Linear Codes over Finite Fields Based On Greedy Algorithms and Their Applications

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Abstract

In this paper we prove that for any ordered basis B_1 of a vector space there is a basis B_2 for which the greedy code generated using the B-ordering is linear with respect to B_2, where B2 is derived from B_1 by a lower triangular matrix P; $B_1 = PB_2$. In Addition we prove a similar result for self-orthogonal greedy codes.

Keywords: greedy algorithm; finite field; linear codes; coding theory

1. Introduction:

A word is a sequence of digits. The length of a word is the number of digits in the word. Let α= {a1, ……, aq} be a finite set called the alphabet, and let αn be the set of all strings of length n over α. Any nonempty subset C of αn is called q-ary block code. Each string in C is called a codeword. If $C \subset$ αn contains M codewords then C has length n and size M or is an (n, M)-code. A code whose α= {0, 1} is called binary code. Let F be a field and n be a positive integer. Let u = {x1, x2,…, xn}. The Hamming weight function is defined as

wh(u) = The cardinality of {i ∈ {1, 2, …., n}: xi ≠ 0}
=The number of non-zero coordinates xi (1, 2, …., n).
Let x and y be two words of the same length. The Hamming distance or
simply the distance between x and y differ. We denote distance between x and y by dh(x, y).
A code C is said to have minimum distance d if $d = \text{minimum}\{dh(x, y)|x, y \in C, x \neq y\}$, and it is denoted by d(C).

1- Orderings:

Lexicographic ordering. If the order of the list is in the "natural" order, then these codes are called lexicode, where the "natural" order means that 0<1, and two binary vectors
(c1, c2……, cn) < (b1, b2……, bn)
if there is anon-negative integer k such that ci = bi for all i = k ≥ 1 and ck+1 = 0, bk+1 = 1. For example (1, 0, 1) < (1, 1, 0).
Lexicodes are proved to be linear codes (c.f. [2]).

2- B-Orderings:

Let V be a finite vector space of a dimension n over a field Zp of prime order. A B-Orderings is generated recursively using an ordered basis B = {b1, b2……, bn} as follows:

The first p vectors are 0, b1, 2b1, …., (p-1)b1. The B-orderings is then generated recursively, where if pk vectors of the ordering have been generated using the basis elements b1, b2, …, bk then the (p-1)pk vectors are generated by adding bk+1 to those vectors already produced, in order, i = 1, 2, …, p-1.

Let d be an integer greater than or equal to one. The greedy code is the set C of vectors that are selected using the following greedy algorithm.

1. Set up some ordering of vectors of a vector space V.
2. The first vector in the ordering is selected and placed in C.
3. We follow the ordering and if we find a vector u with dh(u, c) ≥ d

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for all vectors c ∈ C, then u is selected and placed in C.

4. Continue until the end of the ordering.

Example 1:

Let B={100,010,001} be a basis over F_{3^2}, n=3 where $F_{3^2} = \{b\alpha + a: a, b \in Z3\}$.
={(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)}F_{3^2}
={0, 1, 2, α, α+1, α+2, 2α, 2α+1, 2α+2}, where (0,0)=0, (0,1)=1, (0,2)=2, (1,0)=α, (1,1)=α+1, F_{3^2}
(1,2)= α+2, (2,0)= 2α, (2,1)= 2α+1, (2,2)= 2α+2. Then the greedy code using the B–ordering is not linear over F_{3^2}. [See table (1)].

Theorem 1:

Let d be an integer ≥ 1, and B1 = {b1, b2, .., bn} be an ordered basis for the vector space V over a finite field, where p is prime. Then there exist a basis B2 = {u1, u2, .., un} for which the greedy code using the B-ordering is linear over a finite field with respect to B2. Furthermore there exists an n× n lower triangular non-singular matrix P such that ui = Pbi, i = 1, 2,…,n

Proof.

Let Wk = {b1, b2, .., bk}, C be the greedy code for V and Ck be the greedy code for Wk.

For k = 0, it is clear that W0 = {(0, 0, .., 0)} and C0 = {(0, 0, ..., 0)}, which implies that C0 is linear. Now, let k ≥ 1 and Ck be the greedy code generated by {b1, b2, …, bk}. Also assume that Ck is linear and there exists {u1, u2, …, uk} basis for which the B-ordering using {u1, u2, …, uk}gives Ck, where

$$\text{ui} = \sum_{j=1}^{i} \gamma_j b_j$$

Let j be the smallest integer, and let Ck+j be the greedy code of the vector space Wk+j that is not equal to Ck. Now, let b be the first vector in the Bordering not in Ck that fits the greedy algorithm, i.e., dh(b, c) ≥ d for all c ∈ CK.
(1)

It follows that b ∈ Wk + j

Put uk+1 = bk+1, uk+2 = bk+2, …, uk+j-1 = bk+j-1, uk+j = b (2)

Notice that for i = 1, 2, …., k, ui $= \sum_{j=1}^{i} \gamma_j b_j$ by mathematical induction, and, using (2), for i = k + 1, k + 2, …, k + j – 1, ui $= \sum_{j=1}^{i} \gamma_j b_j$

Finally, for i = k + j, we have set uk+j = b ∈ Wk+j, it follows that $uk + j = b = \sum_{j=1}^{k+j} \gamma_j b_j$. Thus for all i, i = 1, 2, …., k+j, we have ui $= \sum_{j=1}^{i} \gamma_j b_j$. All we need now is to show the linearity of Ck+j.

Now we need to prove the following claim:

For $\gamma \in F$, $\gamma uk + j + v \in Ck + j$ if and only if $v \in Ck$.

Proof of the claim:

Let v ∈ Ck, for γuk+j + v to be in Ck+j it has to satisfy the following: dh(γuk+j + v, u) ≥ d, for all previously chosen vectors u ∈ Ck+j. Now vectors in Ck+j have the form βγuk+j + c for c ∈ Ck . It follows that dh(γuk+j + v, βuk+j + c) = wth (γuk+j + v – βuk+j – c) = wth((γ–β)uk+j – (c – v)) = wth(δuk+j – ω),
where δ = (γ–β) ∈ F, ω = (c – v) is in Ck by linearity of Ck. Since wth (δuk+j – ω) = wth (uk+j – δ-1ω) ≥ d, by (1), it follows that dh (γuk+j+ v, u) ≥ d, for all previously chosen vectors in Ck+j.

Conversely, assume that γuk+j + v ∈ Ck+j, then we have the following:
dh (γuk+j + v, u) ≥ d, for all previously chosen vectors in Ck+j.
In particular dh (γuk+j + v, γuk+j + c) ≥ d, for all previously chosen vectors c ∈ Ck. It follows that dh (v, c) ≥ d, for all previously chosen vectors c ∈ Ck, and so v∈ Ck. This proves our claim.

Our claim showes that αuk+j + v ∈ Ck+j if and only if v ∈ Ck.

Let u = γbk–j + w0, with w0 ∈ Wk+j–1, γ-1u = bk+j + γ-1w0, satisfies our requirements for being in Ck–j . However, by the B-ordering γ-1u comes before u. Therefor, by our choice of u, γ must be 1. Thus
u = bk+j + w0, with w0 ∈ Wk+j–1.

This proves that if γ1 < γ2, then γ1u < γ2u and hence γ1u + c < γ2u + c. It follows that none of the linear combinations will be missed out when we trace the ordering. This completes the proof.

Example 2.

