Orthogonal Semi derivations on Semi prime Γ-Semi rings

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Abstract

In this paper, we introduce the notion of orthogonal semi derivations on Γ-semi rings. Some characterizations of semiprime Γ-semirings are obtained by means of orthogonal semi derivations and obtained necessary and sufficient conditions for two semi derivations to be orthogonal.

Keywords: Γ-Semi rings; Γ-Semi derivation; prime Γ-semi rings; Semi prime Γ-semi rings; Orthogonal semi derivation

1. Introduction

The notion of Γ-ring was introduced by Nobusawa [1] as a generalization of ring in 1964 . Sen [2] introduced the notion of Γ-semigroup in 1981. The notion of Ternary algebraic system was introduced by Lehmer [3], Lister [4] introduced ternary ring. Dutta & Kar [5] introduced the notion of ternary semiring which is a generalization of ternary ring and semiring. In 1995, Murali Krishna Rao [6,7] introduced the notion of Γ-semiring which is a generalization of Γ-ring, ring, ternary semiring and semiring. Bergen [8] introduced the notion of semiderivations in prime rings in the year 1983. Bresar and Vukman [9] introduced the notion of reverse derivations in rings in 1989. Bresar and Vukman [10] obtained some results concerning orthogonal derivations in semiprime rings which are related to the result that is well known to a theorem of Posner [11] for the product of two derivations in prime rings. Chang [12] studied on semi derivations of prime rings. He obtained some results on derivations of prime rings into semiderivations. Chuang [13] studied on the structure of semiderivations in prime rings. He obtained some remarkable results in connection with the semiderivatons. Firat [14] generalized some results of prime rings with derivations to the prime rings with semiderivations. Ali and Khan [15] obtained some results on orthogonal (σ, τ)derivations on semiprime gamma rings. Hamil [16] studied on Orthogonal generalized derivations on Γ – semirings. Rasheed et al., [17] studied on generalized (α, β) derivation on primeSemirings. Several authors have also addressed this topic (see, e.g., [18,19]). Kalyan Kumar Dey, Akhil Chandra Paul, Isamiddin S. Rakhimov [20] studied on semiprime gamma rings with orthogonal reverse derivations. K.Kanak Sindhu , Murugesan and Namasivayam [21] studied on semiprime Semirings with orthogonal semiderivations. B. Venkateswarlu, M. Murali Krishna Rao and Adinarayana [22] studied on Γ−semirings with orthogonal derivation .Motivated by all the above notions, in this paper we extend the results to semiprime Γ −semirings. The notion of orthogonality of two semiderivations is given and conditions of two semiderivations to be orthogonal are provided. We also obtain some characterizations of semiprime Γ −semirings with orthogonal semiderivation. Throughout this paper M will represent a 2-torsion free semiprime Γ −semiring.

2. Preliminaries

In this section, we recall some important definitions which are necessary for this paper.

Definition 2.1

A set M together with two associative binary operations called addition and multiplication (denoted by + and . respectively) will be called a semiring provided

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(i) Addition is a commutative operation.

(ii) Multiplication distributes over addition both from the left and from the right.

(iii) there exists $0 \in M$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in M$.

Definition 2.2

Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. Then we call M as a Γ-semiring, if there exists a mapping $M \times \Gamma \times M \to M$ is written (x, α, y) as $x\alpha y$ such that it satisfying the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$
(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$
(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Every semiring M is a Γ-semiring with $\Gamma = M$ and ternary operation $x\gamma y$ as the usual semiring multiplication.

Definition 2.3
A Γ-semiring M is called a prime if $a\Gamma M\Gamma b = (0)$ implies a =0 or b= 0 where a, b∈ M.

Definition 2.4
A Γ-semiring M is called a semi prime if $a\Gamma M\Gamma a = (0)$ implies a = 0 where a ∊ M.

