(\psi(v_1)\psi(v_2)) = U_j^-(\psi(v_2v_1)) = \delta_{\lambda_j}^-(v_2v_1)$, this implies that $\delta_{\lambda_j}^-(v_1v_2) = \delta_{\lambda_j}^-(v_2v_1)$ and $\delta_{\lambda_j}^+(v_1v_2) = U_j^+(\psi(v_1v_2)) = U_j^+(\psi(v_2)\psi(v_1)) = U_j^+(\psi(v_1)\psi(v_2)) = U_j^+(\psi(v_2v_1)) = \delta_{\lambda_j}^+(v_2v_1)$, which implies that $\delta_{\lambda_j}^+(v_1v_2) = \delta_{\lambda_j}^+(v_2v_1)$. Therefore δ_{λ} is a bipolar valued multi fuzzy normal HX-subgroup of ν_1 . \square

Theorem 1.22. *A homomorphic image of a $(\alpha, \beta) = \langle (\alpha_1, \alpha_2, \dots, \alpha_j), (\beta_1, \beta_2, \dots, \beta_j) \rangle$ -level HX-subgroup of a bipolar valued multi fuzzy HX-subgroup of a HX-group is a $(\alpha, \beta) = \langle (\alpha_1, \alpha_2, \dots, \alpha_j), (\beta_1, \beta_2, \dots, \beta_j) \rangle$ -level HX-subgroup of a bipolar valued multi fuzzy HX-subgroup of a HX-group ν_2 .*

Proof. Let $U = \psi(\delta_{\lambda})$ where $\delta_{\lambda} = \langle \delta_{\lambda_j}^-, \delta_{\lambda_j}^+ \rangle$ is a bipolar valued fuzzy HX-subgroup of ν_1 . By Theorem 1.14 $U = \langle U_j^-, U_j^+ \rangle$ is a bipolar valued fuzzy HX-subgroup of ν_2 . If $v_1, v_2 \in \nu_1$, then $\psi(v_1), \psi(v_2) \in \nu_2$. If $\delta_{\lambda < \alpha, \beta >}$ is a $\langle \alpha, \beta \rangle = \langle (\alpha_1, \alpha_2, \dots, \alpha_j), (\beta_1, \beta_2, \dots, \beta_j) \rangle$ -level HX-subgroup of δ_{λ} , that is, $\delta_{\lambda_j}^-(v_1) \leq \beta_j, \delta_{\lambda_j}^+(v_1) \geq \alpha_j, \delta_{\lambda_j}^-(v_2) \leq \beta_j, \delta_{\lambda_j}^+(v_2) \geq \alpha_j$. Now

$$\begin{aligned} \delta_{\lambda_j}^-(v_1v_2) &\leq \beta_j \\ \delta_{\lambda_j}^+(v_1v_2) &\geq \alpha_j \\ \delta_{\lambda_j}^+(v_1^{-1}) &= \delta_{\lambda_j}^+(v_1) \geq \alpha_j \\ \delta_{\lambda_j}^-(v_1^{-1}) &= \delta_{\lambda_j}^-(v_1) \leq \beta_j \text{ for all } j. \end{aligned}$$

We have to prove that $\psi(\delta_{\lambda < \alpha, \beta >})$ is a $\langle \alpha, \beta \rangle$ -level HX-subgroup of U . Now $U_j^-(\psi(v_1)) \leq \delta_{\lambda_j}^-(v_1) \leq \beta_j$ which implies that $U_j^-(\psi(v_1)) \leq \beta_j \forall j$ and $U_j^-(\psi(v_2)) \leq \delta_{\lambda_j}^-(v_2) \leq \beta_j$ which implies that $U_j^-(\psi(v_2)) \leq$