Let B1 = {100, 010, 001} be a basis over F_{3^2}, where F_{3^2}= {bx+a: a,b∈ Z3}.
= {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)}.F_{3^2}
= {0, 1, 2, α, α+1, α+2, 2α, 2α+1, 2α+2}, where (0, 0) = 0, (0, 1) = 1,F_{3^2}
(0, 2) = 2, (1, 0) = α, (1, 1) = α+1, (1, 2) = α+2, (2, 0) = 2α, (2, 1) = 2α+1,

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(2, 2) = 2α+2. Then there is a basis B2 = {100, 110, 111} over F_{3^2}. The greedy code using the B2–ordering is linear over F_{3^2}.[See table (2)].

Theorem 2: Let B = {b1, b2, .., bn} an ordered basis for the vector space V over a field of prime order p. Then there exist a basis $B_2 = \{u_1, u_2, \dots, u_n\}$ for which the self-orthogonal greedy code using the B-ordering is a linear code over a finite field with respect to the basis B_2. Furthermore there exist n×n lower triangular non-singular matrix P, such that $u_i = pb_i$, i=1,2,…., n.

Proof.

The proof is basically the same as the proof of Theorem 1.
However, we have to make sure that codewords are self-orthogonal.
Let $W_k = \{b_1, b_2, \dots, b_k\}$, C be the self-orthogonal greedy code for V and C_k be the self-orthogonal greedy code for W_k.
For $k = 0$, it is clear that $W_0 = \{(0,0,..,0)\}$ and $C_0 = \{(0,0,..,0)\}$ hence C_0 is linear in this case.
Now, let $k \geq 1$, and assume that C_k is linear for which there exist a basis
for which the B-ordering using $\{u_1, u_2, \dots, u_k\}$gives a linear self-orthogonal greedy $\{u_1, u_2, \dots, u_k\}$ code and $u_i = \sum_{j=1}^{i} \gamma_j b_j$. Let j be the smallest integer such that the self-orthogonal greedy code C_{k+j} of the vector subspace W_{k+j} is not equal to C_k. Now, let b be the first vector in the B-ordering with respect to B_1that is not in C_k that fits greedy algorithm, this means that:
for all $c \in C_k$. (1)$d(b,c) \geq d$
, $c\bullet c = 0$, $b\bullet c = 0 = c\bullet b$, for all $c \in C_k$. (2)$b\bullet b = 0$
Set $u_{k+1} = b_{k+1}, u_{k+2} = b_{k+2}, \dots, u_{k+j-1} = b_{k+j-1}, u_{k+j} = b$ (3)
Notice that for $i = 1, 2, \dots, k$, $u_i = \sum_{j=1}^{i} \gamma_j b_j$ by mathematical induction. And using (3), for $i = k + 1, k + 2, \dots, k + j - 1$, we get

$$u_i = \sum_{j=1}^{i} \gamma_j b_j$$

Finally, for $i = k + j$, we have set $u_{k+j} = b$, it follows that $u_{k+j} = b = \sum_{j=1}^{k+j} \gamma_j b_j$. Thus for all i, $i = 1, 2, \dots, k+j$, we have and we need to show now the linearity of C_{k+j}.
This can be achieved by proving the following claim:
For all α ∈ F then $\alpha b + v \in C_{k+j}$ if and only if $v \in C_k$.

Proof of the claim:

Let $v \in Ck$, for $\gamma b + v$ to be in C_{k+j} then the following properties should be satisfied:

1. $d_h(\gamma b + v, u) \geq d$, for all previously chosen vectors $u \in C_{k+j}$.
2. $(\gamma b + v)\bullet(\gamma b + v) = 0$.
3. $(\gamma b + v)\bullet u = 0$for all chosen vectors $u \in C_{k+j}$.

Now vectors in C_{k+j} has the form $\beta b + c$ where $c \in C_k$, β ∈ F. Let $u = \beta b + c$, it follows that:

$$\begin{aligned} d_h(\alpha b + v, u) &= \mathrm{wt}_h(\gamma b + v - \beta b - c) \\ &= \mathrm{wt}_h(\gamma - \beta)b - (c - v)) \\ &= \mathrm{wt}_h(\delta b - \omega) \end{aligned}$$

where $\delta = (\gamma - \beta) \in F$, $\omega = (c - v) \in C_k$
We have $\mathrm{wt}_h(\delta b - \omega) = \mathrm{wt}_h(b - \delta^{-1}\omega) \geq d$, by (1).
It follows that $d_h(\gamma b + v, u) \geq d$, for all chosen vectors $u \in C_{k+j}$.
, since, $b\bullet b = 0$, $b\bullet v = 0$, $v\bullet b = 0$, $v\bullet v =(\gamma b + v)\bullet(\gamma b + v) = \gamma^2 b\bullet b + \gamma b\bullet v + \gamma v\bullet b + v\bullet v = 0$
0 by (2).
for the same reasons by (2). Conversely, assume that γb $+(\gamma b + v)\bullet u = (\gamma b + v)\bullet(\beta b + c) = 0$
$v \in C_{k+j}$, then it satisfies the following:

1. $d_h(\gamma b + v, u) \geq d$, for all chosen vectors in C_{k+j}.
2. $(\gamma b + v)\bullet(\gamma b + v) = 0$ (4)

In particular $d_h(\gamma b + v, \gamma b + c) \geq d$, for vectors $c \in C_k$.
It follows that $d_h(v, c) \geq d$, for all vectors $c \in C_k$.
Using equation (4), we get
, and since $\gamma b + v$ comes after γb in the$\gamma^2 b\bullet b + \gamma b\bullet v + \gamma v\bullet b + v\bullet v = 0$
B-ordering, then by the choice of $\gamma b + v$, and using equation (3),
Therefore $b\bullet v = 0 = v\bullet b$. Thus $v\bullet v = 0$. It follows that $v \in C_k$. Our claim showed that $\gamma b + v \in C_{k+j}$ if and only if $v \in C_k$.
This means that $C_{k+j} = \langle b, C_k \rangle$

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Let $u = \gamma b_{k-j} + w_0$, with $w_0 \in W_{k+j-1}$, $\gamma^{-1}u = b_{k+j} + \gamma^{-1}w_0$, satisfies our requirements for being in C_{k-j}. However, by the B-ordering $\gamma^{-1}u$ comes before u. Therefor, by our choice of u, γ must be 1. Thus $u = b_{k+j} + w_0$, with $w_0 \in W_{k+j-1}$.
This proves that if $\gamma_1 < \gamma_2$, then $\gamma_1 u < \gamma_2 u$ and $\gamma_1 u + c < \gamma_2 u + c$ hence. It follows that none of the linear combinations will be missed out when we trace the ordering. This completes the proof.

Example:

In the last example $B_1 = \{100, 010, 001\}$ is a basis over F_{3^2}. There is a basis $B_2 = \{u_1, u_2, u_3\} = \{b_1, b_1 + b_2, b_1 + b_2 + b_3\} = \{100, 110, 111\}$ basis over F_{3^2}. The self-orthogonal greedy code using B- ordering is linear over F_{3^2}, where the self-orthogonal greedy code used $d = 3$ is {000, 111, 222, α α α, α+1 α+1 α+1, α+2 α+2 α+2, 2α 2α 2α, 2α+1 2α+1 2α+1, 2α+2 2α+2 2α+2} linear code.

Conclusion:

We study greedy code generated by the B-ordering over finite fields. We show that the greedy code over the finite field is not always linear, when generated by the B-ordering. To have linear codes we have prove that for any ordered basis $B_1$of a vector space over finite fields there is a basis B_2 for which the greedy code generated using the B-ordering is linear with respect to B_2, where B_2 is derived from B_1 by a lower triangular matrix P; $B_2 = PB_1$. In Addition we prove a similar result for self-orthogonal greedy codes.