Definition 2.5
Let M be a Γ-semi ring, then M is called a 2-torsion free if 2a = 0 implies a = 0 for every a ∈ M.
Definition 2.6
Let (M, Γ) be a gamma semiring, then an additive mapping f: M → M is called a Gamma-semi derivation associated with a function g: M → M if for all x, y ∈ M *and* $\alpha \in \Gamma$,

f $(x\alpha y)$ = f$(x)\alpha g(y)$ + $x\alpha$f(y) = f$(x)\alpha y$ + $g(x)\alpha$f(y)

and f$(g(x))$ = g(f(x))

If g = 1 i.e the identity mapping on M, then all Gamma-semi derivations associated with g are merely ordinary Gamma-derivations. If g is an endomorphism of M, then other examples of semiderivatons are of the form f (x) = $x - g(x)$.

3. Orthogonal Semi derivation of Γ-semirings

Definition 3.1

Let M be a Γ-semiring. Let f_1 and f_2 be two semiderivations of M associated with functions $g_1: M \to M$ and $g_2: M \to M$ respectively. Then f_1 and f_2 are said to be orthogonal if $f_1(x)\Gamma M\Gamma f_2(y) = 0 = f_2(y)\Gamma M\Gamma f_1(x)$ for all $x, y \in M$.

Example 3.2

Let M_1 be a Γ_1 semiring and M_2 be a Γ_2-semiring. Consider $M = M_1 \times M_2$ and $\Gamma = \Gamma_1 \times \Gamma_2$. Define addition and multiplication on M and Γ by

$$(m_1, m_2) + (m_3, m_4) = (m_1 + m_3, m_2 + m_4)$$
$$(\alpha_1, \alpha_2) + (\alpha_3, \alpha_4) = (\alpha_1 + \alpha_3, \alpha_2 + \alpha_4)$$
$$(m_1, m_2)(\alpha_1, \alpha_2)(m_3, m_4) = (m_1\alpha_1 m_3, m_2\alpha_2 m_4)$$

for every $(m_1, m_2), (m_3, m_4) \in M$ and $(\alpha_1, \alpha_2), (\alpha_3, \alpha_4) \in \Gamma$.

Under these addition and multiplication M is a Γ-semiring. Let $\delta_1: M_1 \to M_1$ be an additive map, $\delta_2: M_1 \to M_1$ be a left and right M_1 module which is not a derivation. Define $f_1: M \to M$ such that $f_1(x_1, x_2) = (0, \delta_2(x_2))$ and

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$g_1: M \to M$ such that $g_1(x_1, x_2) = (\delta_1(x_1), 0)$ for all $x_1, x_2 \in M_1$. Also define $f_2: M \to M$ such that $f_2(x_1, x_2) = (\delta_2(x_1), 0)$ and $g_2: M \to M$ such that $g_2(x_1, x_2) = \big(0, \delta_1(x_2)\big)$ for all $x_1, x_2 \in M_1$.

Then it can easily be seen that f_1 and f_2 are semiderivations of M (with associated mappings δ_1 and δ_2 respectively) which are not derivations. Also it is clear that f_1 and f_2 are orthogonal semiderivations of M.

4. Results

Theorem 4.1

Let M be a semiprime Γ -semiring, and $a \in M$.If M admits a semiderivation f of M into M associated with function $g: M \to M$ such that $a\,\alpha\, f\,(x) = 0$ or $f(x)\alpha a = 0$ for all $x \in M, \alpha \in \Gamma$,then $a = 0$ or $d = 0$.

Proof

By the given hypothesis, we have

$$a\alpha f(b) = 0, \text{ for all } b \in M, \alpha \in \Gamma. \qquad (1)$$

Replace b by $b\beta c$, where $c \in M, \beta \in \Gamma$ in (1), we have

$$a\alpha f(b\beta c) = 0, \text{ for all of the } b, c \in M, \alpha, \beta \in \Gamma.$$

Since f a semiderivationof M,we get

$$a\alpha f(b)\beta g(c) + a\alpha b\beta f(c) = 0, \text{ for all of the } b, c \in M, \alpha, \beta \in \Gamma$$

Using (1), we get

$$a\alpha b\beta f(c) = 0, \text{ For all of the } b, c \in M, \alpha, \beta \in \Gamma.$$

