A Review on the Structure of Fuzzy Regular Proper Mappings in Fuzzy Topological Spaces and Their Properties

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Abstract

The purpose of this paper is construct the concept of fuzzy regular proper mapping in fuzzy topological spaces. We give some characterization of fuzzy regular compact mapping and fuzzy regular coercive mapping. We study the relation among the concepts of fuzzy regular proper mapping, fuzzy regular compact mapping and fuzzy regular coercive mapping and we obtained several properties.

Keywords: Fuzzy set; regular mapping; Fuzzy topological space; fuzzy subset.

1. Introduction

The concept of fuzzy sets and fuzzy set operation were first introduced by L. A. Zadeh [5]. Several other authors applied fuzzy sets to various branches of mathematics. One of these objects is a topologically space. From [3] at the first time in 1968, C.L. Chang introduced and developed the concept of fuzzy topological spaces and investigated how some of the basic ideas and theorems of point – set topology behave in this generalized setting. Moreover, many properties on a fuzzy topologically space were prove them by Chang 's definition.

In this paper we introduce and discuss the concepts of fuzzy regular proper mapping correspondence from a fuzzy topological space to another fuzzy topological space and we obtained several properties and characterization of these mappings by comparing with the other mappings.

2. Preliminaries

First, we present some fundamental definitions and proposition which are needed in the next sections.

Definition 2.1. [6] Let X be a non – empty set and let I be the unit interval, i.e., I=[0,1]. A fuzzy set in X is a function from X into the unit interval I (i.e., $A: X \to [0,1]$ be a function).

A fuzzy set A in X can be represented by the set of pairs: $= \{(x, A(x)): x \in X\}$. The family of all fuzzy sets in X is denoted by I^X.

Remark 2.2.

(i) 0_X(**the empty set**) is a fuzzy set which has membership defined by $0_X(x) = 0$ for all $x \in X$

(ii) 1_X (**the universal set**) is a fuzzy set which has membership defined by $1_X(x) = 1$ for all $x \in X$.

Definition 2.3. [3,4,7] let A,B and $A_i, i \in I$ be any fuzzy sets in X. Then we put:

(i) $A \leq B$ if and only if $A(x) \leq B(x), \forall x \in X$;

(ii) $A = B$ if and only if $A(x) = B(x), \forall x \in X$;

(iii) $Z = A \wedge B$ if and only if $Z(x) = min\{ A(x), B(x)\}, \forall x \in X$; ($Z$ is a fuzzy set in X);

(iv) $Z = A \vee B$ if and only if $Z(x) = max\{ A(x), B(x)\}, \forall x \in X$; ($Z$ is a fuzzy set in X);

(v) $Z = \vee_{i \in I} A_i$ **if and only if** $Z(x) = \sup\{A_i(x) / i \in I\}, \forall x \in X$ **(Z is a fuzzy set in X);**

(vi) $Z = \wedge_{i \in I} A_i$ **if and only if** $Z(x) = \inf\{A_i(x) / i \in I\}, \forall x \in X$ **(Z is a fuzzy set in X);**

(vii) $E = A^C$(the complement of A) if and only if $E(x) = 1 - A(x), \forall x \in X$

(viii) $(A \backslash B)(x) = A(x) \wedge B^C(x), \forall x \in X$.

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Definition 2.4. [6] Let X and Y be two non –empty sets $f: X \to Y$ be function. For a fuzzy set B in Y, the inverse image of B under f is the fuzzy set $f^{-1}(B)$ in X with membership function denoted by the rule**:**

for $x \in X f^{-1}(B)(x) = B(f(x))$

.$(\text{i.e.}, f^{-1}(B) = B \circ f)$

For a fuzzy set A in X, the image of A under f is the fuzzy set $f(A)$ in Y with membership function $f(A)$(y), $y \in Y$ defined by

$$f(A)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} A(x) & if\ f^{-1}(y) \neq \emptyset \\ 0 & if\ f^{-1}(y) = \emptyset \end{cases}$$

Where $f^{-1}(y) = \{x: f(x) = y\}$**.**

Theorem 2.5. [7] Let X,Y be two non-empty sets and $f: X \to Y$ be a function. Let $\{A_j\}_{j\in J}$, $\{B_j\}_{j\in J}$ be family of fuzzy sets in X and Y respectively, then

(i) $f(\vee_{j\in J} A_j) = \vee_{j\in J} f(A_j)$.

(ii) $f(\wedge_{j\in J} A_j) \leq \wedge_{j\in J} f(A_j)$.

(iii) $f^{-1}(\vee_{j\in J} B_j) = \vee_{j\in J} f^{-1}(B_j)$.

(iv) $f^{-1}(\wedge_{j\in J} B_j) = \wedge_{j\in J} f^{-1}(B_j)$

Theorem 2.6.[3,7] Let X,Y and Z be non-empty sets and $f: X \to Y$, $g: Y \to Z$ be a functions, then the following statements are the holds:

(i) $f^{-1}(B^c) = (f^{-1}(B))^C$ **for any fuzzy set *B* in *Y*.**

(ii) For any fuzzy set A in X:

(a) $(f(A))^C \leq f(A^c)$**; (b)** $(f(A))^C = f(A^c)$**,** if is a bijective function**.**

(iii) **If** $B_1 \leq B_2$**, then** $f^{-1}(B_1) \leq f^{-1}(B_2)$**,** B_1**and** B_2 **are fuzzy sets in *X*.**

(iv) **If** $A_1 \leq A_2$**, then** $f(A_1) \leq f(A_2)$**,** A_1**and** A_2 **are fuzzy sets in *X*.**

(v) For any fuzzy set A in X:

(a) $A \leq f^{-1}(f(A))$**; (b)** $f^{-1}(f(A)) = A$**,** if f is an injection function.

(vi) For any fuzzy set B in ***Y*:**

(a) $f(f^{-1}(B)) \leq B$**; (b)** $f(f^{-1}(B)) = B$**,** if f is a surjective function.

(vii) $f(f^{-1}(B) \wedge A) = B \wedge f(A)$**.**

(viii) If A is fuzzy set in X and B is a fuzzy set in Y, then $f(A) \leq B$ if and only if $A \leq f^{-1}(B)$**.**

(ix) If $g \circ f: X \to Z$ is the composition between g and f, then:

(a) $(g \circ f)(A) = g(f(A))$, for any fuzzy set A in X.

(b) $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$, for any fuzzy set C in Z.

Definition 2.7. [2,6] A fuzzy point x_α in X is a fuzzy set defined as follows

$$x_\alpha(y) = \begin{cases} \alpha & if\ y = x \\ 0 & if\ y \neq x \end{cases}$$

Where $0 \leq \alpha \leq 1$**;** α **is called its value and *x* is support of** x_α**.**

The set of all fuzzy points in X will be denoted by $FP(X)$**.**

Definition 2.8. [1,6] A fuzzy point x_α is said to belong to a fuzzy set A in X (denoted by: $x_\alpha \in A$) if and only if $\alpha \leq A(x)$**.**

Definition 2.9. [1,6] A fuzzy set A in X is called quasi – coincident with a fuzzy set B in X, denoted by AqB if and only if $A(x) + B(x) > 1$, for some $x \in X$. If A is not quasi –coincident with B, then $A(x) + B(x) \leq 1$, for every $x \in X$ and denoted by $A\tilde{q}B$**.**

Lemma 2.10. [2] Let A and B are fuzzy sets in X. Then**:**

(i) **If** $A \wedge B = 0_X$**, then** $A\tilde{q}B$**.**

(ii) $A\tilde{q}B$ **if and only if** $A \leq B^C$**.**

Proposition 2.11. [2] If A is a fuzzy set in ***X*,** then $x_\alpha \in A$ if and only if $x_\alpha \tilde{q} A^C$

Definition 2.12. [3] A fuzzy topology on a set X is a collection T of fuzzy sets in X satisfying**:**

(i) $0_X \in T$ **and** $1_X \in T$**,**

(ii) **If *A* and *B* belong to *T*, then** $A \wedge B \in T$**,**

(iii) **If** A_i**belongs to *T* for each** $i \in I$ **then so does** $\vee_{i\in I} A_i$**.**

If T is a fuzzy topology on X, then the pair (X,T) is called a fuzzy topological space. And X is called fuzzy space. Members of T are called fuzzy open sets. Fuzzy sets of the forms $1_X - A$, where A is fuzzy open set are called fuzzy closed sets**.**

Definition 2.13. [6] A fuzzy set A in a fuzzy topological space (X,T) is called quasi-neighborhood of a fuzzy point x_α in X if and only if there exists $B \in T$ such that $x_\alpha q B$ and $B \leq A$**.**