There are many problems:

1- The case of a ring: The linearity of the greedy codes for Boolean rings, using B-ordering.
2- The case of finite chain rings: The linearity (as a left module) of the greedy codes for finite chain rings, See [3], using B-ordering.

Tables:

Table 1: The B-ordering

1	0 0 0	34	2α α 0	67	α 2α+1 0
2	1 0 0	35	2α+1 α 0	68	α+1 2α+1 0
3	2 0 0	36	2α+2 α 0	69	α+2 2α+1 0
4	α 0 0	37	0 α+1 0	70	2α 2α+1 0
5	α+1 0 0	38	1 α+1 0	71	2α+1 2α+1 0
6	α+2 0 0	39	2 α+1 0	72	2α+2 2α+1 0
7	2α 0 0	40	α α+1 0	73	0 2α+2 0
8	2α+1 0 0	41	α+1 α+1 0	74	1 2α+2 0
9	2α+2 0 0	42	α+2 α+1 0	75	2 2α+2 0
10	0 1 0	43	2α α+1 0	76	α 2α+2 0
11	1 1 0	44	2α+1 α+1 0	77	α+1 2α+2 0
12	2 1 0	45	2α+2 α+1 0	78	α+2 2α+2 0
13	α 1 0	46	0 α+2 0	79	2α 2α+2 0
14	α+1 1 0	47	1 α+2 0	80	2α+1 2α+2 0
15	α+2 1 0	48	2 α+2 0	81	2α+2 2α+2 0
16	2α 1 0	49	α α+2 0	82	0 0 1
17	2α+1 1 0	50	α+1 α+2 0	83	1 0 1
18	2α+2 1 0	51	α+2 α+2 0	84	2 0 1
19	0 2 0	52	2α α+2 0	85	α 0 1
20	1 2 0	53	2α+1 α+2 0	86	α+1 0 1
21	2 2 0	54	2α+2 α+2 0	87	α+2 0 1
22	α 2 0	55	0 2α 0	88	2α 0 1
23	α+1 2 0	56	1 2α 0	89	2α+1 0 1
24	α+2 2 0	57	2 2α 0	90	2α+2 0 1
25	2α 2 0	58	α 2α 0	91	0 1 1
26	2α+1 2 0	59	α+1 2α 0	92	1 1 1
27	2α+2 2 0	60	α+2 2α 0	93	2 1 1
28	0 α 0	61	2α 2α 0	94	α 1 1

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29	1 α 0	62	2α+1 2α 0	95	α+1 1 1
30	2 α 0	63	2α+2 2α 0	96	α+2 1 1
31	α α 0	64	0 2α+1 0	97	2α 1 1
32	α+1 α 0	65	1 2α+1 0	98	2α+1 1 1
33	α+2 α 0	66	2 2α+1 0	99	2α+2 1 1

100	0 2 1	135	2α+2 α+2 1	170	2α+1 0 2
101	1 2 1	136	0 2α 1	171	2α+2 0 2
102	2 2 1	137	1 2α 1	172	0 1 2
103	α 2 1	138	2 2α 1	173	1 1 2
104	α+1 2 1	139	α 2α 1	174	2 1 2
105	α+2 2 1	140	α+1 2α 1	175	α 1 2
106	2α 2 1	141	α+2 2α 1	176	α+1 1 2
107	2α+1 2 1	142	2α 2α 1	177	α+2 1 2
108	2α+2 2 1	143	2α+1 2α 1	178	2α 1 2
109	0 α 1	144	2α+2 2α 1	179	2α+1 1 2
110	1 α 1	145	0 2α+1 1	180	2α+2 1 2
111	2 α 1	146	1 2α+1 1	181	0 2 2
112	α α 1	147	2 2α+1 1	182	1 2 2
113	α+1 α 1	148	α 2α+1 1	183	2 2 2
114	α+2 α 1	149	α+1 2α+1 1	184	α 2 2
115	2α α 1	150	α+2 2α+1 1	185	α+1 2 2
116	2α+1 α 1	151	2α 2α+1 1	186	α+2 2 2
117	2α+2 α 1	152	2α+1 2α+1 1	187	2α 2 2
118	0 α+1 1	153	2α+2 2α+1 1	188	2α+1 2 2
119	1 α+1 1	154	0 2α+2 1	189	2α+2 2 2
120	2 α+1 1	155	1 2α+2 1	190	0 α 2
121	α α+1 1	156	2 2α+2 1	191	1 α 2
122	α+1 α+1 1	157	α 2α+2 1	192	2 α 2
123	α+2 α+1 1	158	α+1 2α+2 1	193	α α 2
124	2α α+1 1	159	α+2 2α+2 1	194	α+1 α 2
125	2α+1 α+1 1	160	2α 2α+2 1	195	α+2 α 2
126	2α+2 α+1 1	161	2α+1 2α+2 1	196	2α α 2
127	0 α+2 1	162	2α+2 2α+2 1	197	2α+1 α 2
128	1 α+2 1	163	0 0 2	198	2α+2 α 2
129	2 α+2 1	164	1 0 2	199	0 α+1 2
130	α α+2 1	165	2 0 2	200	1 α+1 2
131	α+1 α+2 1	166	α 0 2	201	2 α+1 2
132	α+2 α+2 1	167	α+1 0 2	202	α α+1 2
133	2α α+2 1	168	α+2 0 2	203	α+1 α+1 2
134	2α+1 α+2 1	169	2α 0 2	204	α+2 α+1 2

205	2α α+1 2	240	α+2 2α+2 2	275	α+1 α α
206	2α+1 α+1 2	241	2α 2α+2 2	276	α+2 α α
207	2α+2 α+1 2	242	2α+1 2α+2 2	277	2α α α

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208	0 α+2 2	243	2α+2 2α+2 2	278	2α+1 α α
209	1 α+2 2	244	0 0 α	279	2α+2 α α
210	2 α+2 2	245	1 0 α	280	0 α+1 α
211	α α+2 2	246	2 0 α	281	1 α+1 α
212	α+1 α+2 2	247	α 0 α	282	2 α+1 α
213	α+2 α+2 2	248	α+1 0 α	283	α α+1 α
214	2α α+2 2	249	α+2 0 α	284	α+1 α+1 α
215	2α+1 α+2 2	250	2α 0 α	285	α+2 α+1 α
216	2α+2 α+2 2	251	2α+1 0 α	286	2α α+1 α
217	0 2α 2	252	2α+2 0 α	287	2α+1 α+1 α
218	1 2α 2	253	0 1 α	288	2α+2 α+1 α
219	2 2α 2	254	1 1 α	289	0 α+2 α
220	α 2α 2	255	2 1 α	290	1 α+2 α
221	α+1 2α 2	256	α 1 α	291	2 α+2 α
222	α+2 2α 2	257	α+1 1 α	292	α α+2 α
223	2α 2α 2	258	α+2 1 α	293	α+1 α+2 α
224	2α+1 2α 2	259	2α 1 α	294	α+2 α+2 α
225	2α+2 2α 2	260	2α+1 1 α	295	2α α+2 α
226	0 2α+1 2	261	2α+2 1 α	296	2α+1 α+2 α
227	1 2α+1 2	262	0 2 α	297	2α+2 α+2 α
228	2 2α+1 2	263	1 2 α	298	0 2α α
229	α 2α+1 2	264	2 2 α	299	1 2α α
230	α+1 2α+1 2	265	α 2 α	300	2 2α α
231	α+2 2α+1 2	266	α+1 2 α	301	α 2α α
232	2α 2α+1 2	267	α+2 2 α	302	α+1 2α α
233	2α+1 2α+1 2	268	2α 2 α	303	α+2 2α α
234	2α+2 2α+1 2	269	2α+1 2 α	304	2α 2α α
235	0 2α+2 2	270	2α+2 2 α	305	2α+1 2α α
236	1 2α+2 2	271	0 α α	306	2α+2 2α α
237	2 2α+2 2	272	1 α α	307	0 2α+1 α
238	α 2α+2 2	273	2 α α	308	1 2α+1 α
239	α+1 2α+2 2	274	α α α	309	2 2α+1 α