Thus,

$$a\Gamma M\Gamma d(c) = 0, \text{ for all } c \in M.$$

By primness of M, we have

$$\text{Either } a = 0 \text{ or } d = 0.$$

Similarly we can prove for $f(x)\alpha a = 0$

Theorem 4.2

Let a and b be two elements of 2 -torsion free semiprime Γ-semiring M. Then the following are equivalent
(i) $a\Gamma x\Gamma b = 0$.
(ii) $b\Gamma x\Gamma a = 0$.
(iii) $a\Gamma x\Gamma b + b\Gamma x\Gamma a = 0$, for all $x \in M$.
If one of these conditions are fulfilled then $a\Gamma b = b\Gamma a = 0$

Proof.

Let a and b be two elements of 2-torsion free semi prime Γ-semi ring M.

(i) ⇒ (ii)

Assume $a\Gamma x\Gamma b = 0$, for all $x \in M$.
Pre and post multiplying by $b\Gamma x\Gamma$ and $\Gamma x\Gamma a$ then

$$(b\Gamma x\Gamma a)\Gamma x\Gamma(b\Gamma x\Gamma a) = 0.$$

Since M is a semiprime, for all $x \in M$. Therefore$b\Gamma x\Gamma a = 0$,.

(ii) ⇒ (i)

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Suppose $b\Gamma x\Gamma a = 0$, for all $x \in M$.

Pre and post multiplying by $a\Gamma x\Gamma$ and $\Gamma x\Gamma b$ then

$$(a\Gamma x\Gamma b)\Gamma M\Gamma(a\Gamma x\Gamma b) = 0.$$

Since M is a semiprime, for all $x \in M$.Thereforea$\Gamma x\Gamma b = 0$, for all $x \in M$.

(ii) ⇒ (iii)

Suppose $b\Gamma x\Gamma a = 0$, for all $x \in M$.

Then $a\Gamma x\Gamma b = 0$, for all $x \in M$Therefore $a\Gamma x\Gamma b + b\Gamma x\Gamma a = 0$, for all $x \in M$.

(iii) ⇒ (i)

Suppose $a\Gamma x\Gamma b + b\Gamma x\Gamma a = 0$, for all $x \in M$ (1)
Pre multiplying by $b\Gamma x\Gamma$ then

$$b\Gamma x\Gamma a\Gamma x\Gamma b + b\Gamma x\Gamma b\Gamma x\Gamma a = 0.$$

Again pre multiplying by $a\Gamma x\Gamma$ then

$$(a\Gamma x\Gamma b)\Gamma x\Gamma(a\Gamma x\Gamma b) + (a\Gamma x\Gamma b)\Gamma x\Gamma(b\Gamma x\Gamma a) = 0. \quad (2)$$

Post multiplying (1) by $\Gamma x\Gamma a$ then

$$a\Gamma x\Gamma b\Gamma x\Gamma a + b\Gamma x\Gamma a\Gamma x\Gamma a = 0.$$

Again post multiplying by $\Gamma x\Gamma b$ then
$$(a\Gamma x\Gamma b)\Gamma x\Gamma(a\Gamma x\Gamma b) + (b\Gamma x\Gamma a)\Gamma x\Gamma(a\Gamma x\Gamma b) = 0. \quad (3)$$

Adding (2) and (3) then using (1), we get $2(a\Gamma x\Gamma b)\Gamma x\Gamma(a\Gamma x\Gamma b) = 0$, for all $x \in M$

Since M is a 2 -torsion free and semiprime then $a\Gamma x\Gamma b = 0$, for all $x \in M$.

Let $a\Gamma x\Gamma b = 0$, for all $x \in M$.
Pre and post multiplying by $b\Gamma$ and Γa respectively then $b\Gamma a\Gamma x\Gamma b\Gamma a = 0$.
Since M is a semiprime,we get

$$b\Gamma a = 0.$$

Similarly, from $b\Gamma x\Gamma a = 0$, we can show that $b\Gamma a = 0$.