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Definition 2.14. [6] Let (X,T) be a fuzzy topological space and x_α be a fuzzy point in X. Then the family $N^Q_{x_\alpha}$ consisting of all quasi-neighborhood (q-neighborhood) of x_α is called the system of quasi-neighborhood of x_α.
Remark 2.15. Let (X,T) be a fuzzy topological space and $A \in FP(X)$. Then A is fuzzy open if and only if A is q – neighbourhood of each its fuzzy point**.**
Definition 2.16. [1] A fuzzy topological spaces (X,T) is called a fuzzy hausdorff (fuzzy T_2- space **)** if and only if for pair of fuzzy points x_r, y_s such that $x \neq y$ in X, there exists $A \in N^Q_{x_r}$, $B \in N^Q_{y_s}$ and $A \wedge B = 0_X$**.**
Definition 2.17. [4] Let be a fuzzy set in X and T be a fuzzy topology on X. Then the induced fuzzy topology on A is the family of fuzzy subsets of A which are the intersection with A of fuzzy open set in X. The induced fuzzy topology is denoted by T_A, and the pair (A, T_A) is called **a fuzzy subspace** of ***X*.**
Proposition 2.18. Let $A \leq Y \leq X$.Then:

(i) If A is a fuzzy open set in Y and Y is a fuzzy open set in X, then A is a fuzzy open set in X.
(ii) If A is a fuzzy closed set in Y and Y is a fuzzy closed set in X, then A is a fuzzy closed set in ***X*.**

Definition 2.19. [10 , 7] Let (X,T) be a fuzzy topological space and $A \in I^X$. Then:
(i) The union of all fuzzy open sets contained in A is called the fuzzy interior of A and denoted by A°. i. e., $A^\circ = \sup\{B : B \leq A, B \in T\}$
(ii) The intersection of all fuzzy closed sets containing A is called the fuzzy closure of A and denoted by $\bar{A}$. i. e., $\bar{A} = inf\ \{B : A \leq B, B^C \in T\}$**.**

Remarks 2.20. [7]
(i) The interior of a fuzzy set A is the largest open fuzzy set contained in A and trivially, a fuzzy set A is fuzzy open if and only if $A = A^\circ$.
(ii) The closure of a fuzzy set A is the smallest closed fuzzy set containing A and trivially, a fuzzy set A is a fuzzy closed if and only if $A = \bar{A}$.

Theorem 2.21 . [10 ,7] Let (X,T) be a fuzzy topological space and A,B are two fuzzy sets in X. Then**:**
(i) $0_X = \overline{0_X}, 1_X = \overline{1_X}$**.**
(ii) $\overline{A \vee B} = \bar{A} \vee \bar{B}, \overline{A \wedge B} \leq \bar{A} \wedge \bar{B}$
(iii) $(A \wedge B)^\circ = A^\circ \wedge B^\circ, A^\circ \vee B^\circ \leq (A \vee B)^\circ$
(iv) $\bar{\bar{A}} = \bar{A}$**,** $(A^\circ)^\circ = A^\circ$
(v) $A^\circ \leq A \leq \bar{A}$**.**
(vi) **If** $A \leq B$ **Then** $A^\circ \leq B^\circ$
(vii) **If** $A \leq B$ **Then** $\bar{A} \leq \bar{B}$ **.**

Proposition 2.22. Let (X,T) be a fuzzy topological space and A be a fuzzy set in X. A fuzzy point $x_\alpha \in \bar{A}$ if and only if for every fuzzy open set B in X, if $x_\alpha q B$ then ***AqB*.**
Proof:$\Rightarrow$ Suppose that B be a fuzzy open set in X such that $x_\alpha q B$ and $A \tilde{q} B$. Then $A \leq B^C$. But $x_\alpha \notin B^C$ (since $x_\alpha q B$, then $\alpha > B^C(x)$) and B^C be a fuzzy closed set in X. Thus $x_\alpha \notin \bar{A}$
Let $x_\alpha \notin \bar{A}$, then there exists a fuzzy closet set B in X such that $A \leq B$ and $x_\alpha \notin B$, hence by $\Leftarrow$ proposition (2.11), we have $x_\alpha q B^C$. Since $A \leq B$, then by lemma (2.10. ii) , $A \tilde{q} B^C$. This complete the proof**.**
Definition 2.23. [7] A fuzzy subset A of a fuzzy topological space X is called fuzzy regular open (fuzzy r- open) if $A = \bar{A}^\circ$. The complement of fuzzy r- open is called fuzzy regular closed (fuzzy r- closed) .Then fuzzy subset of a fuzzy space is fuzzy r- closed if $A = \overline{A^\circ}$
Remark 2.24. [7] Every fuzzy r- open set is a fuzzy open set and every fuzzy r- closed set is a fuzzy closed set .

The converse of remark (2.24), is not true in general as the following example shows**:**

Example 2.25. Let $X = \{a, b\}$ be a set and
, be a fuzzy$T = \{0_X, \{a_{0.3}, b_{0.5}\}, \{a_{0.5}, b_{0.5}\}, \{a_{0.3}, b_{0.7}\}, \{a_{0.5}, b_{0.7}\} 1_X\}$
topology on X.
Notice that $A = \{a_{0.3}, b_{0.5}\}$ is a fuzzy open set in X, but its not fuzzy r- open set and $A = \{a_{0.7}, b_{0.5}\}$ is a fuzzy closed set in X, but its not fuzzy r- closed set**.**
Proposition 2.26. Let $A \leq Y \leq X$. Then:
(i) If A is a fuzzy r- open set in Y and Y is a fuzzy r- open set in X, then A is a fuzzy r - open set in X.
(ii) If A is a fuzzy r - closed set in Y and Y is a fuzzy r - closed set in X, then A is a fuzzy r - closed set in ***X*.**
Corollary 2.27. A fuzzy subset B of a fuzzy space X is fuzzy clopen (fuzzy open and fuzzy closed) if and only if B is fuzzy r - clopen (fuzzy r – open and fuzzy r – closed**).**

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Definition 2.28. [8] The collection of all fuzzy r - open sets of the fuzzy space (X,T) forms a base for a fuzzy topology on X say T^r and its called the fuzzy semi – regularization of T.

Definition 2.29. [7] Let X and Y be fuzzy topological spaces . A map $f: X \to Y$ is fuzzy continuous if and only if for every fuzzy point x_α in X and for every fuzzy open set A in Y, such that $f(x_\alpha) \in A$, there exists fuzzy open set B of X such that $x_\alpha \in B$ and $f(B) \leq A$.

Theorem 2.30. [6] Let X,Y are fuzzy topological spaces and let $f: X \to Y$ be a mapping. Then the following statements are equivalent:

(i) f is fuzzy continuous.

(ii) For each fuzzy open set B in Y, $f^{-1}(B)$ is a fuzzy open set in ***X*.**

(iii) For each fuzzy closed set B in Y, then $f^{-1}(B)$ is a fuzzy closed set in X.

(iv) For each fuzzy set B in Y, $\overline{f^{-1}(B)} \leq f^{-1}(\bar{B})$.

(v) For each fuzzy set A in X, $f(\bar{A}) \leq \overline{f(A)}$.

(vi) For each fuzzy set B in Y, $f^{-1}(B^{\circ}) \leq (f^{-1}(B))^{\circ}$.

Proposition 2.31. [7] If $f: X \to Y$ and
are fuzzy continuous, then$g: Y \to Z$
is fuzzy continuous mapping.$f \circ g: X \to Z$

Proposition 2.32 . Let (X, T) be a fuzzy topological space and A be a non-empty fuzzy subset of X, then the fuzzy inclusion $i_A: (A, T_A) \to (X, T)$ is a fuzzy continuous mapping.

Proof : Let $B \in T$. Since $i_A^{-1}(B) = B \wedge A$, then $i_A^{-1}(B) \in T_A$. Therefore i_A is fuzzy continuous.

Proposition 2.33. Let X,Y be fuzzy topological spaces and A be a fuzzy subset of X.
If $f: X \to Y$ is fuzzy continuous, then the restriction $f_{|A}: A \to Y$ is fuzzy continuous **Proof:** Since f is fuzzy continuous and $f \circ i_A = f_{|A}$.
Then by proposition (2. 32) and proposition (2.31), $f_{|A}$ is fuzzy continuous.

Definition 2.34. Let $f: X \to Y$ be a map from a fuzzy topological space to a fuzzy topological space Y. Then f is called fuzzy r-irresolute mapping if $f^{-1}(A)$ is a fuzzy r- open set in X for every fuzzy r - open set A in Y.