310	α 2α+1 α	345	2 2 α+1	380	1 2α α+1
311	α+1 2α+1 α	346	α 2 α+1	381	2 2α α+1
312	α+2 2α+1 α	347	α+1 2 α+1	382	α 2α α+1
313	2α 2α+1 α	348	α+2 2 α+1	383	α+1 2α α+1
314	2α+1 2α+1 α	349	2α 2 α+1	384	α+2 2α α+1
315	2α+2 2α+1 α	350	2α+1 2 α+1	385	2α 2α α+1
316	0 2α+2 α	351	2α+2 2 α+1	386	2α+1 2α α+1
317	1 2α+2 α	352	0 α α+1	387	2α+2 2α α+1
318	2 2α+2 α	353	1 α α+1	388	0 2α+1 α+1
319	α 2α+2 α	354	2 α α+1	389	1 2α+1 α+1
320	α+1 2α+2 α	355	α α α+1	390	2 2α+1 α+1
321	α+2 2α+2 α	356	α+1 α α+1	391	α 2α+1 α+1
322	2α 2α+2 α	357	α+2 α α+1	392	α+1 2α+1 α+1
323	2α+1 2α+2 α	358	2α α α+1	393	α+2 2α+1 α+1
324	2α+2 2α+2 α	359	2α+1 α α+1	394	2α 2α+1 α+1
325	0 0 α+1	360	2α+2 α α+1	395	2α+1 2α+1 α+1
326	1 0 α+1	361	0 α+1 α+1	396	2α+2 2α+1 α+1

Doi: https://doi.org/10.54216/PMTCS.030202
Received: June 11, 2023 Revised: September 24, 2023 Accepted: December 17, 2023

327	2 0 α+1	362	1 α+1 α+1	397	0 2α+2 α+1
328	α 0 α+1	363	2 α+1 α+1	398	1 2α+2 α+1
329	α+1 0 α+1	364	α α+1 α+1	399	2 2α+2 α+1
330	α+2 0 α+1	365	α+1 α+1 α+1	400	α 2α+2 α+1
331	2α 0 α+1	366	α+2 α+1 α+1	401	α+1 2α+2 α+1
332	2α+1 0 α+1	367	2α α+1 α+1	402	α+2 2α+2 α+1
333	2α+2 0 α+1	368	2α+1 α+1 α+1	403	2α 2α+2 α+1
334	0 1 α+1	369	2α+2 α+1 α+1	404	2α+1 2α+2 α+1
335	1 1 α+1	370	0 α+2 α+1	405	2α+2 2α+2 α+1
336	2 1 α+1	371	1 α+2 α+1	406	0 0 α+2
337	α 1 α+1	372	2 α+2 α+1	407	1 0 α+2
338	α+1 1 α+1	373	α α+2 α+1	408	2 0 α+2
339	α+2 1 α+1	374	α+1 α+2 α+1	409	α 0 α+2
340	2α 1 α+1	375	α+2 α+2 α+1	410	α+1 0 α+2
341	2α+1 1 α+1	376	2α α+2 α+1	411	α+2 0 α+2
342	2α+2 1 α+1	377	2α+1 α+2 α+1	412	2α 0 α+2
343	0 2 α+1	378	2α+2 α+2 α+1	413	2α+1 0 α+2
344	1 2 α+1	379	0 2α α+1	414	2α+2 0 α+2

415	0 1 α+2	450	2α+2 α+1 α+2	485	2α+1 2α+2 α+2
416	1 1 α+2	451	0 α+2 α+2	486	2α+2 2α+2 α+2
417	2 1 α+2	452	1 α+2 α+2	487	0 0 2α
418	α 1 α+2	453	2 α+2 α+2	488	1 0 2α
419	α+1 1 α+2	454	α α+2 α+2	489	2 0 2α
420	α+2 1 α+2	455	α+1 α+2 α+2	490	α 0 2α
421	2α 1 α+2	456	α+2 α+2 α+2	491	α+1 0 2α
422	2α+1 1 α+2	457	2α α+2 α+2	492	α+2 0 2α
423	2α+2 1 α+2	458	2α+1 α+2 α+2	493	2α 0 2α
424	0 2 α+2	459	2α+2 α+2 α+2	494	2α+1 0 2α
425	1 2 α+2	460	0 2α α+2	495	2α+2 0 2α
426	2 2 α+2	461	1 2α α+2	496	0 1 2α
427	α 2 α+2	462	2 2α α+2	497	1 1 2α
428	α+1 2 α+2	463	α 2α α+2	498	2 1 2α
429	α+2 2 α+2	464	α+1 2α α+2	499	α 1 2α
430	2α 2 α+2	465	α+2 2α α+2	500	α+1 1 2α
431	2α+1 2 α+2	466	2α 2α α+2	501	α+2 1 2α
432	2α+2 2 α+2	467	2α+1 2α α+2	502	2α 1 2α
433	0 α α+2	468	2α+2 2α α+2	503	2α+1 1 2α
434	1 α α+2	469	0 2α+1 α+2	504	2α+2 1 2α
435	2 α α+2	470	1 2α+1 α+2	505	0 2 2α
436	α α α+2	471	2 2α+1 α+2	506	1 2 2α
437	α+1 α α+2	472	α 2α+1 α+2	507	2 2 2α
438	α+2 α α+2	473	α+1 2α+1 α+2	508	α 2 2α
439	2α α α+2	474	α+2 2α+1 α+2	509	α+1 2 2α
440	2α+1 α α+2	475	2α 2α+1 α+2	510	α+2 2 2α
441	2α+2 α α+2	476	2α+1 2α+1 α+2	511	2α 2 2α
442	0 α+1 α+2	477	2α+2 2α+1 α+2	512	2α+1 2 2α
443	1 α+1 α+2	478	0 2α+2 α+2	513	2α+2 2 2α
444	2 α+1 α+2	479	1 2α+2 α+2	514	0 α 2α
445	α α+1 α+2	480	2 2α+2 α+2	515	1 α 2α

Doi: https://doi.org/10.54216/PMTCS.030202
Received: June 11, 2023 Revised: September 24, 2023 Accepted: December 17, 2023

446	α+1 α+1 α+2	481	α 2α+2 α+2	516	2 α 2α
447	α+2 α+1 α+2	482	α+1 2α+2 α+2	517	α α 2α
448	2α α+1 α+2	483	α+2 2α+2 α+2	518	α+1 α 2α
449	2α+1 α+1 α+2	484	2α 2α+2 α+2	519	α+2 α 2α