Theorem 4.3.

Let M be a 2-torsion free semiprime Γ-semiring. If additive mappings f_1 and $f_2$of M into itself satisfy $f_1(x)\Gamma M\Gamma f_2(x) = 0$, for all $x \in M$ then $f_1(x)\Gamma M\Gamma f_2(y) = 0$, for all $x, y \in M$

Proof

By the given hypothesis, we have
$$f_1(x)\Gamma m\Gamma f_2(x) = 0, \text{ for all } x, m \in M. \quad (1)$$

Replacing x by x+y, where $y \in M$ in (1), we get

$$f_1(x + y)\Gamma m\Gamma f_2(x + y) = 0$$

Since M is Γ-semiring ,we get

$$f_1(x)\Gamma m\Gamma f_2(x) + f_1(x)\Gamma m\Gamma f_2(y) + f_1(y)\Gamma m\Gamma f_2(x) + f_1(y)\Gamma m\Gamma f_2(y) = 0$$

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By using equation (1),we get

$$f_1(x)\Gamma m\Gamma f_2(y) + f_1(y)\Gamma m\Gamma f_2(x) = 0$$

Pre multiplying above equation by $f_1(x)\Gamma m\Gamma f_2(y)\Gamma s\Gamma$, where $s \in M$,we get

$$\begin{aligned} 0 &= [f_1(x)\Gamma m\Gamma f_2(y)]\Gamma s\Gamma[f_1(x)\Gamma m\Gamma f_2(y)] \\ &+f_1(x)\Gamma m\Gamma[f_2(y)\Gamma s\Gamma f_1(y)]\Gamma m\Gamma f_2(x) \end{aligned}$$

By using (1)and by Theorem 4.2 we get

$$0 = [f_1(x)\Gamma m\Gamma f_2(y)]\Gamma s\Gamma[f_1(x)\Gamma m\Gamma f_2(y)]$$

Hence,

$$0 = f_1(x)\Gamma m\Gamma f_2(x)\Gamma M\Gamma f_1(x)\Gamma m\Gamma f_2(x)$$

Since M is a semiprime, we get $f_1(x)\Gamma m\Gamma f_2(x) = 0$.

Theorem 4.4

Let M be a 2-torsion free semiprime Γ-semiring, and let f_1 and f_2 be semiderivations of M into M associated with functions $g_1: M \to M$ and $g_2: M \to M$ respectively. Then f_1 and f_2 are orthogonal if and only if $f_1(x)\alpha f_2(y) + f_2(x)\alpha f_1(y) = 0$, for all $x, y \in M, \alpha \in \Gamma$.

Proof

(⟹)Suppose

$$f_1(x)\alpha f_2(y) + f_2(x)\alpha f_1(y) = 0, \text{ for all } x, y \in M, \alpha \in \Gamma. \qquad (1)$$

Replace y by $y\beta x$, where $\beta \in \Gamma$, we get

$$0 = f_1(x)\alpha f_2(y\beta x) + f_2(x)\alpha f_1(y\beta x) \text{ for all } x, y \in M.$$

Since f_1 and f_2 are semiderivations of M ,we get

$$0 = f_1(x)\alpha(f_2(y)\beta g_2(x) + y\beta f_2(x)) + f_2(x)\alpha(f_1(y)\beta g_1(x) + y\beta f_1(x)).$$

Hence,

$$0= f_1(x)\alpha f_2(y)\beta g_2(x) + f_1(x)\alpha y\beta f_2(x) + f_2(x)\alpha f_1(y)\beta g_1(x) + f_2(x)\alpha y\beta f_1(x)$$

Since g_1 and g_2 are surjective we have

$$0 = f_1(x)\alpha f_2(y) + f_1(x)\alpha y\beta f_2(x) + f_2(x)\alpha f_1(y) + f_2(x)\alpha y\beta f_1(x)$$

$$\text{for all } x, y \in M, \alpha, \beta \in \Gamma.$$

Using (1), we get

$$0 = 2f_1(x)\Gamma M\Gamma f_2(x) \text{ for all } x \in M$$

Since M is 2 -torsion free ,we get

$$f_1(x)\Gamma M\Gamma f_2(x) = 0 \text{ for all } x \in M.$$

By Theorem 4.3, we have

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$$f_1(x)\Gamma M\Gamma f_2(y) = 0 \text{ for all } x, y \in M.$$