$\beta_j \forall j$. Then $U_j^-(\psi(v_1)\psi(v_2)) = U_j^-(\psi(v_1v_2)) \leq \delta_{\lambda_j}^-(v_1v_2) \leq \beta_j$ which implies that $U_j^-(\psi(v_1)\psi(v_2)) \leq \beta_j \forall j$ and $U_j^-(\psi(v_1^{-1})) = U_j^-(\psi^{-1}(v_1)) \leq \delta_{\lambda_j}^-(\psi^{-1}(v_1)) = \delta_{\lambda_j}^-(v_1) \leq \beta_j \forall j$. This implies that $U_j^-(\psi(v_1^{-1})) \leq \beta_j \forall j$. Also, $U_j^+(\psi(v_1)) \geq \delta_{\lambda_j}^+(v_1) \geq \alpha_j$ this implies that $U_j^+(\psi(v_1)) \geq \alpha_j \forall j$ and $U_j^+(\psi(v_2)) \geq \delta_{\lambda_j}^+(v_2) \geq \alpha_j$ this implies that $U_j^+(\psi(v_2)) \geq \alpha_j, \forall j$. Then $U_j^+(\psi(v_1)\psi(v_2)) = U_j^+(\psi(v_1v_2)) \geq \delta_{\lambda_j}^+(v_1v_2) \geq \alpha_j$ and hence $U_j^+(\psi(v_1)\psi(v_2)) \geq \alpha_j \forall j$ and $U_j^+(\psi(v_1^{-1})) = U_j^+(\psi^{-1}(v_1)) \geq \delta_{\lambda_j}^+(\psi^{-1}(v_1)) = \delta_{\lambda_j}^+(v_1) \geq \alpha_j \forall j$. Hence $U_j^+(\psi(v_1^{-1})) \geq \alpha_j \forall j$. Therefore $\psi(\delta_{\lambda < \alpha, \beta >})$ is a $< \alpha, \beta >$ -level HX-subgroup of a bipolar valued fuzzy HX-subgroup of ν_2 . \square

Theorem 1.23. *A homomorphic preimage of a $< \alpha, \beta >$ -level HX-subgroup of a bipolar valued fuzzy HX-subgroup of a HX-group ν_2 is a $< \alpha, \beta >$ -level HX-subgroup of a bipolar valued fuzzy HX-subgroup of a HX-group ν_1 .*

Proof. If $U = \psi(\delta_\lambda)$ such that $U = \langle U_j^-, U_j^+ \rangle$ is a bipolar valued fuzzy HX-subgroup of ν . By theorem 1.15 $\delta_\lambda = \langle \delta_{\lambda_j}^-, \delta_{\lambda_j}^+ \rangle$ is a bipolar valued fuzzy HX-subgroup of ν_1 . If $\psi(u_1), \psi(u_2) \in \nu_2$, then $v_1, v_2 \in \nu_2$. Let $\psi(\delta_{\lambda < \alpha, \beta >})$ be a $< \alpha, \beta >$ -level HX-subgroup of U . That is, $U_j^-(\psi(v_1)), U_j^-(\psi(v_2)) \leq \beta_j$ and $U_j^+(\psi(v_1)), U_j^+(\psi(v_2)) \geq \alpha_j$, so $U_j^-(\psi(v_1)\psi(v_2)) \leq \beta_j$ and $U_j^+(\psi(v_1)\psi(v_2)) \geq \alpha_j$ then $U_j^-(\psi(v_1^{-1})) = U_j^-(v_1) \leq \beta_j$ and $U_j^+(\psi(v_1^{-1})) = U_j^+(v_1) \geq \alpha_j$. We have to prove that $\delta_\lambda = \langle \delta_{\lambda_j}^-, \delta_{\lambda_j}^+ \rangle$ is a $< \alpha, \beta >$ -level HX-subgroup of δ_λ . Now $\delta_{\lambda_j}^-(v_1) \leq U_j^-(\psi(v_1)) \leq \beta_j$. Thus $\delta_{\lambda_j}^-(v_1) \leq \beta_j \forall j$ and $\delta_{\lambda_j}^-(v_2) \leq U_j^-(\psi(v_2)) \leq \beta_j$ hence $\delta_{\lambda_j}^-(v_2) \leq \beta_j \forall j$. Then $\delta_{\lambda_j}^-(v_1v_2) = U_j^-(\psi(v_1v_2)) = U_j^-(\psi(v_1)\psi(v_2)) \leq \beta_j$ this implies that $U_j^-(v_1v_2) \leq \beta_j \forall j$. And $\delta_{\lambda_j}^-(v_1^{-1}) = U_j^-(\psi(v_1^{-1})) = U_j^-(\psi^{-1}(v_1)) = U_j^-(\psi(v_1)) \leq \beta_j$. Hence $\delta_{\lambda_j}^-(v_1^{-1}) \leq \beta_j, \forall j$ and $\delta_{\lambda_j}^+(v_1) = U_j^+(\psi(v_1)) \geq \alpha_j$ thus $\delta_{\lambda_j}^+(v_1) \geq \alpha_j \forall j$ and $\delta_{\lambda_j}^+(v_2) = U_j^+(\psi(v_2)) \geq \alpha_j$ hence $\delta_{\lambda_j}^+(v_2) \geq \alpha_j$. Then $\delta_{\lambda_j}^+(v_1v_2) = \delta_{\lambda_j}^+(\psi(v_1v_2)) \geq U_j^+(\psi(v_1)\psi(v_2)) \geq \alpha_j$ this implies that $\delta_{\lambda_j}^+(\psi(v_1v_2)) \geq \alpha_j \forall j$ and $\delta_{\lambda_j}^+(v_1^{-1}) = U_j^+(\psi(v_1^{-1})) = U_j^+(\psi^{-1}(v_1)) = U_j^+(\psi(v_1)) \geq \alpha_j$ and therefore $\delta_{\lambda_j}^+(v_1^{-1}) \geq \alpha_j \forall j$. We get $\psi(\delta_{\lambda < \alpha, \beta >})$ is a $< \alpha, \beta >$ -level HX-subgroup of a bipolar valued fuzzy HX-subgroup U . \square