Definition 2.35. A mapping $f: X \to Y$ is called a fuzzy r - closed mapping if the image of each fuzzy closed subset of X is a fuzzy r - closed set in ***Y*.**

Proposition 2.36. If $f: X \to Y$ and $g: Y \to Z$ are fuzzy r - closed mapping, then is a fuzzy r – closed mapping.

Proposition 2.37. If (X, T) is a fuzzy topological space and A is a fuzzy r- closed subset of X, then the fuzzy inclusion $i_A: A \to X$ is a fuzzy r- closed mapping.

Proof : Let F be a fuzzy r - closed set in A. Since A is a fuzzy r- closed set in X and $i_A(F) = A \wedge F$, then $i_A(F)$ is a fuzzy r- closed set in X. Hence the inclusion mapping $i_A: A \to X$ is fuzzy r- closed.

Proposition 2.38. Let $f: X \to Y$ be a fuzzy r - closed mapping . If F is a fuzzy r-closed subset of X, then the restriction mapping $f_{|A}: F \to Y$ is a fuzzy r – closed mapping.

Proof : Since F is a fuzzy r- closed set in ***X*,** then by proposition (2.37), the inclusion mapping $i_F: F \to X$ is a fuzzy r- closed mapping, since f is a fuzzy r - closed, then by proposition (2.36), $f \circ i_F: F \to Y$ is a fuzzy r-closed mapping, but $f \circ i_F = f_{|F}$, thus the restriction mapping $f_{|A}: F \to Y$ is a fuzzy r - closed mapping.

Definition 2.39. [1] A fuzzy filter base on X is a nonempty subset $\mathcal{F}$ of I^X Such that

(i) $0_X \notin \mathcal{F}$

(ii) If $A_1, A_2 \in \mathcal{F}$,

Then $\exists A_3 \in \mathcal{F}$ such that $A_3 \leq A_1 \wedge A_2$.

Definition 2.40. A fuzzy point x_α in a fuzzy topological space X is said to be a fuzzy cluster point of a fuzzy filter base $\mathcal{F}$ on X if $x_\alpha \in \bar{B}$, for all $B \in \mathcal{F}$.

Definition 2. 41 . [1] A mapping $S: D \to FP(X)$ is called a fuzzy net in X and is denoted by $\{S(n): n \in D\}$, where D is a directed set. If $S(n) = x_{\alpha_n}^n$ for each $n \in D$ where $x \in X, n \in D$ and $\alpha_n \in (0,1]$ - then the fuzzy net S is denoted as $\{\mathrm{x}_{\alpha_\mathrm{n}}^\mathrm{n}, \mathrm{n} \in \mathrm{D}\}$ or simply $\{\mathrm{x}_{\alpha_\mathrm{n}}^\mathrm{n}\}$.

Definition 2.42. [1] A fuzzy net $\mathfrak{I} = \{y_{\alpha_\mathrm{m}}^m : m \in E\}$ in X is called a fuzzy subnet of fuzzy net $\{x_{\alpha_n}^n, n \in D\}$ if and only if there is a mapping $f: E \to D$ such that

(i) $\mathfrak{I} = S \circ f$, **that is,** $y_{\alpha_i}^i = x_{\alpha_{f(i)}}^{f(i)}$ **for each** $i \in E$.

(ii) For each $n \in D$ there exists some $m \in E$ such that $f(m) \geq n$.

We shall denote a fuzzy subnet of a fuzzy net
by $\{x_{\alpha_{f(m)}}^{f(m)}, m \in E\}$.$\{x_{\alpha_n}^n, n \in D\}$

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Definition 2.43. [1] Let (X, T) be a fuzzy topological space and let $S = \{x_{\alpha_n}^n, n \in D\}$ be a fuzzy net in X and $A \in I^X$. Then S is said to be:
(i) Eventually with A if and only if $\exists m \in D$ such that $x_{\alpha_n}^n qA, \forall n \geq m$.
(ii) Frequently with A if and only if $\forall n \in D, m \geq n$, and $x_{\alpha_m}^m qA$.

Definition 2.44. [1] Let (X,T) be a fuzzy topological space and $S = \{x_{\alpha_n}^n : n \in D\}$ be a fuzzy net in X and $x_\alpha \in FP(X)$. Then S is said to be:
(i) Convergent to x_α and denoted by $S \to x_\alpha$, if S is eventually with A, $\forall A \in N_{x_\alpha}^Q$
(ii) Has a cluster point x_α and denoted by $S \propto x_\alpha$, if S is frequently with A, $\forall A \in N_{x_\alpha}^Q$.

Proposition 2.45. A fuzzy point x_α is a cluster point of a fuzzy net $\{x_{\alpha_n}^n : n \in D\}$, where $(D, \geq)$ is a directed set, in a fuzzy topological space X if and only if it has a fuzzy subnet which converges to x_α.
Proof
⇒
Let $x_\alpha$be a cluster point of the fuzzy net $\{x_{\alpha_n}^n : n \in D\}$, with the directed set $(D, \geq)$ as the domain. Then for any $U \in N_{x_\alpha}^Q$, there exists $n \in D$ such that $x_{\alpha_n}^n qU$. Let $E = \{(n, U): n \in D, U \in N_{x_\alpha}^Q \text{ and } x_{\alpha_n}^n qU\}$. Then $(E, \geq)$ is directed set where $(m, U) \geq (n, V)$ if and only if $m \geq n$ in D and $U \leq V \text{ in } N_{x_\alpha}^Q$. Then $\mathfrak{J}: E \to FP(X)$ given by $\mathfrak{J}(m, U) = x_{\alpha_m}^m$is a fuzzy subnet of fuzzy net $\{x_{\alpha_n}^n : n \in D\}$. To show that $\mathfrak{J} \to x_\alpha$. Let $B \in N_{x_\alpha}^Q$. Then there exists $n \in D$ such that $(n, B) \in E$ and $x_{\alpha_n}^n qB$. Thus for any $(m, U) \in E$ such that $(n, U) \geq (n, B)$, we have $\mathfrak{J}(m, U) = x_{\alpha_m}^m qU \leq B$. Hence $\mathfrak{J} \to x_\alpha$.
If a fuzzy net $\{x_{\alpha_n}^n \in D\}$, has not a cluster point.⇐
Then for every fuzzy point x_α there is q-neighborhood of $x_\alpha$and $n \in D$ such that $x_{\alpha_m}^m \tilde{q} U$, for all $m \geq n$. Then obviously no fuzzy net converge to x_α.

Theorem 2.46. Let (X,T) be a fuzzy topological space , $x_\alpha \in FP(X)$ and $A \in I^X$.
Then $x_\alpha \in \bar{A}$ if and only if there exists a fuzzy net in A convergent to x_α.
Proof ⇒ **Let** $x_\alpha \in \bar{A}$**, then for every** $B \in N_{x_\alpha}^Q$ **there exists**

$$x_B(y) = \begin{cases} A(x_\alpha) & if \quad y = x_B \\ 0 & if \quad y \neq x_B \end{cases}$$

Such that $B(x_B) + A(x_B) > 1$ **notice that** $(N_{x_\alpha}^Q, \geq)$ **is a directed set, then** $S: N_{x_\alpha}^Q \to FP(X)$ is defined as $S(B) = x_B^A$ is a fuzzy net in A. To prove that $S \to x_\alpha$. Let $D \in N_{x_\alpha}^Q$. Then there exists $F \in T$ such that $x_\alpha qF$ and $F \leq D$.
Since $F(x_F^A) + x_F^A > 1$ and $F \leq D$. Then $D(x_F^A) + x_F^A > 1$. Thus $x_F^A qD$. Let $E \geq F$, then $E \leq F$. Since $E(x_E^A) + x_E^A > 1$ and $F \leq D$, then $D(x_F^A) + x_F^A > 1$. Thus $x_E^A qD$, $\forall E \geq F$
Therefore $S \to x_\alpha$.
Let $\{x_{\alpha_n}^n : n \in D\}$ be a fuzzy net in A where $(D, \geq)$ is a directed set such that $x_{\alpha_n}^n \to x_\alpha$. Then for ⇐ every $B \in N_{x_\alpha}^Q$, there exists $m \in D$ such that $x_{\alpha_n}^n qB$ for all $n \geq m$. Since $x_{\alpha_n}^n \in A$, then by proposition (2.11), $x_{\alpha_n}^n \tilde{q} A^c$. Thus AqB. Therefore $x_\alpha \in \bar{A}$.