Hence ,by Theorem 4.2, we have

$$f_1(x)\Gamma M\Gamma f_2(y) = 0 \ = f_2(y)\Gamma M\Gamma f_1(x) \text{ for all } x, y \in M.$$

Thus, f_1 and f_2 are orthogonal.

(⟵)Conversely ,

Assume that , $f_1$and $f_2(y)$ are orthogonal.

Then,

$$f_1(x)\Gamma m\Gamma f_2(y) = 0 \ = f_1(y)\Gamma m\Gamma f_2(x) \text{ , for all } x, y, m \in M$$

By Theorem 4.2,we get

$$f_1(x)\Gamma f_2(y) = 0 \ = f_2(x)\Gamma f_1(y)$$

Hence,

$$f_1(x)\Gamma f_2(y) + f_2(x)\Gamma f_1(y) \ = 0, \text{ for all } x, y \in M.$$

Theorem 4.5

Let M be a 2-torsion free semi prime Γ-semiring.and let f_1 and f_2 be semiderivations of M into M associated with functions $g_1 : M \to M$ and $g_2 : M \to M$ respectively. Then f_1 and f_2 are orthogonal if and only if $f_1 f_2 = 0 = f_2 f_1$.

Proof

(⟹) Suppose that f_1 and f_2 are orthogonal , we have

$$f_1(x)\Gamma m\Gamma f_2(y) = 0 \text{ , for all } x, y, m \in M. \qquad (1)$$

Pre multiply by f_1 in (1), we have

$$0 = f_1\big(f_1(x)\alpha m\beta f_2(y)\big) \text{ for all } x, y \in M, \alpha, \beta \in \Gamma.$$

Since f_1 is semi derivations of M , we get

$$0 = f_1(f_1(x))\alpha g_1(m)\beta g_1(f_2(y)) + f_1(x)\alpha f_1(m)\beta g_1(f_2(y)) + f_1(x)\alpha m\beta f_1\big(f_2(y)\big)$$

Since g_1 is surjective, we get

$$0 = f_1\big(f_1(x)\big)\alpha m\beta f_2(y) + f_1(x)\alpha f_1(m)\beta f_2(y) + f_1(x)\alpha m\beta f_1\big(f_2(y)\big)$$
$$\text{for all } x, y \in M, \alpha, \beta \in \Gamma.$$

Sincef_1 and f_2 are orthogonal ,we get

$$f_1(x)\alpha m\beta f_1\big(f_2(y)\big) \text{ for all } x, y \in M, \alpha, \beta \in \Gamma. \qquad (2)$$

Replacing x by $f_2(y)$ in (2), we get

$$0 = f_1(f_2(y))\Gamma M\Gamma f_1\big(f_2(y)\big) \text{ for all } y \in M.$$

By the semiprimeness of M we get

$$f_1 f_2(y) = 0 \text{ for all } y \in M .$$

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Hence , $f_1 f_2 = 0$.Similarly we can prove $f_2 f_1 = 0$.

(⟵) Conversely assume $f_1 f_2 = 0$, we get

$$0 = f_1(f_2(x\alpha y)) \text{ for all } x, y \in M, \alpha \in \Gamma.$$

Since f_2 is semiderivations of M , we get

$$0 = f_1(f_2(x)\alpha g_2(y) + x\alpha f_2(y)) \text{ for all } x, y \in M$$

Since f_1 is semiderivations of M , we get

$$0 = f_1(f_2(x))\alpha g_1(g_2(y)) + f_2(x)\alpha f_1(g_2(y)) + f_1(x)\alpha g_1(f_2(y) + x\alpha f_1(f_2(y)) \text{ for all } x, y \in M$$

Since g_1 and g_2 are surjective and since $f_1 f_2 = 0$, we get

$$0 = f_2(x)\alpha f_1(y) + f_1(x)\alpha f_2(y) \text{ for all } x, y \in M.$$

By Theorem 4.4 we have f_1 and f_2 are orthogonal.