Theorem 1.24. *The anti homomorphic image of a $< \alpha, \beta >$ -level HX-subgroup of a bipolar valued fuzzy HX-subgroup of a HX-group ν_1 is a $< \alpha, \beta >$ -level HX-subgroup of a bipolar valued fuzzy HX-subgroup of a HX-group ν_2 .*

Proof. Suppose that $U = \psi(\delta_\lambda)$ where $\delta_\lambda = \langle \delta_{\lambda_j}^-, \delta_{\lambda_j}^+ \rangle$ is a bipolar valued fuzzy HX-subgroup of ν_1 . By theorem 1.16 $U = \langle U_j^-, U_j^+ \rangle$ be a bipolar valued fuzzy HX-subgroup of ν_2 . If $v_1, v_2 \in \nu_1$, then $\psi(v_1), \psi(v_2) \in \nu_2$ and if $\delta_{\lambda < \alpha, \beta >}$ is a $< \alpha, \beta >$ -level HX-subgroup of δ_λ . That is, $\delta_{\lambda_j}^-(v_1) \cdot \delta_{\lambda_j}^-(v_2) \leq \beta_j, \delta_{\lambda_j}^+(v_1) \cdot \delta_{\lambda_j}^+(v_2) \geq \alpha_j, \delta_{\lambda_j}^-(v_1v_2) \leq \beta_j, \delta_{\lambda_j}^+(v_1v_2) \geq \alpha_j, \delta_{\lambda_j}^-(v_1^{-1}) = \delta_{\lambda_j}^-(v_1) \leq \beta_j$ and $\delta_{\lambda_j}^+(v_1^{-1}) = \delta_{\lambda_j}^+(v_1) \geq \alpha_j$. We have to prove that $\psi(\delta_{\lambda < \alpha, \beta >})$ be a $< \alpha, \beta >$ -level HX-subgroup U . Now $U_j^-(\psi(v_1)) \leq \lambda_j^-(v_1) \leq \beta_j$, thus $U_j^-(\psi(v_1)) \leq \beta_j \forall j$ and $U_j^-(\psi(v_2)) \leq \lambda_j^-(v_2) \leq \beta_j$, thus $U_j^-(\psi(v_2)) \leq \beta_j \forall j$, that is, $U_j^-(\psi(v_1)\psi(v_2)) = U_j^-(\psi(v_1v_2)) \leq \delta_{\lambda_j}^-(v_1v_2) \leq \beta_j$ hence $U_j^-(\psi(v_1)\psi(v_2)) \leq \beta_j$ also $U_j^-(\psi(v_1^{-1})) = U_j^-(\psi^{-1}(v_1)) \leq \delta_{\lambda_j}^-(v_1^{-1}) = \delta_{\lambda_j}^-(v_1) \leq \beta_j \forall j. U_j^+(\psi(v_1)\psi(v_2)) = U_j^+(\psi(v_1v_2)) \geq \delta_{\lambda_j}^+(v_1v_2) \geq \alpha_j$ thus $U_j^+(\psi(v_1)\psi(v_2)) \geq \alpha_j \forall$ and $U_j^+(\psi(v_1^{-1})) = U_j^+(\psi^{-1}(v_1)) \geq \delta_{\lambda_j}^+(v_1^{-1}) = \delta_{\lambda_j}^+(v_1) \geq \alpha_j \forall j$. Therefore $\psi(\delta_{\lambda < \alpha, \beta >})$ is $< \alpha, \beta >$ -level HX-subgroup of a bipolar valued fuzzy HX-subgroup of ν_2 . \square