Proposition 2.47. If is a fuzzy T_2-space, then convergent fuzzy net on X has a unique limit point.
Proof: Let $x_{\alpha_n}^n$ be a fuzzy net on X such that $x_{\alpha_n}^n \to x_\alpha$, $x_{\alpha_n}^n \to y_\beta$ and $x \neq y$. Since $x_{\alpha_n}^n \to x_\alpha$, We have $\forall A \in N_{x_\alpha}^Q, \exists\, m_1 \in D$, such that $x_{\alpha_n}^n qA, \forall\, n \geq m_1$. Also $x_{\alpha_n}^n \to y_\beta$, We have $\forall B \in N_{x_\beta}^Q, \exists\, m_2 \in D$, such that $x_{\alpha_n}^n qB, \forall\, n \geq m_2$. Since D is a directed set, then there exists $m \in D$, such that $m_1 \geq m$ and $m_2 \geq m$, then $x_{\alpha_n}^n q(A \wedge B)$, $\forall n \geq m$. Thus $A \wedge B \neq 0_X$, a contradiction.
Let X be a not fuzzy T_2 space, then there exists $x_\alpha, y_\beta \in FP(X)$ such that $x \neq y$ and $A \wedge B \neq 0_X$, ⇐ $\forall A \in N_{x_\alpha}^Q, B \in N_{y_\beta}^Q$. Put $N_{x_\alpha, y_\beta}^Q = \{A \wedge B / A \in N_{x_\alpha}^Q, B \in N_{y_\beta}^Q\}$. Thus $\forall D \in N_{x_\alpha, y_\beta}^Q$, there exists $x_D qD$, then $\{x_D\}_{D \in N_{x_\alpha, y_\beta}^Q}$ is a fuzzy net in X.To prove that $x_D \to x_\alpha$ and $x_D \to y_\beta$. Let $E \in N_{x_\alpha}^Q$, then $E \in N_{x_\alpha, y_\beta}^Q$ (since $E = E \wedge X$). Thus $x_D qD, \forall D \geq E$, thus $x_D \to x_\alpha$. Also $x_D \to y_\beta$, So $\{x_D\}_{D \in N_{x_\alpha, y_\beta}^Q}$ has two limit point.

3. Fuzzy compact space.

This section contains the definitions, proportions and theorems about fuzzy compact space and we give a new results

Definition 3.1. [3,6] A family Λ of fuzzy sets is called a cover of a fuzzy set A if and only if $\Lambda \leq \vee\{B_i : B_i \in \Lambda\}$ and Λ is called fuzzy open cover if each member B_i is a fuzzy open set. A sub cover of Λ is a subfamily of Λ which is also a cover of A.

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Definition 3.2. [3,6] Let (X, T) be a fuzzy topological space and let $A \in I^X$. Then A is said to be a fuzzy compact set if for every fuzzy open cover of A has a finite sub cover of A. Let $A = X$, then is called a fuzzy compact space that is $A_i \in T$ for every $i \in I$ and $V_{i \in I} A_i = 1_X$, then there are finitely many indices $i_1, i_2, \dots, i_n \in I$ such that $V_{j=1}^{n} A_{ij} = 1_X$.

Example 3.3. If (X, T) is a fuzzy topological space such that T is finite then X is fuzzy compact.

Remark 3.4 Not every fuzzy point set of a fuzzy space is X fuzzy compact in general. See the following example:

Example 3.5 Let $X = \{a\}$ be a set and $T = \{0_X, 1_X, a_{\frac{1}{2}-\frac{1}{n}} / n \in Z^+, n \geq 3\}$, where $a \in X$ be a fuzzy topology on X.

Notice that is a $\{a_{\frac{1}{2}-\frac{1}{n}} / n \geq 3\}$ fuzzy open cover of $a_{\frac{1}{2}}$, but its has no finite sub cover for $a_{\frac{1}{2}}$. Thus $a_{\frac{1}{2}}$ is not fuzzy compact.

Then we will give the following definition.

Definition 3.6 A fuzzy topological space (X, T) is called fuzzy singleton compact space (fuzzy sc – space) if every fuzzy point of X is fuzzy compact.

Example 3.7. Every fuzzy topological space with finite fuzzy topology is fuzzy sc – space.

Proposition 3.8. Let Y be a fuzzy subspace of a fuzzy topological space X and let $A \in I^Y$. Then A is fuzzy compact relative to X if and only if A is fuzzy compact relative to Y.

Proof $\Rightarrow$ Let A be a fuzzy compact relative to X and let $\{V_\lambda : \lambda \in \Lambda\}$ be a collection of fuzzy open sets relative to Y, which covers A so that $A \leq V_{\lambda \in \Lambda} V_\lambda$, then there exist G_λ fuzzy open relative to X, such that $V_\lambda = Y \wedge G_\lambda$ for any $\lambda \in \Lambda$. It then follows that $A \leq V_{\lambda \in \Lambda} G_\lambda$. So that $\{G_\lambda : \lambda \in \Lambda\}$ is fuzzy open cover of A relative to X. Since A is fuzzy compact relative to X, then there exists a finitely many indices $\lambda_1, \lambda_2, \dots, \lambda_n \in \Lambda$ such that $A \leq V_{j=1}^{n} G_{\lambda_i}$. Since $A \leq Y$, we have $A = Y \wedge A \leq Y \wedge \left(G_{\lambda_1} \vee G_{\lambda_2} \vee \dots \vee G_{\lambda_n}\right) = \left(Y \wedge G_{\lambda_1}\right) \dots (Y \wedge G_n)$, since $Y \wedge G_{\lambda_i} = V_{\lambda_i} (i = 1, 2, \dots n)$ we obtain $A \leq V_{j=1}^{n} G_{\lambda_i}$. Thus show that A is fuzzy compact relative to Y.

Let A be fuzzy compact relative to Y and let $\{G_\lambda : \lambda \in \Lambda\}$ be a collection of fuzzy open cover of X, so $\Leftarrow$ that $A \leq V_{\lambda \in \Lambda} G_\lambda$. Since $A \leq Y$, we have $A = Y \wedge A \leq Y \wedge (V_{\lambda \in \Lambda} G_\lambda) = V_{\lambda \in \Lambda}(Y \wedge G_\lambda)$. Since $Y \wedge G_\lambda$ is fuzzy open relative to Y, then the collection $\{Y \wedge G_\lambda : \lambda \in \Lambda\}$ is a fuzzy open cover relative to Y. Since A is fuzzy compact relative to Y, we must have $A \leq$
$(Y \wedge G_{\lambda_1}) \vee (Y \wedge G_{\lambda_2}) \vee \dots \vee (Y \wedge G_{\lambda_n})$……………………………….(*)
for some choice of finitely many indices $\lambda_1, \lambda_2, \dots, \lambda_n$. But (*) implies that $A \leq \bigvee_{i=1}^{n} G_{\lambda_i}$. It follows that A is fuzzy compact relative to X.

Theorem 3.9. [3] A fuzzy topological space (X, T) is fuzzy compact if and only if for every collection $\left\{A_j : j \in J\right\}$ of fuzzy closed sets of X having the finite intersection property $\bigwedge_{j \in J} A_j \neq 0_X$.

Proof $\Rightarrow$ Let $\left\{A_j : j \in J\right\}$ be a collection of fuzzy closed sets of X with the finite intersection property. Suppose that $\bigwedge_{j \in J} A_j \neq 0_X$, then $\bigvee_{j \in J} A_j^C = 1_X$. Since X is fuzzy compact, then there exists $j_1, j_2, \dots, j_n$ such that $A \leq \bigvee_{i=1}^{n} A_{ji}^C = 1_X$.

Then $\bigwedge_{i=1}^{n} A_{ji} = 0_X$. Which gives a contradiction and therefore $\bigwedge_{j \in J} A_j \neq 0_X$.

let $\left\{A_j : j \in J\right\}$ be a fuzzy open cover of X. Suppose that for every finite $j_1, j_2, \dots, j_n$, we have $\Leftarrow$ $\bigvee_{i=1}^{n} A_{ji} \neq 1_X$. then $\bigwedge_{i=1}^{n} A_{ji}^C \neq 0_X$. Hence $\left\{A_j^C : j \in J\right\}$ satisfies the finite intersection property . Then from the hypothesis we have $\bigwedge_{j \in J} A_j^C \neq 0_X$. Which implies $\bigvee_{j \in J} \mathrm{A}_j \neq 1_X$ and this contradicting that $\left\{\mathrm{A}_j : j \in J\right\}$ is a fuzzy open cover of X. Thus X is fuzzy compact.

Theorem 3.10. A fuzzy closed subset of a fuzzy compact space is fuzzy compact.