Theorem4.6

Let M be a 2-torsion free semiprime Γ-semiring and let f_1 and f_2 be a semiderivations of M into Massociated with functions $g_1 : M \to M$ and $g_2 : M \to M$ respectively. Then f_1 and f_2 are orthogonal if and only if $f_1 f_2 + f_2 f_1 = 0$

Proof

(⟹) Suppose $0 = f_1 f_2 + f_2 f_1$, we have

$$0 = (f_1 f_2 + f_2 f_1)(x\alpha y) \text{ for all } x, y \in M, \alpha \in \Gamma.$$

Since f_1 and f_2 be an additive mapping ,we get

$$0 = f_1 f_2(x\alpha y) + f_2 f_1(x\alpha y) \text{ for all } x, y \in M$$

Since f_1 and f_2 are semi derivations of M , we get

$$0 = f_1\big(f_2(x)\alpha g_2(y) + x\alpha f_2(y)\big) + f_2\big(f_1(x)\alpha g_1(y) + x\alpha f_1(y)\big) \text{ for all } x, y \in M.$$

$$0 = f_1(f_2(x))\alpha g_1(g_2(y)) + f_2(x)\alpha f_1(g_2(y)) + f_1(x)\alpha g_1(f_2(y)) + x\alpha f_1(f_2(y))$$

$$+ f_2(f_1(x))\alpha g_2(g_1(y)) + f_1(x)\alpha f_2(g_1(y)) + f_2(x)\alpha g_2(f_1(y)) + x\alpha f_2(f_1(y))$$

$$\text{for all } x, y \in M, \alpha \in \Gamma.$$

Since g_1 and g_2 are surjective and using hypothesis ,we get

$$0 = 2(f_1(x)\alpha f_2(y) + f_2(x)\alpha f_1(y)) \text{ for all } x, y \in M.$$

By Theorem 4.4 ,we get

f_1 and f_2 are orthogonal.

(⟵) Conversely Suppose that f_1 and f_2 are orthogonal.

By Theorem 4.5 ,we get

$$f_1 f_2 = 0 = f_2 f_1.$$

Hence,

$$f_1 f_2 + f_2 f_1 = 0.$$

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Theorem 4.7

Let M be a 2-torsion free semiprime Γ-semiring.and let f_1 and f_2 be semiderivations of M into M associated with functions $g_1: M \to M$ and $g_2: M \to M$ respectively. Then f_1 and f_2 are orthogonal if and only if $f_1 f_2$ or $f_2 f_1$ is a semiderivation associated with the function $g_1 g_2: M \to M$.

Proof

(⟹) Suppose that f_1 and f_2 are orthogonal.

By Theorem 4.6 ,we get

$$f_1 f_2 + f_2 f_1 = 0 \qquad (1)$$

Also by Theorem 4.5, we get

$$0 = f_1 f_2(x\alpha y) \text{ for all } x, y \in M, \alpha \in \Gamma.$$

Since$f_1$and f_2 are semiderivations of M , we get

$$0 = f_1(f_2(x))\alpha g_1(g_2(y)) + f_2(x)\alpha f_1(g_2(y)) + f_1(x)\alpha g_1(f_2(y)) + x\alpha f_1(f_2(y))$$
$$\text{for all } x, y \in M, \alpha \in \Gamma.$$

Since g_1 and g_2 are surjective ,we get

$$0 = f_1 f_2(x)\alpha g_1 g_2(y) + f_2(x)\alpha f_1(y) + f_1(x)\alpha f_2(y) + x\alpha f_1 f_2(y)$$
$$\text{for all } x, y \in M, \alpha \in \Gamma.$$

By Theorem 4.4,we get

$$0 = f_1 f_2(x)\alpha g_1 g_2(y) + x\alpha f_1 f_2(y) \text{ for all } x, y \in M, \alpha \in \Gamma.$$

Also, Since g_1 and g_2 are surjective ,we get

$g_1 g_2(f_1 f_2(x)) = f_1 f_2(x) = f_1 f_2(g_1 g_2(x))$,

Hence,

$f_1 f_2$ is a semiderivation associated with the function $g_1 g_2: M \to M$.