Theorem 1.25. *The anti homomorphic preimage of a $< \alpha, \beta >$ -level HX-subgroup of a bipolar valued fuzzy HX-subgroup of ν_2 is a $< \alpha, \beta >$ -level HX-subgroup of a bipolar valued fuzzy HX-subgroup of ν_1 .*

Proof. If $U = \psi(\delta_\lambda)$ such that $U = \langle U_j^-, U_j^+ \rangle$ is a bipolar valued fuzzy HX-subgroup of ν_2 . By theorem 1.17 $\delta_\lambda = \langle \delta_{\lambda_j}^-, \delta_{\lambda_j}^+ \rangle$ is a bipolar valued fuzzy HX-subgroup of ν_1 . If $\psi(v_1), \psi(v_1) \in \nu_2$ then $v_1, v_2 \in \nu_1$. Let $\psi(\delta_{\lambda < \alpha, \beta >})$ be a $< \alpha, \beta >$ -level HX-subgroup of U . That is $U_j^-(\psi(v_1)), U_j^-(\psi(v_2)) \leq \beta_j, U_j^+(\psi(v_1)), U_j^+(\psi(v_2)) \geq \alpha_j$ and $U_j^-(\psi(v_1)\psi(v_2)) \leq \beta_j, U_j^+(\psi(v_1)\psi(v_2)) \geq \alpha_j$. And $U_j^-(\psi(v_1^{-1})) = U_j^-(\psi(v_1)) \leq \beta_j$ and $U_j^+(\psi(v_1^{-1})) = U_j^+(\psi(v_1)) \geq \alpha_j$. We have to prove that $\delta_{\lambda < \alpha, \beta >}$ is a $< \alpha, \beta >$ -level HX-subgroup of a bipolar valued fuzzy HX-subgroup δ_λ . Now $\delta_{\lambda_j}^-(v_1) \leq U_j^-(\psi(v_1)) \leq \beta_j$ that $\delta_{\lambda_j}^-(v_1) \leq \beta_j$ and $\delta_{\lambda_j}^-(v_2) \leq U_j^-(\psi(v_2)) \leq \beta_j$ thus $\delta_{\lambda_j}^-(v_2) \leq \beta_j \forall j$. Then $\delta_{\lambda_j}^-(v_1v_2) = U_j^-(\psi(v_1v_2)) \leq U_j^-(\psi(v_1)\psi(v_2)) \leq \beta_j$ thus $\delta_{\lambda_j}^-(v_1v_2) \leq \beta_j \forall j$ and $\delta_{\lambda_j}^-(v_1^{-1}) = U_j^-(\psi(v_1^{-1})) = U_j^-(\psi^{-1}(v_1)) \leq \beta_j$ then

$\delta_{\lambda_j}^-(v_1^{-1}) \leq \beta_j \forall j$ and $\delta_{\lambda_j}^+(v_1 v_2) = U_j^+(\psi(v_1 v_2)) \geq U_j^+(\psi(v_1)\psi(v_2)) \geq \alpha_j$ thus $\delta_{\lambda_j}^+(v_1 v_2) \geq \alpha_j \forall j$ and $\delta_{\lambda_j}^+(v_1^{-1}) = U_j^+(\psi(v_1^{-1})) = U_j^+(\psi^{-1}(v_1)) \geq \alpha_j \forall j$. Therefore $\delta_{\lambda < \alpha, \beta >}$ is a $< \alpha, \beta >$ -level HX-subgroup of a bipolar valued fuzzy HX-subgroup of δ of ν_1 . \square

conclusion

In this article, we introduced and discussed the concept of bipolar valued multi fuzzy, multi normal fuzzy HX subgroup, homomorphism and anti homomorphism with suitable theorems. We extended these ideas to the bipolar multi fuzzy quotient HX subgroup and discussed few important theorems and properties of them.

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