Proof : Let A be a fuzzy closed subset of a fuzzy space X and let $\{B_i : i \in I\}$ be any family of fuzzy closed in A with finite intersection property, since A is fuzzy closed in X, then by proposition (2. 18 . ii) , B_i are also fuzzy closed in X, since X is fuzzy compact, then by proposition (3 . 89) , $\bigwedge_{i \in I} B_i \neq 0_X$. Therefore A is fuzzy compact**.**

Theorem 3.11. A fuzzy topological space (X, T) is a fuzzy compact if and only if every fuzzy filter base on has a fuzzy cluster point**.**

Proof $\Rightarrow$ Let X be fuzzy compact and let $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a fuzzy filter base on X having no a fuzzy cluster point. Let $x \in X$. Corresponding to each $n \in N$ (N denoted the set of natural numbers), there exists a fuzzy q neigh bour hood U_x^n of the fuzzy point $x_{\frac{1}{n}}$ and an $F_x^n \in \mathcal{F}$ such that $U_x^n \tilde{q} F_x^n$**. Since** $1 - \frac{1}{n} < U_x^n(x)$, we have $U_x(x) = 1$, where $U_x = \bigvee\{U_{x_n} : n \in N\}$. Thus $\mathcal{U} = \{U_x^n : n \in N, x \in X\}$ is a fuzzy open cover of X. Since X is fuzzy

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compact, then there exists finitely many members $U_{x_1}^{n_1}, U_{x_2}^{n_2}, \dots, U_{x_k}^{n_k}$ of $\mathcal{U}$ **such that** $\vee_{i-1}^{k} U_{x_i}^{n_i} = 1_X$.
Since $\mathcal{F}$ is fuzzy filter base, then there exists $F \in \mathcal{F}$ such that $F \leq F_{x_{n1}} \wedge F_{x_{n2}} \wedge \dots \wedge F_{x_{nk}}$. But $U_{x_i}^{n_i} \tilde{q} F_{x_i}^{n_i}$, then $F \tilde{q} 1_X$.
Consequently, $F = 0_X$ and this contradicts the definition of a fuzzy filter base.
Let $\beta = \{F_\alpha : \alpha \in \Lambda\}$ be a family of fuzzy closed sets having finite intersection property. Then the $\Leftarrow$ set of finite intersections of members
of β forms a fuzzy filter base $\mathcal{F}$ on X. So by the condition $\mathcal{F}$ has a fuzzy cluster point say x_s.
Thus $x_s \in F_\alpha$. So $x_s \in \wedge_{\alpha \in \Lambda} \overline{F_\alpha} = \wedge_{\alpha \in \Lambda} F_\alpha$.
Thus $\wedge\{F, F \in \mathcal{F}\} \neq 0_X$. Hence by theorem (3 . 9) , X is fuzzy compact.
Theorem 3.12. A fuzzy topological space (X, T) is fuzzy compact if and only if every fuzzy net in has a cluster point.
Proof $\Rightarrow$ Let X be fuzzy compact. Let $\{S(n) : n \in D\}$ be a fuzzy net in X which has no cluster point, then for each fuzzy point x_α, there is a fuzzy q– neigh bour hood $U_{x_\alpha}$of x_α and an $x_{U_{x_\alpha}} \in D$ such that $S_m \tilde{q} U_{x_\alpha}$ for all $m \in D$ with $m \geq n_{U_{x_\alpha}}$. Since $x_\alpha q U_{x_\alpha}$, then $S_m \neq 0$, $\forall\, m \geq n_{U_{x_\alpha}}$. Let $\mathcal{U}$ denoted the collection of all U_{x_α}, where x_α runs over all fuzzy points in X. Now to prove that the collection $V = \{1_X - U_{x_\alpha} : U_{x_\alpha} \in \mathcal{U}\}$ is a family of fuzzy closed sets in X possessing finite intersection property. First notice that there exists $k \geq n_{U_{\alpha_1}}, \dots, n_{U_{\alpha m}}$ such that $S_p \tilde{q} U_{x_{\alpha_i}}$ for i=1,2,….., m and for all $p \geq k$ ($p \in D$), **i.e.** $S_P \in 1_X - \vee_{i=1}^{m} U_{x_{\alpha_i}} = \wedge_{i=1}^{m} (1_X - U_{x_{\alpha_i}})$ for all $p \geq k$. Hence $\wedge\{1_X - U_{x_{\alpha_i}} : i = 1,2, \dots, m\} \neq 0_X$. Since X is fuzzy compact, by theorem (3 . 9), there exists a fuzzy point y_β in X such that $y_\beta \in \wedge\{1_X - U_{x_\alpha} : U_{x_\alpha} \in \mathcal{U}\} = 1_X \vee \{U_{x_\alpha} : U_{x_\alpha} \in \mathcal{U}\}$. Thus $y_\beta \in 1_X - U_{x_\alpha}$, for all $U_{x_\alpha} \in \mathcal{U}$ and hence in particular, $y_\beta \in 1_X - U_{y_\beta}$, i.e., $y_\beta \tilde{q} U_{y_\beta}$. But
by construction, for each fuzzy point x_α, there exists $U_{x_\alpha} \in \mathcal{U}$ Such that $x_\alpha q U_{x_\alpha}$, and we arrive at a contradiction.
To prove that converse by theorem (3 . 11), that every fuzzy filter base on X has a cluster point. Let $\Leftarrow$ $\mathcal{F}$ be a fuzzy filter base on X. Then each $F \in \mathcal{F}$ is non empty set, we choose a fuzzy point $x_F \in F$. Let $S = \{x_F : F \in \mathcal{F}\}$ and let a relation " $\geq$ " be defined in $\mathcal{F}$ as follows $F_\alpha \geq F_\beta$ if and only if $F_\alpha \geq F_\beta$ in X, for $F_\alpha, F_\beta \in \mathcal{F}$. Then $(\mathcal{F}, \geq)$ is directed set. Now S is a fuzzy net with the directed set $(\mathcal{F}, \geq)$. By hypothesis the fuzzy net S has a cluster point x_t. Then for every fuzzy q – neighbourhood W of x_t and for each $F \in \mathcal{F}$, there exists $G \in \mathcal{F}$ with $G \geq F$ such that $x_G q W$. As $x_G \leq G \leq F$. It
follows that $F q W$ for each $F \in \mathcal{F}$, then by proposition (2 . 28) , $x_t \in \bar{F}$. Hence x_t is a cluster point of $\mathcal{F}$.
Corollary 3.13. A fuzzy topological space (X, T) is fuzzy compact if and only if every fuzzy net in has a convergent fuzzy subnet.
Proof : By proposition (2.45), and theorem (3.12).
Theorem 3.14. Every fuzzy compact subset of a fuzzy Hausdroff topological space is fuzzy closed.
Proof: Let $x_\alpha \in \bar{A}$, then by theorem (2. 46), there exists fuzzy net $x_{\alpha_n}^{n}$ such that $x_{\alpha_n}^{n} \to x_\alpha$. Since is fuzzy compact and X is fuzzy T_2 space, then by corollary (3. 13) and proposition (2 .47), we have $x_\alpha \in A$. Hence A is fuzzy closed set.
Theorem 3.15. In any fuzzy space, the intersection of a fuzzy compact set with a fuzzy closed set is fuzzy compact.
Proof Let A be a fuzzy compact set and B be a fuzzy closed set. To prove that $A \wedge B$ is a fuzzy compact set. Let $x_{\alpha_n}^{n}$ be a fuzzy net in $A \wedge B$.
Then $x_{\alpha_n}^{n}$ is fuzzy net in A, since A is fuzzy compact, then by corollary (3.13), $x_{\alpha_n}^{n} \to x_\alpha$ for some $x_\alpha \in FP(X)$ and by proposition (2.22), $x_\alpha \in \bar{B}$. Since B is fuzzy closed, then. Hence $x_\alpha \in A \wedge B$ and $x_{\alpha_n}^{n} \to x_\alpha$.
Thus $A \wedge B$ is fuzzy compact.
Proposition 3.16. Let X and Y be fuzzy spaces and $f : X \to Y$ be a fuzzy continuous mapping. If U is a fuzzy compact set in X, then $f(U)$ is a fuzzy compact set in Y.
Proof : Let $\{V_i : i \in I\}$ be a fuzzy open cover of $f(U)$ in Y, i.e., $(f(U) \leq \vee_{i \in I} G_i)$. Since f is a fuzzy continuous, then $f^{-1}(G_i)$ is a fuzzy open
set in X, $\forall i \in I$. Hence the collection $\{f^{-1}(G_i) : i \in I\}$ be a fuzzy open cover of U in X, i.e., $U \leq f^{-1}\big(f(U)\big) \leq f^{-1}(\vee_{i \in I} G_i) = \vee_{i \in I} f^{-1}(G_i)$

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Since U is a fuzzy compact set in X, then there exists finitely many indices $i_1, i_2, \dots \dots, i_n$ Such that $U \leq \bigvee_{j=1}^{n} f^{-1}(\mathrm{G}_{i_j})$, so that $f(U) \leq f(\bigvee_{j=1}^{n}(f^{-1}(\mathrm{G}_{i_j})) = \bigvee_{j=1}^{n}(f(f^{-1}(\mathrm{G}_{i_j}))) \leq \bigvee_{j=1}^{n}\mathrm{G}_{i_j}$. Hence $f(U)$ is a fuzzy compact

4. Compactly fuzzy closed set

The section will contain the definition of compactly fuzzy closed set and we give new results.