520	2α α 2α	555	α+2 2α+1 2α	590	α+1 2 2α+1
521	2α+1 α 2α	556	2α 2α+1 2α	591	α+2 2 2α+1
522	2α+2 α 2α	557	2α+1 2α+1 2α	592	2α 2 2α+1
523	0 α+1 2α	558	2α+2 2α+1 2α	593	2α+1 2 2α+1
524	1 α+1 2α	559	0 2α+2 2α	594	2α+2 2 2α+1
525	2 α+1 2α	560	1 2α+2 2α	595	0 α 2α+1
526	α α+1 2α	561	2 2α+2 2α	596	1 α 2α+1
527	α+1 α+1 2α	562	α 2α+2 2α	597	2 α 2α+1
528	α+2 α+1 2α	563	α+1 2α+2 2α	598	α α 2α+1
529	2α α+1 2α	564	α+2 2α+2 2α	599	α+1 α 2α+1
530	2α+1 α+1 2α	565	2α 2α+2 2α	600	α+2 α 2α+1
531	2α+2 α+1 2α	566	2α+1 2α+2 2α	601	2α α 2α+1
532	0 α+2 2α	567	2α+2 2α+2 2α	602	2α+1 α 2α+1
533	1 α+2 2α	568	0 0 2α+1	603	2α+2 α 2α+1
534	2 α+2 2α	569	1 0 2α+1	604	0 α+1 2α+1
535	α α+2 2α	570	2 0 2α+1	605	1 α+1 2α+1
536	α+1 α+2 2α	571	α 0 2α+1	606	2 α+1 2α+1
537	α+2 α+2 2α	572	α+1 0 2α+1	607	α α+1 2α+1
538	2α α+2 2α	573	α+2 0 2α+1	608	α+1 α+1 2α+1
539	2α+1 α+2 2α	574	2α 0 2α+1	609	α+2 α+1 2α+1
540	2α+2 α+2 2α	575	2α+1 0 2α+1	610	2α α+1 2α+1
541	0 2α 2α	576	2α+2 0 2α+1	611	2α+1 α+1 2α+1
542	1 2α 2α	577	0 1 2α+1	612	2α+2 α+1 2α+1
543	2 2α 2α	578	1 1 2α+1	613	0 α+2 2α+1
544	α 2α 2α	579	2 1 2α+1	614	1 α+2 2α+1
545	α+1 2α 2α	580	α 1 2α+1	615	2 α+2 2α+1
546	α+2 2α 2α	581	α+1 1 2α+1	616	α α+2 2α+1
547	2α 2α 2α	582	α+2 1 2α+1	617	α+1 α+2 2α+1
548	2α+1 2α 2α	583	2α 1 2α+1	618	α+2 α+2 2α+1
549	2α+2 2α 2α	584	2α+1 1 2α+1	619	2α α+2 2α+1
550	0 2α+1 2α	585	2α+2 1 2α+1	620	2α+1 α+2 2α+1
551	1 2α+1 2α	586	0 2 2α+1	621	2α+2 α+2 2α+1
552	2 2α+1 2α	587	1 2 2α+1	622	0 2α 2α+1
553	α 2α+1 2α	588	2 2 2α+1	623	1 2α 2α+1
554	α+1 2α+1 2α	589	α 2 2α+1	624	2 2α 2α+1

625	α 2α 2α+1	660	2 1 2α+2	695	1 α+2 2α+2
626	α+1 2α 2α+1	661	α 1 2α+2	696	2 α+2 2α+2
627	α+2 2α 2α+1	662	α+1 1 2α+2	697	α α+2 2α+2
628	2α 2α 2α+1	663	α+2 1 2α+2	698	α+1 α+2 2α+2
629	2α+1 2α 2α+1	664	2α 1 2α+2	699	α+2 α+2 2α+2
630	2α+2 2α 2α+1	665	2α+1 1 2α+2	700	2α α+2 2α+2

Doi: https://doi.org/10.54216/PMTCS.030202
Received: June 11, 2023 Revised: September 24, 2023 Accepted: December 17, 2023

631	0 2α+1 2α+1	666	2α+2 1 2α+2	701	2α+1 α+2 2α+2
632	1 2α+1 2α+1	667	0 2 2α+2	702	2α+2 α+2 2α+2
633	2 2α+1 2α+1	668	1 2 2α+2	703	0 2α 2α+2
634	α 2α+1 2α+1	669	2 2 2α+2	704	1 2α 2α+2
635	α+1 2α+1 2α+1	670	α 2 2α+2	705	2 2α 2α+2
636	α+2 2α+1 2α+1	671	α+1 2 2α+2	706	α 2α 2α+2
637	2α 2α+1 2α+1	672	α+2 2 2α+2	707	α+1 2α 2α+2
638	2α+1 2α+1 2α+1	673	2α 2 2α+2	708	α+2 2α 2α+2
639	2α+2 2α+1 2α+1	674	2α+1 2 2α+2	709	2α 2α 2α+2
640	0 2α+2 2α+1	675	2α+2 2 2α+2	710	2α+1 2α 2α+2
641	1 2α+2 2α+1	676	0 α 2α+2	711	2α+2 2α 2α+2
642	2 2α+2 2α+1	677	1 α 2α+2	712	0 2α+1 2α+2
643	α 2α+2 2α+1	678	2 α 2α+2	713	1 2α+1 2α+2
644	α+1 2α+2 2α+1	679	α α 2α+2	714	2 2α+1 2α+2
645	α+2 2α+2 2α+1	680	α+1 α 2α+2	715	α 2α+1 2α+2
646	2α 2α+2 2α+1	681	α+2 α 2α+2	716	α+1 2α+1 2α+2
647	2α+1 2α+2 2α+1	682	2α α 2α+2	717	α+2 2α+1 2α+2
648	2α+2 2α+2 2α+1	683	2α+1 α 2α+2	718	2α 2α+1 2α+2
649	0 0 2α+2	684	2α+2 α 2α+2	719	2α+1 2α+1 2α+2
650	1 0 2α+2	685	0 α+1 2α+2	720	2α+2 2α+1 2α+2
651	2 0 2α+2	686	1 α+1 2α+2	721	0 2α+2 2α+2
652	α 0 2α+2	687	2 α+1 2α+2	722	1 2α+2 2α+2
653	α+1 0 2α+2	688	α α+1 2α+2	723	2 2α+2 2α+2
654	α+2 0 2α+2	689	α+1 α+1 2α+2	724	α 2α+2 2α+2
655	2α 0 2α+2	690	α+2 α+1 2α+2	725	α+1 2α+2 2α+2
656	2α+1 0 2α+2	691	2α α+1 2α+2	726	α+2 2α+2 2α+2
657	2α+2 0 2α+2	692	2α+1 α+1 2α+2	727	2α 2α+2 2α+2
658	0 1 2α+2	693	2α+2 α+1 2α+2	728	2α+1 2α+2 2α+2
659	1 1 2α+2	694	0 α+2 2α+2	729	2α+2 2α+2 2α+2

Table (2) The B-ordering

1	0 0 0	36	2 α 0	71	α+2 2α+1 0
2	1 0 0	37	α+1 α+1 0	72	α 2α+1 0
3	2 0 0	38	α+2 α+1 0	73	2α+2 2α+2 0
4	α 0 0	39	α α+1 0	74	2α 2α+2 0
5	α+1 0 0	40	2α+1 α+1 0	75	2α+1 2α+2 0
6	α+2 0 0	41	2α+2 α+1 0	76	2 2α+2 0
7	2α 0 0	42	2α α+1 0	77	0 2α+2 0
8	2α+1 0 0	43	1 α+1 0	78	1 2α+2 0
9	2α+2 0 0	44	2 α+1 0	79	α+2 2α+2 0
10	1 1 0	45	0 α+1 0	80	α 2α+2 0
11	2 1 0	46	α+2 α+2 0	81	α+1 2α+2 0
12	0 1 0	47	α α+2 0	82	1 1 1
13	α+1 1 0	48	α+1 α+2 0	83	2 1 1
14	α+2 1 0	49	2α+2 α+2 0	84	0 1 1
15	α 1 0	50	2α α+2 0	85	α+1 1 1
16	2α+1 1 0	51	2α+1 α+2 0	86	α+2 1 1
17	2α+2 1 0	52	2 α+2 0	87	α 1 1
18	2α 1 0	53	0 α+2 0	88	2α+1 1 1
19	2 2 0	54	1 α+2 0	89	2α+2 1 1

Doi: https://doi.org/10.54216/PMTCS.030202
Received: June 11, 2023 Revised: September 24, 2023 Accepted: December 17, 2023

20	0 2 0	55	2α 2α 0	90	2α 1 1
21	1 2 0	56	2α+1 2α 0	91	2 2 1
22	α+2 2 0	57	2α+2 2α 0	92	0 2 1
23	α 2 0	58	0 2α 0	93	1 2 1
24	α+1 2 0	59	1 2α 0	94	α+2 2 1
25	2α+2 2 0	60	2 2α 0	95	α 2 1
26	2α 2 0	61	α 2α 0	96	α+1 2 1
27	2α+1 2 0	62	α+1 2α 0	97	2α+2 2 1
28	α α 0	63	α+2 2α 0	98	2α 2 1
29	α+1 α 0	64	2α+1 2α+1 0	99	2α+1 2 1
30	α+2 α 0	65	2α+2 2α+1 0	100	0 0 1
31	2α α 0	66	2α 2α+1 0	101	1 0 1
32	2α+1 α 0	67	1 2α+1 0	102	2 0 1
33	2α+2 α 0	68	2 2α+1 0	103	α 0 1
34	0 α 0	69	0 2α+1 0	104	α+1 0 1
35	1 α 0	70	α+1 2α+1 0	105	α+2 0 1

106	2α 0 1	141	0 2α+1 1	176	α+1 0 2
107	2α+1 0 1	142	α+1 2α+1 1	177	α+2 0 2
108	2α+2 0 1	143	α+2 2α+1 1	178	2α 0 2
109	α+1 α+1 1	144	α 2α+1 1	179	2α+1 0 2
110	α+2 α+1 1	145	2α+2 2α+2 1	180	2α+2 0 2
111	α α+1 1	146	2α 2α+2 1	181	1 1 2
112	2α+1 α+1 1	147	2α+1 2α+2 1	182	2 1 2
113	2α+2 α+1 1	148	2 2α+2 1	183	0 1 2
114	2α α+1 1	149	0 2α+2 1	184	α+1 1 2
115	1 α+1 1	150	1 2α+2 1	185	α+2 1 2
116	2 α+1 1	151	α+2 2α+2 1	186	α 1 2
117	0 α+1 1	152	α 2α+2 1	187	2α+1 1 2
118	α+2 α+2 1	153	α+1 2α+2 1	188	2α+2 1 2
119	α α+2 1	154	2α 2α 1	189	2α 1 2
120	α+1 α+2 1	155	2α+1 2α 1	190	α+2 α+2 2
121	2α+2 α+2 1	156	2α+2 2α 1	191	α α+2 2
122	2α α+2 1	157	0 2α 1	192	α+1 α+2 2
123	2α+1 α+2 1	158	1 2α 1	193	2α+2 α+2 2
124	2 α+2 1	159	2 2α 1	194	2α α+2 2
125	0 α+2 1	160	α 2α 1	195	2α+1 α+2 2
126	1 α+2 1	161	α+1 2α 1	196	2 α+2 2
127	α α 1	162	α+2 2α 1	197	0 α+2 2
128	α+1 α 1	163	2 2 2	198	1 α+2 2
129	α+2 α 1	164	0 2 2	199	α α 2
130	2α α 1	165	1 2 2	200	α+1 α 2
131	2α+1 α 1	166	α+2 2 2	201	α+2 α 2
132	2α+2 α 1	167	α 2 2	202	2α α 2
133	0 α 1	168	α+1 2 2	203	2α+1 α 2
134	1 α 1	169	2α+2 2 2	204	2α+2 α 2
135	2 α 1	170	2α 2 2	205	0 α 2
136	2α+1 2α+1 1	171	2α+1 2 2	206	1 α 2
137	2α+2 2α+1 1	172	0 0 2	207	2 α 2
138	2α 2α+1 1	173	1 0 2	208	α+1 α+1 2
139	1 2α+1 1	174	2 0 2	209	α+2 α+1 2

Doi: https://doi.org/10.54216/PMTCS.030202
Received: June 11, 2023 Revised: September 24, 2023 Accepted: December 17, 2023

140	2 2α+1 1	175	α 0 2	210	α α+1 2

211	2α+1 α+1 2	246	α+2 α α	281	2α+2 2α+1 α
212	2α+2 α+1 2	247	2α α α	282	2α 2α+1 α
213	2α α+1 2	248	2α+1 α α	283	1 2α+1 α
214	1 α+1 2	249	2α+2 α α	284	2 2α+1 α
215	2 α+1 2	250	0 α α	285	0 2α+1 α
216	0 α+1 2	251	1 α α	286	α+1 2α+1 α
217	2α+2 2α+2 2	252	2 α α	287	α+2 2α+1 α
218	2α 2α+2 2	253	α+1 α+1 α	288	α 2α+1 α
219	2α+1 2α+2 2	254	α+2 α+1 α	289	2α+2 2α+2 α
220	2 2α+2 2	255	α α+1 α	290	2α 2α+2 α
221	0 2α+2 2	256	2α+1 α+1 α	291	2α+1 2α+2 α
222	1 2α+2 2	257	2α+2 α+1 α	292	2 2α+2 α
223	α+2 2α+2 2	258	2α α+1 α	293	0 2α+2 α
224	α 2α+2 2	259	1 α+1 α	294	1 2α+2 α
225	α+1 2α+2 2	260	2 α+1 α	295	α+2 2α+2 α
226	2α 2α 2	261	0 α+1 α	296	α 2α+2 α
227	2α+1 2α 2	262	α+2 α+2 α	297	α+1 2α+2 α
228	2α+2 2α 2	263	α α+2 α	298	0 0 α
229	0 2α 2	264	α+1 α+2 α	299	1 0 α
230	1 2α 2	265	2α+2 α+2 α	300	2 0 α
231	2 2α 2	266	2α α+2 α	301	α 0 α
232	α 2α 2	267	2α+1 α+2 α	302	α+1 0 α
233	α+1 2α 2	268	2 α+2 α	303	α+2 0 α
234	α+2 2α 2	269	0 α+2 α	304	2α 0 α
235	2α+1 2α+1 2	270	1 α+2 α	305	2α+1 0 α
236	2α+2 2α+1 2	271	2α 2α α	306	2α+2 0 α
237	2α 2α+1 2	272	2α+1 2α α	307	1 1 α
238	1 2α+1 2	273	2α+2 2α α	308	2 1 α
239	2 2α+1 2	274	0 2α α	309	0 1 α
240	0 2α+1 2	275	1 2α α	310	α+1 1 α
241	α+1 2α+1 2	276	2 2α α	311	α+2 1 α
242	α+2 2α+1 2	277	α 2α α	312	α 1 α
243	α 2α+1 2	278	α+1 2α α	313	2α+1 1 α
244	α α α	279	α+2 2α α	314	2α+2 1 α
245	α+1 α α	280	2α+1 2α+1 α	315	2α 1 α