Definition 4.1. Let X be a fuzzy space. Then a fuzzy subset W of X is called compactly fuzzy closed set if $W \wedge K$ is fuzzy compact, for every fuzzy compact set K in X.

Example 4.2. Every fuzzy subset of indiscrete fuzzy topological space is compactly fuzzy closed set**.**

Proposition 4.3. Every fuzzy closed subset of a fuzzy space X is compactly fuzzy closed

Proof Let A be a fuzzy closed subset of a fuzzy space X and let K be a fuzzy compact set. Then by theorem (3.15)**,** $A \wedge K$ is a fuzzy compact. Thus A is a compactly fuzzy closed set .

The converse of proposition (4.3), is not true in general as the following example show :

Example 4.4. Let $X = \{a, b\}$ be a set and T be the indiscrete fuzzy space on X. Notice that $A = \{0.2, 0.3\}$ is compactly fuzzy closed set, but its not fuzzy closed set.

Theorem 4.5. Let X be a fuzzy T_2-space. A fuzzy subset A of X is compactly fuzzy closed if and only if A is fuzzy closed.

Proof ⇒ Let A be a compactly fuzzy closed set in X and $x_\alpha \in \bar{A}$. Then by proposition (2.46), there exists a fuzzy net $x_{\alpha_n}^{n}$ in A, such that $x_{\alpha_n}^{n} \to x_\alpha$, then by corollary (3.13), $F = \{x_{\alpha_n}^{n}, x_\alpha\}$ is a fuzzy compact set. Since A is
compactly fuzzy closed, then $A \wedge F$ is a fuzzy compact set. But X is a fuzzy T_2-space, then by theorem (3.14), $A \wedge F$ is fuzzy closed. Since $x_{\alpha_n}^{n} \to x_\alpha$ and $x_{\alpha_n}^{n} \in A \wedge F$, then by proposition (2.46), $x_\alpha \in A \wedge F$ so $x_\alpha \in A$.

Hence $\bar{A} \leq A$. Therefore is a fuzzy closed set.

By proposition (4.3).⇐

5. Fuzzy regular compact space

This section contains the definitions, proportions about fuzzy regular compact space and we give a new results .

Definition 5.1. Let (X, T) be a fuzzy space. A family δ of fuzzy subset of X is called a fuzzy r- open cover of X if δ covers X and δ is subfamily of T^r.

Definition 5.2. A fuzzy space X is calledfuzzy r - compact if every fuzzy r- open of cover has a finite sub cover .

Example 5.3. The indiscrete fuzzy topological space is a fuzzy r - compact.

Proposition 5.4. Every fuzzy compact space is a fuzzy r - compact space.

The converse of proposition (5.4), is not true in general as the following example shows**:**

Example 5.5. Let $X = \{a, b\}$ and $T = \{0_x, 1_x, f_n\}$ where $f_n : X \to [0,1]$ such that $f_n(x) = 1 - \frac{1}{n}, \forall x \in X, n = 1.2.3, \dots \dots \dots$

Notice that the fuzzy topological space (X, T) is fuzzy r - compact, but its not fuzzy compact.

Remark 5.6. The fuzzy space (X, T) is fuzzy r- compact if and only if the fuzzy space (X, T^r) is fuzzy compact**.**

Proposition 5.7. Every fuzzy r- closed subset of a fuzzy r- compact space is fuzzy r - compact .

Proof : By remark (5.6), and theorem (3.10).

Remarks 5.8

(i) Every fuzzy r - closed subset of a fuzzy compact space is fuzzy r- compact.

(ii) Every fuzzy r- compact subset of a fuzzy T_2 space is fuzzy r –closed**.**

Proposition 5.9. Let X be a fuzzy compact set of a fuzzy T_2-space and $A \in I^X$. Then:

(i) is fuzzy closed if and only if A is fuzzy r-closed**.**

(ii) A is fuzzy compact if and only if A is fuzzy r- compact.

Proof: (i) Let A be a fuzzy closed set in X. Since X is fuzzy compact, then by theorem (3.10), A is a fuzzy compact set, so its fuzzy r-compact. Since X is a fuzzy T_2-space, then by remark (5 . 8 . ii) , A is a fuzzy r-closed set**.**

By remark (2 . 24)⇐

(iii) ⇒ By proposition (5 . 4).

Let be a fuzzy r- compact set in X. Since X is a fuzzy T_2-space, then by remark (5 . 8 .ii), A is fuzzy ⇐ r- closed in X, and then its fuzzy closed set.

Since X is a fuzzy compact space, then by theorem (3.11), A is a fuzzy compact set in X**.**

Proposition 5.10 Let X be a fuzzy space and Y be a fuzzy regular open sub space of X, $K \leq Y$. Then K is a fuzzy regular compact set in Y if and only if K is a fuzzy regular compact set in X**.**

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Proof: ⇒ Let K be a fuzzy regular compact set in Y. To prove that K is a fuzzy regular compact set in X. Let $\{U_\lambda : \lambda \in \Delta\}$ be a fuzzy regular open cover in X of K, let $V_\lambda = U_\lambda \wedge Y, \forall \lambda \in \Delta$. Then V_λ is fuzzy regular open in X, $\forall \lambda \in \Delta$.. But $V_\lambda \leq Y$, thus V_λ is fuzzy regular open in Y, $\forall \lambda \in \Delta$. Since $K \leq \vee_{\lambda \in \Delta} V_\lambda$, then $\{V_\lambda : \lambda \in \Delta\}$ is a fuzzy regular open cover in Y of K, and by hypothesis this cover has finite sub cover $\{V_{\lambda_1}, V_{\lambda_2}, \dots, V_{\lambda_N}\}$ of K, thus the cover $\{U_\lambda : \lambda \in \Delta\}$ has a finite sub cover of K. Hence K is a fuzzy regular compact set in X.

Let K be a fuzzy regular compact set in X. To prove that K is a fuzzy regular compact set in Y. Let ⇐ $\{U_\lambda : \lambda \in \Delta\}$ be a fuzzy regular open cover in Y of K. Since Y is a fuzzy regular open subspace of X, then by proposition (2.22.i), $\{U_\lambda : \lambda \in \Delta\}$ is a fuzzy regular open cover in X of K. Then by hypothesis there exists $\{\lambda_1, \lambda_2, \dots \dots, \lambda_m\}$, such that $K \leq \vee_{\lambda=1}^{n} U_\lambda$, thus the cover $\{U_\lambda : \lambda \in \Delta\}$ has a finite sub cover of . Hence K is a fuzzy r – compact set in Y.

Proposition 5.11 Let $f: X \to Y$ be a fuzzy regular irresolute mapping. If A is a fuzzy regular compact set in X, then $f(A)$ is a fuzzy regular compact set in Y.

Proof: Let $\{G_i : i \in I\}$ be a fuzzy regular open of $f(A)$ in Y(i.e., $f(A) \leq \vee_{i \in I} G_i$). Since f is fuzzy regular irresolute, then $f^{-1}(G_i)$ is fuzzy regular open set in X, $\forall i \in I$. Hence the collection $\{f^{-1}(G_i) : i \in I\}$ be a fuzzy regular open cover of A in X. i.e., $\leq f^{-1}\big(f(A)\big) \leq f^{-1}(\vee_{i \in I} G_i) = \vee_{i \in I} f^{-1}(G_i)$, since A is fuzzy regular compact set in X, there exists finitely many indices $i_1, i_2, \dots, i_n$

Such that $A \leq \vee_{j=1}^{n} f^{-1}(G_{i_j})$, so that $f(A) \leq f(\vee_{j=1}^{n} f^{-1}(G_{i_j})) = \vee_{j=1}^{n} f(f^{-1}(G_{i_j})) \leq \vee_{j=1}^{n} G_{i_j}$. Hence $f(A)$ is a fuzzy regular compact set.

6- Fuzzy regular compact mapping.

The section will contain the concept of fuzzy regular compact mapping and we give new results.