316	2 2 α	351	2 α α+1	386	2α+2 1 α+1
317	0 2 α	352	2α+1 2α+1 α+1	387	2α 1 α+1
318	1 2 α	353	2α+2 2α+1 α+1	388	2 2 α+1
319	α+2 2 α	354	2α 2α+1 α+1	389	0 2 α+1
320	α 2 α	355	1 2α+1 α+1	390	1 2 α+1
321	α+1 2 α	356	2 2α+1 α+1	391	α+2 2 α+1
322	2α+2 2 α	357	0 2α+1 α+1	392	α 2 α+1

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Received: June 11, 2023 Revised: September 24, 2023 Accepted: December 17, 2023

323	2α 2 α	358	α+1 2α+1 α+1	393	α+1 2 α+1
324	2α+1 2 α	359	α+2 2α+1 α+1	394	2α+2 2 α+1
325	α+1 α+1 α+1	360	α 2α+1 α+1	395	2α 2 α+1
326	α+2 α+1 α+1	361	2α+2 2α+2 α+1	396	2α+1 2 α+1
327	α α+1 α+1	362	2α 2α+2 α+1	397	0 0 α+1
328	2α+1 α+1 α+1	363	2α+1 2α+2 α+1	398	1 0 α+1
329	2α+2 α+1 α+1	364	2 2α+2 α+1	399	2 0 α+1
330	2α+1 α+1 α+1	365	0 2α+2 α+1	400	α 0 α+1
331	0 α+1 α+1	366	1 2α+2 α+1	401	α+1 0 α+1
332	1 α+1 α+1	367	α+2 2α+2 α+1	402	α+2 0 α+1
333	2 α+1 α+1	368	α 2α+2 α+1	403	2α 0 α+1
334	α+2 α+2 α+1	369	α+1 2α+2 α+1	404	2α+1 0 α+1
335	α α+2 α+1	370	2α 2α α+1	405	2α+2 0 α+1
336	α+1 α+2 α+1	371	2α+1 2α α+1	406	α+2 α+2 α+2
337	2α+2 α+2 α+1	372	2α+2 2α α+1	407	α α+2 α+2
338	2α α+2 α+1	373	0 2α α+1	408	α+1 α+2 α+2
339	2α+1 α+2 α+1	374	1 2α α+1	409	2α+2 α+2 α+2
340	2 α+2 α+1	375	2 2α α+1	410	2α α+2 α+2
341	0 α+2 α+1	376	α 2α α+1	411	2α+1 α+2 α+2
342	1 α+2 α+1	377	α+1 2α α+1	412	2 α+2 α+2
343	α α α+1	378	α+2 2α α+1	413	0 α+2 α+2
344	α+1 α α+1	379	1 1 α+1	414	1 α+2 α+2
345	α+2 α α+1	380	2 1 α+1	415	α α α+2
346	2α α α+1	381	0 1 α+1	416	α+1 α α+2
347	2α+1 α α+1	382	α+1 1 α+1	417	α+2 α α+2
348	2α+2 α α+1	383	α+2 1 α+1	418	2α α α+2
349	0 α α+1	384	α 1 α+1	419	2α+1 α α+2
350	1 α α+1	385	2α+1 1 α+1	420	2α+2 α α+2

421	0 α α+2	456	0 2α+1 α+2	491	1 2α 2α
422	1 α α+2	457	α+1 2α+1 α+2	492	2 2α 2α
423	2 α α+2	458	α+2 2α+1 α+2	493	α 2α 2α
424	α+1 α+1 α+2	459	α 2α+1 α+2	494	α+1 2α 2α
425	α+2 α+1 α+2	460	2 2 α+2	495	α+2 2α 2α
426	α α+1 α+2	461	0 2 α+2	496	2α+1 2α+1 2α
427	2α+1 α+1 α+2	462	1 2 α+2	497	2α+2 2α+1 2α
428	2α+2 α+1 α+2	463	α+2 2 α+2	498	2α 2α+1 2α
429	2α α+1 α+2	464	α 2 α+2	499	1 2α+1 2α
430	1 α+1 α+2	465	α+1 2 α+2	500	2 2α+1 2α
431	2 α+1 α+2	466	2α+2 2 α+2	501	0 2α+1 2α
432	0 α+1 α+2	467	2α 2 α+2	502	α+1 2α+1 2α
433	2α+2 2α+2 α+2	468	2α+1 2 α+2	503	α+2 2α+1 2α
434	2α 2α+2 α+2	469	0 0 α+2	504	α 2α+1 2α
435	2α+1 2α+2 α+2	470	1 0 α+2	505	2α+2 2α+2 2α
436	2 2α+2 α+2	471	2 0 α+2	506	2α 2α+2 2α
437	0 2α+2 α+2	472	α 0 α+2	507	2α+1 2α+2 2α
438	1 2α+2 α+2	473	α+1 0 α+2	508	2 2α+2 2α
439	α+2 2α+2 α+2	474	α+2 0 α+2	509	0 2α+2 2α
440	α 2α+2 α+2	475	2α 0 α+2	510	1 2α+2 2α
441	α+1 2α+2 α+2	476	2α+1 0 α+2	511	α+2 2α+2 2α
442	2α 2α α+2	477	2α+2 0 α+2	512	α 2α+2 2α

Doi: https://doi.org/10.54216/PMTCS.030202
Received: June 11, 2023 Revised: September 24, 2023 Accepted: December 17, 2023

443	2α+1 2α α+2	478	1 1 α+2	513	α+1 2α+2 2α
444	2α+2 2α α+2	479	2 1 α+2	514	0 0 2α
445	0 2α α+2	480	0 1 α+2	515	1 0 2α
446	1 2α α+2	481	α+1 1 α+2	516	2 0 2α
447	2 2α α+2	482	α+2 1 α+2	517	α 0 2α
448	α 2α α+2	483	α 1 α+2	518	α+1 0 2α
449	α+1 2α α+2	484	2α+1 1 α+2	519	α+2 0 2α
450	α+2 2α α+2	485	2α+2 1 α+2	520	2α 0 2α
451	2α+1 2α+1 α+2	486	2α 1 α+2	521	2α+1 0 2α
452	2α+2 2α+1 α+2	487	2α 2α 2α	522	2α+2 0 2α
453	2α 2α+1 α+2	488	2α+1 2α 2α	523	1 1 2α
454	1 2α+1 α+2	489	2α+2 2α 2α	524	2 1 2α
455	2 2α+1 α+2	490	0 2α 2α	525	0 1 2α

526	α+1 1 2α	561	α+1 α+2 2α	596	2 1 2α+1
527	α+2 1 2α	562	2α+2 α+2 2α	597	0 1 2α+1
528	α 1 2α	563	2α α+2 2α	598	α+1 1 2α+1
529	2α+1 1 2α	564	2α+1 α+2 2α	599	α+2 1 2α+1
530	2α+2 1 2α	565	2 α+2 2α	600	α 1 2α+1
531	2α 1 2α	566	0 α+2 2α	601	2α+1 1 2α+1
532	2 2 2α	567	1 α+2 2α	602	2α+2 1 2α+1
533	0 2 2α	568	2α+1 2α+1 2α+1	603	2α 1 2α+1
534	1 2 2α	569	2α+2 2α+1 2α+1	604	2 2 2α+1
535	α+2 2 2α	570	2α 2α+1 2α+1	605	0 2 2α+1
536	α 2 2α	571	1 2α+1 2α+1	606	1 2 2α+1
537	α+1 2 2α	572	2 2α+1 2α+1	607	α+2 2 2α+1
538	2α+2 2 2α	573	0 2α+1 2α+1	608	α 2 2α+1
539	2α 2 2α	574	α+1 2α+1 2α+1	609	α+1 2 2α+1
540	2α+1 2 2α	575	α+2 2α+1 2α+1	610	2α+2 2 2α+1
541	α α 2α	576	α 2α+1 2α+1	611	2α 2 2α+1
542	α+1 α 2α	577	2α+2 2α+2 2α+1	612	2α+1 2 2α+1
543	α+2 α 2α	578	2α 2α+2 2α+1	613	0 0 2α+1
544	2α α 2α	579	2α+1 2α+2 2α+1	614	1 0 2α+1
545	2α+1 α 2α	580	2 2α+2 2α+1	615	2 0 2α+1
546	2α+2 α 2α	581	0 2α+2 2α+1	616	α 0 2α+1
547	0 α 2α	582	1 2α+2 2α+1	617	α+1 0 2α+1
548	1 α 2α	583	α+2 2α+2 2α+1	618	α+2 0 2α+1
549	2 α 2α	584	α 2α+2 2α+1	619	2α 0 2α+1
550	α+1 α+1 2α	585	α+1 2α+2 2α+1	620	2α+1 0 2α+1
551	α+2 α+1 2α	586	2α 2α 2α+1	621	2α+2 0 2α+1
552	α α+1 2α	587	2α+1 2α 2α+1	622	α+1 α+1 2α+1
553	2α+1 α+1 2α	588	2α+2 2α 2α+1	623	α+2 α+1 2α+1
554	2α+2 α+1 2α	589	0 2α 2α+1	624	α α+1 2α+1
555	2α α+1 2α	590	1 2α 2α+1	625	2α+1 α+1 2α+1
556	1 α+1 2α	591	2 2α 2α+1	626	2α+2 α+1 2α+1
557	2 α+1 2α	592	α 2α 2α+1	627	2α α+1 2α+1
558	0 α+1 2α	593	α+1 2α 2α+1	628	1 α+1 2α+1
559	α+2 α+2 2α	594	α+2 2α 2α+1	629	2 α+1 2α+1
560	α α+2 2α	595	1 1 2α+1	630	0 α+1 2α+1

Doi: https://doi.org/10.54216/PMTCS.030202
Received: June 11, 2023 Revised: September 24, 2023 Accepted: December 17, 2023

631	α+2 α+2 2α+1	666	α+2 2α 2α+2	701	2α+2 1 2α+2
632	α α+2 2α+1	667	2α+1 2α+1 2α+2	702	2α 1 2α+2
633	α+1 α+2 2α+1	668	2α+2 2α+1 2α+2	703	α+2 α+2 2α+2
634	2α+2 α+2 2α+1	669	2α 2α+1 2α+2	704	α α+2 2α+2
635	2α α+2 2α+1	670	1 2α+1 2α+2	705	α+1 α+2 2α+2
636	2α+1 α+2 2α+1	671	2 2α+1 2α+2	706	2α+2 α+2 2α+2
637	2 α+2 2α+1	672	0 2α+1 2α+2	707	2α α+2 2α+2
638	0 α+2 2α+1	673	α+1 2α+1 2α+2	708	α+1 α+2 2α+2
639	1 α+2 2α+1	674	α+2 2α+1 2α+2	709	2 α+2 2α+2
640	α α 2α+1	675	α 2α+1 2α+2	710	0 α+2 2α+2
641	α+1 α 2α+1	676	2 2 2α+2	711	1 α+2 2α+2
642	α+2 α 2α+1	677	0 2 2α+2	712	α α 2α+2
643	2α α 2α+1	678	1 2 2α+2	713	α+1 α 2α+2
644	2α+1 α 2α+1	679	α+2 2 2α+2	714	α+2 α 2α+2
645	2α+2 α 2α+1	680	α 2 2α+2	715	2α α 2α+2
646	0 α 2α+1	681	α+1 2 2α+2	716	2α+1 α 2α+2
647	1 α 2α+1	682	2α+2 2 2α+2	717	2α+2 α 2α+2
648	2 α 2α+1	683	2α 2 2α+2	718	0 α 2α+2
649	2α+2 2α+2 2α+2	684	2α+1 2 2α+2	719	1 α 2α+2
650	2α 2α+2 2α+2	685	0 0 2α+2	720	2 α 2α+2
651	2α+1 2α+2 2α+2	686	1 0 2α+2	721	α+1 α+1 2α+2
652	2 2α+2 2α+2	687	2 0 2α+2	722	α+2 α+1 2α+2
653	0 2α+2 2α+2	688	α 0 2α+2	723	α α+1 2α+2
654	1 2α+2 2α+2	689	α+1 0 2α+2	724	2α+1 α+1 2α+2
655	α+2 2α+2 2α+2	690	α+2 0 2α+2	725	2α+2 α+1 2α+2
656	α 2α+2 2α+2	691	2α 0 2α+2	726	2α α+1 2α+2
657	α+1 2α+2 2α+2	692	2α+1 0 2α+2	727	1 α+1 2α+2
658	2α 2α 2α+2	693	2α+2 0 2α+2	728	2 α+1 2α+2
659	2α+1 2α 2α+2	694	1 1 2α+2	729	0 α+1 2α+2
660	2α+2 2α 2α+2	695	2 1 2α+2		
661	0 2α 2α+2	696	0 1 2α+2		
662	1 2α 2α+2	697	α+1 1 2α+2		
663	2 2α 2α+2	698	α+2 1 2α+2		
664	α 2α 2α+2	699	α 1 2α+2		
665	α+1 2α 2α+2	700	2α+1 1 2α+2		

The greedy code for *d*=2 is C={000, 110, 220, α α 0, α+1 α+1 0, α+2 α+2 0, 2α 2α 0, 2α+1 2α+1 0, 2α+2 2α+2 0, 211, 021, 101, α+2 α+1 1, α α+2 1, α+1 α 1, 2α+2 2α+1 1, 2α 2α+2 1, 2α+1 2α 1, 122, 202, 012, α+1 α+2 2, α+2 α 2, α α+1 2, 2α+1 2α+2 2, 2α+2 2α 2, 2α 2α+1 2, 2α α α, 2α+1 α+1 α, 2α+2 α+2 α, 0 2α α, 1 2α+1 α, 2 2α+2 α, α 0 α, α+1 1 α, α+2 2 α, 2α+2 α+1 α+1, 2α α+2 α+1, 2α+1 α α+1, 2 2α+1 α+1, 0 2α+2 α+1, 1 2α α+1, α+2 1 α+1, α 2 α+1, α+1 0 α+1, 2α+1 α+2 α+2, 2α+2 α α+2, 2α α+1 α+2, 1 2α+2 α+2, 2 2α α+2, 0 2α+1 α+2, α+1 2 α+2, α+2 0 α+2, α 1 α+2, α 2α 2α, α+1 2α+1 2α, α+2 2α+2 2α, 2α 0 2α, 2α+1 1 2α, 2α+2 2 2α, 0 α 2α, 1 α+1 2α, 2 α+2 2α, α+2 2α+1 2α+1, α 2α+2 2α+1, α+1 2α 2α+1, 2α+2 1 2α+1, 2α 2 2α+1, 2α+1 0 2α+1, 2 α+1 2α+1, 0 α+2 2α+1, 1 α 2α+1, α+1 2α+2 2α+2, α+2 2α 2α+2, α 2α+1 2α+2, 2α+1 2 2α+2, 2α+2 0 2α+2, 2α 1 2α+2, 1 α+2 2α+2, 2 α 2α+2, 0 α+1 2α+2} is a linear code.

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