Definition 6.1. Let X and Y be fuzzy spaces. A mapping $f: X \to Y$ is called a fuzzy r-compact mapping if the inverse image of each fuzzy r- compact set in Y, is a fuzzy compact set in X.

Example 6.2. Let $X = \{a\}$, $Y = \{b\}$ be sets and $T = \{0_X, 1_X, b_{1-\frac{1}{n}} / \, n \geq 2, n \in Z^+\}$ be fuzzy topology on X and Y respectively.

Let $f: X \to Y$ be a mapping which is defined by: $f(a) = b$. Notice that is fuzzy r- compact

Proposition 6.3. If $f: X \to Y$ is a fuzzy r-compact, fuzzy continuous, mapping and A is a fuzzy clopen subset of Y, then $f_A: f^{-1}(A) \to A$ is a fuzzy r- compact mapping.

Proof: Let K be a fuzzy r- compact subset of A. Since A is a fuzzy open set in Y, then by corollary (2.27), A is a fuzzy r- open, and by proposition (4.10), K is a fuzzy r- compact set in Y. Since f is a fuzzy r- compact mapping, then $f^{-1}(K)$ is fuzzy compact in X. Now, since A is a fuzzy closed set in Y, and f is a fuzzy continuous mapping, then $f^{-1}(A)$ is a fuzzy closed set in X, thus by theorem (3.15), $f^{-1}(A) \wedge f^{-1}(K)$ is a fuzzy compact set. But $f_A^{-1}(K) = f^{-1}(A) \wedge f^{-1}(K)$, then $f_A^{-1}(K)$ is a fuzzy compact set in $f^{-1}(A)$. Therefore f_A is a fuzzy r - compact mapping.

Proposition 6.4. If $f: X \to Y$, $g: Y \to Z$ are fuzzy continuous mapping. Then:

(i) If f and g are fuzzy r- compact mappings, then is a fuzzy r- compact mapping .

(ii) If $g \circ f$ is a fuzzy r- compact mapping and f is onto, then g is a fuzzy r- compact mapping.

(iii) If $g \circ f$ is a fuzzy r- compact mapping and g is fuzzy r - irresolute and one to one, then f is a fuzzy r- compact mapping.

Proof:

(i) Let A be a fuzzy r- compact set in Z, then $g^{-1}(A)$ is a fuzzy compact set in Y, hence $g^{-1}(A)$ is a fuzzy r- compact set and then $f^{-1}\big(g^{-1}(A)\big) = (g \circ f)^{-1}(A)$ is a fuzzy compact set in X. Hence $g \circ f: X \to Z$ is a fuzzy r- compact mapping.

(ii) Let A be a fuzzy r- compact set in Z, then $(g \circ f)^{-1}(A)$ is a fuzzy compact set in X, and so $f(g \circ f)^{-1}(A)$ is a fuzzy compact set in Y.

Now, since f is onto, then $f(g \circ f)^{-1}(A) = (g)^{-1}(A)$, therefore g is a fuzzy r- compact mapping.

(iii) Let A be a fuzzy r- compact set in Y. Since g is a fuzzy r- irresolute mapping, then $g(A)$ is a fuzzy r- compact set in Z. Since $g \circ f$ is a fuzzy r-compact mapping, then $(g \circ f)^{-1}(g(A))$ is a fuzzy compact set in X. Since g is one to one, then $(g \circ f)^{-1}\big(g(A)\big) = f^{-1}(A)$. Hence $f^{-1}(A)$ is a fuzzy compact set in X. Then f is a fuzzy r- compact mapping.

Proposition 6.5. For any fuzzy r- closed subset F of a fuzzy space X, the inclusion $i_F: F \to X$ is a fuzzy r- compact mapping.

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Proposition 6.6. If $f: X \to Y$ is a fuzzy r-compact mapping and F is a fuzzy r- closed subset of X, then $f_{|F}: F \to Y$ is a fuzzy r-compact mapping.

Proof : Since F is a fuzzy r- closed subset of X, then by proposition (6.5), the inclusion $i_F: F \to X$ is a fuzzy r- compact mapping. But $f_{|F}: f \circ i_F$, then by proposition (6.4. i), $f_{|F}$ is a fuzzy r- compact mapping.

7- Fuzzy regular coercive mapping .

The section will contain the definition of a fuzzy regular coercive mapping and the relation between fuzzy regular compact mapping and the fuzzy regular coercive mapping.

Definition 7.1. Let X and Y be fuzzy spaces. We say that a mapping $f: X \to Y$ is **fuzzy r-coercive** if for every fuzzy r- compact set $G \leq Y$, there exists a fuzzy compact set $K \leq X$ such that $f(1_X \backslash K) \leq (1_Y \backslash G)$.

Example 7.2 If $f: (X, T) \to (Y, \tau)$ is a mapping, such that $X = \{a\}$, $T = \{0_X, 1_X, b_{\frac{1}{2}-\frac{1}{n}} / \, n \geq 3, n \in Z^+\}$ is a fuzzy compact space and τ any fuzzy topology on Y, then f is a fuzzy r-coercive.

Proposition 7.3 If $f: X \to Y$ and $g: Y \to Z$ are fuzzy r- coercive mappings, then $g \circ f$ is a fuzzy r - coercive mapping .

Proof : Let G be a fuzzy r- compact set in Z. Since g is a fuzzy r- coercive mapping, then there exists a fuzzy compact set K in Y, such that $g(1_Y \backslash K) \leq (1_Z \backslash G)$, then by proposition (5.4), K is fuzzy r- compact in Y. Since $f: X \to Y$ is a fuzzy r- coercive mapping.

Then there exists a fuzzy compact set H in X, such that

. Hence $g \circ f(1_X \backslash H) \leq (1_Y \backslash K) \to g\big(f(1_X \backslash H)\big) \leq g(1_Y \backslash K) \leq (1_Z \backslash G) \to (g \circ f)(1_X \backslash H) \leq (1_Z \backslash G)$ f is a fuzzy r- coercive mapping.

Proposition 7.4 Every fuzzy r- compact mapping is fuzzy r- coercive .

The converse of proposition (7.4), is not true in general as the following example shows:

Example 7.5. Let $X = \{a\}$, $Y = \{b\}$ be sets and $T = \{0_X, 1_X, a_{\frac{1}{2}-\frac{1}{n}} / \, n \geq 3, n \in Z^+\}$, $\tau = \{0_X, b_{\frac{1}{2}}, 1_X\}$ be fuzzy topology on X and Y respectively.

Let $f: X \to Y$ be a mapping which is defined by : $f(a) = b$. Notice that f is a fuzzy r-coercive mapping, but its not fuzzy r- compact mapping.

Proposition 7.6. Let X and Y be fuzzy spaces, such that Y is a fuzzy T_2 - space, and $f: X \to Y$ is a fuzzy continuous mapping. Then f is fuzzy r- coercive if and only if f is fuzzy r-compact.

Proof:⇒ Let G be a fuzzy r- compact set in Y. To prove that $f^{-1}(G)$ is a fuzzy r-compact set in X. Since Y is a fuzzy T_2 – space, then by remark (5.8 . ii), G is a fuzzy r-closed set in Y, so it's a fuzzy closed set . Since f

is a fuzzy continuous mapping, then $f^{-1}(G)$ is a fuzzy closed set in X. Since f is a fuzzy r-coercive mapping, then there exists a fuzzy compact set K in X, such that $f(1_X \backslash K) \leq (1_Y \backslash G)$. Then $f(K^C) \leq G^C$, therefore $f^{-1}(G) \leq K$. Thus $f^{-1}(G)$ is a fuzzy compact set in X. Hence f is a fuzzy r- compact mapping.

By proposition (5.10).⇐

Corollary 7.7 . If $f: X \to Y$ is fuzzy r-compact and g $g: X \to Z$ is fuzzy r-coercive, then $g \circ f: X \to Z$ is a fuzzy r- coercive mapping.

Proposition 7.8. Let X and Y be fuzzy spaces and $f: X \to Y$ be a fuzzy r- coercive mapping. If F is a fuzzy r- closed subset of X, then the restriction mapping $f_{|F}: F \to Y$ is a fuzzy r- coercive mapping.

Proof: Since F is a fuzzy r- closed subset of X, then by proposition (6.5), the inclusion mapping $i_F: F \to X$ is a fuzzy r- compact mapping. But $f_{|F} = f \circ i_F$, then by corollary (7.7), $f_{|F}$ is a fuzzy r- coercive mapping.

8 – Fuzzy regular proper mapping .

The section will contain the definition of fuzzy regular proper mapping and addition to studying relation among fuzzy regular proper mapping, fuzzy regular compact mapping and fuzzy regular coercive mapping.

Definition 8.1. A fuzzy continuous mapping $f: X \to Y$ is called fuzzy r - proper if

(i) $(f)$is fuzzy r – closed.

(ii) $f^{-1}(y_\alpha)$ is fuzzy compact, for all $y_\alpha \in FP(Y)$.

Corollary 8.2. If $f: X \to Y$ and $g: Y \to Z$ are fuzzy r - proper mappings, then $g \circ f: X \to Z$ is a fuzzy r- proper mapping .

Proof : Clearly.

Proposition 8.3. Let X, Y and Z be fuzzy spaces and $f: X \to Y$, $g: Y \to Z$ be fuzzy continuous mappings, such that $g \circ f: X \to Z$ is a fuzzy r - proper mapping. If f is onto, then g is fuzzy r – proper.

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Proof: (i) Let F be a fuzzy closed subset of Y, since f is fuzzy continuous, then $f^{-1}(F)$ is fuzzy closed in X. Since $g \circ f$ is a fuzzy r - proper mapping, then $(g \circ f)(f^{-1}(F))$ is fuzzy r - closed in Z. But f is onto, then $(g \circ f)\left(f^{-1}(F)\right) = g(F)$. Hence $g(F)$. is fuzzy r - closed in Z. Thus g is a fuzzy r – closed mapping.

(ii)Let $Z_\alpha \in FP(Z)$. Since $g \circ f$ is a fuzzy r -proper mapping, then $(g \circ f)^{-1}(Z_\alpha) = f^{-1}(g^{-1}(Z_\alpha))$ is fuzzy compact. But f is fuzzy continuous, then $f(f^{-1}(g^{-1}(Z_\alpha)))$ is a fuzzy compact set. Since f is onto, then $f\left(f^{-1}\left(g^{-1}(Z_\alpha)\right)\right) = g^{-1}(Z_\alpha)$ is fuzzy compact. Thus g is fuzzy r -proper.

Proposition 8.4. Let X, Y and Z be fuzzy spaces and $f: X \to Y$, $g: Y \to Z$ be fuzzy continuous mappings, such that $g \circ f: X \to Z$ is a fuzzy r - proper mapping . If g is one to one, then f is fuzzy r – proper.

Proof: (i) Let F be a fuzzy closed subset of X. Then $(g \circ f)(F)$ is a fuzzy r-closed set in Z. Since $g: Y \to Z$ is a one to one, fuzzy r-irresolute, mapping, then $g^{-1}\left(g\left(f(F)\right)\right) = f(F)$ is fuzzy r - closed in Y. Hence $X: Y \to Y$ is fuzzy r – closed.

(ii)Let $y_\alpha \in FP(Y)$, then $g(y_\alpha) \in Z$. Now, since $g \circ f: X \to Z$ is fuzzy r - proper and g is one to one, then $(g \circ f)^{-1}\left(g(y_\alpha)\right) = f^{-1}\left(g^{-1}\left(g(y_\alpha)\right)\right) = f^{-1}(y_\alpha)$ is fuzzy compact. Therefore the mapping f is fuzzy r – proper.

Proposition 8.5. If $f: X \to Y$ is a fuzzy r - proper mapping, and A is a fuzzy r - closed set in X, then the restriction mapping $f_{|F}: A \to Y$ is a fuzzy r - proper mapping.

Proof (i) Since is fuzzy r - closed , thus by proposition (2.38), $f_{|A}$ is a fuzzy r – closed mapping.

(ii) Let $y_\alpha \in FP(Y)$. Since f is a fuzzy r -proper mapping, then $f^{-1}(y_\alpha)$ is a fuzzy compact set in X. Since A is a fuzzy closed set in X, then by theorem (3.15), $A \wedge f^{-1}(y_\alpha)$is a fuzzy compact set. But $(f_{|A})^{-1}(y_\alpha) = A \wedge f^{-1}(y_\alpha)$, then $(f_{|A})^{-1}(y_\alpha)$ is a fuzzy compact set in A. Since by proposition (2.33), $f_{|A}$ is fuzzy continuous, thus $f_{|A}: A \to Y$ is a fuzzy proper mapping.

Proposition 8.6. If $f: X \to Y$ is a fuzzy r -proper mapping, then f is a fuzzy r- compact mapping.

Proof: Let K be a fuzzy r - compact subset of Y and let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a fuzzy open cover of $f^{-1}(K)$. Since f is a fuzzy r - proper mapping, then $f^{-1}(k_\alpha)$ is a fuzzy compact set, $\forall\, k_\alpha \in K$. But $f^{-1}(k_\alpha) \leq f^{-1}(K) \leq \bigvee_{\lambda\in\Lambda} U_\lambda$, thus there exists n_k, such that $f^{-1}(k_\alpha) \leq \bigvee_{\lambda=1}^{n_k} U_\lambda$. Let $U_{n_k} = \bigvee_{\lambda=1}^{n_k} U_\lambda$ Thus, for all $k_\alpha \in K$, there exists n_k such that $f^{-1}(k_\alpha) \leq U_{n_k}$. Notice that for all $k_\alpha \in K$, $k_\alpha \leq \left(1_Y \backslash f\left(1_X \backslash U_{n_k}\right)\right) \to K \leq \bigvee_{k_\alpha\in K}\left(1_Y \backslash f\left(1_X \backslash U_{n_k}\right)\right)$, but the sets $\left(1_Y \backslash f\left(1_X \backslash U_{n_k}\right)\right)$ are fuzzy open sets. Then by remark (5.6) , K is fuzzy compact. So there exists $n_{1k}, n_{2k}, \dots \dots, n_{jk}$, such that $K \leq \bigvee_{\lambda=1}^{j}\left(1_Y \backslash f\left(1_X \backslash U_{n_{\lambda k}}\right)\right) \to f^{-1}(K) \leq \bigvee_{\lambda=1}^{j} U_{n_{\lambda k}}$. Therefore $f^{-1}(K)$ is a fuzzy compact set in X. Hence the mapping $f: X \to Y$ is a fuzzy r - compact mapping.

Proposition 8.7. Let X and Y be fuzzy spaces, such that Y is a fuzzy T_2-space, fuzzy compact and fuzzy sc - space . If $f: X \to Y$ is a fuzzy continuous mapping, then f is fuzzy r - proper if and only if f is fuzzy r-compact.

Proof: $\Rightarrow$ By proposition (8.6).

To prove that f is a fuzzy r- proper mapping$\Leftarrow$

(i) Let F be a fuzzy closed subset of X. To prove that $f(F)$ is a fuzzy r – closed set in Y, let K be a fuzzy compact set in Y. Then $f^{-1}(K)$ is a fuzzy compact set in X, then by theorem (2 . 1 . 22), $F \wedge f^{-1}(K)$is a fuzzy compact set in X. Since f is fuzzy continuous, then $f(F \wedge f^{-1}(K))$ is a fuzzy compact set in Y. But $f(F \wedge f^{-1}(K)) = f(F) \wedge K$, then $f(F) \wedge K$ is fuzzy compact, thus $f(F)$ is a compactly fuzzy closed set in Y. Since Y is a fuzzy T_2-space, then by theorem (4.5), $f(F)$ is a fuzzy closed set in Y. Hence by proposition (5.9), f is a fuzzy r – closed mapping.

(ii) Let $y_\alpha \in FP(Y)$, since Y is a fuzzy sc -space, then y_α is fuzzy compact in Y. Then its fuzzy r - compact. Since f is a fuzzy r - compact mapping , then $f^{-1}(y_\alpha)$ is fuzzy compact in X. Thus f is fuzzy r – proper mapping.

Proposition 8.8. Let X and Y be fuzzy spaces, such that Y is a fuzzy compact space, fuzzy sc - space and fuzzy T_2-space and $f: X \to Y$is a fuzzy continuous mapping. Then the following statements are equivalent:

(i) f is a fuzzy r - coercive mapping .

(ii) f is a fuzzy r - compact mapping .

is a fuzzy r - proper mapping.f (iii)

(iv)

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Proof :

By proposition (7.6).$(i) \rightarrow (ii)$

By proposition (8.7).$(ii) \rightarrow (iii)$

Let G be a fuzzy r - compact set in Y. Since f is fuzzy r - proper, then by proposition (8.6), $(iii) \rightarrow (i)$ f is a fuzzy r – compact mapping, then $f^{-1}(G)$ is a fuzzy compact set in X. But $f\left(1_X \backslash f^{-1}(G)\right) \leq (1_Y \backslash G)$. Then $f: X \rightarrow Y$ is a fuzzy r - coercive mapping.

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