(⟵) Conversely ,

Assume that $f_1 f_2$ is a semiderivation associated with the function

$g_1 g_2: M \to M$.

Then,

$$f_1 f_2(x\alpha y) = f_1 f_2(x)\alpha g_1 g_2(y) + x\alpha f_1 f_2(y) \text{ for all } x, y \in M, \alpha \in \Gamma. \qquad (2)$$

Also,

$$f_1 f_2(x\alpha y) = f_1(f_2(x)\alpha g_2(y) + x\alpha f_2(y))$$
$$= f_1(f_2(x))\alpha g_1(g_2(y)) + f_2(x)\alpha f_1(g_2(y)) + f_1(x)\alpha g_1(f_2(y)) + x\alpha f_1(f_2(y))$$
$$\text{for all } x, y \in M, \alpha \in \Gamma.$$

Since g_1 and g_2 are surjective, we get

$$f_1 f_2(x\alpha y) = f_1 f_2(x)\alpha g_1 g_2(y) + f_2(x)\alpha f_1(y) + f_1(x)\alpha f_2(y) + x\alpha f_1 f_2(y)$$
$$\text{for all } x, y \in M, \alpha \in \Gamma. \qquad (3)$$

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Comparing (2) and (3), we get

$$f_2(x)\alpha f_1(y) + f_1(x)\alpha f_2(y) = 0 \quad \text{for all } x, y \in M, \alpha \in \Gamma.$$

By Theorem 4.4,we get

$$f_1 \text{ and } f_2 \text{ are orthogonal.}$$

Similarly, we can prove if $f_2 f_1$ is a semiderivation associated with the function $g_1 g_2: M \to M$ then f_1 and f_2 are orthogonal.

Theorem 4.8

Let M be a 2-torsion free semiprime Γ-semiring and let f_1 and f_2 be semiderivations of M into M associated with functions $g_1: M \to M$ and $g_2: M \to M$ respectively. Then f_1 and f_2 are orthogonal if and only if there exists $a, b \in M$ such that $f_1 f_2(x) = a\alpha x + x\alpha b$, for all $x \in M, \alpha \in \Gamma$.

Proof

(⟹) Suppose that f_1 and f_2 are orthogonal.
By Theorem 4.5 ,we get

$$f_1 f_2 = 0.$$

Choosing $a = b = 0$,we get

$$f_1 f_2(x) = a\alpha x + x\alpha b, \text{ for all } x \in M, \alpha \in \Gamma.$$

(⟵) Conversely , let $a, b \in M$

Assume that,

$$f_1 f_2(x) = a\alpha x + x\alpha b, \text{ for all } x \in M, \alpha \in \Gamma. \qquad (1)$$

Replace x by xβy, we get

$$f_1 f_2(x\beta y) = a\alpha x\beta y + x\beta y\alpha b \text{ for all } x \in M, \alpha, \beta \in \Gamma.$$

Since f_1 and f_2 are semiderivations of M , we get

$$f_1(f_2(x)\beta g_2(y) + x\beta f_2(y)) = a\alpha x\beta y + x\beta y\alpha b$$

$$f_1(f_2(x))\beta g_1(g_2(y)) + f_2(x)\beta f_1(g_2(y)) + f_1(x)\beta g_1(f_2(y)) + x\beta f_1(f_2(y)) = a\alpha x\beta y + x\beta y\alpha b$$

$$\text{, for all } x \in M, \alpha, \beta \in \Gamma.$$

Since g_1 and g_2 are surjective ,we get

$$f_1 f_2(x)\beta y + f_2(x)\beta f_1(y) + f_1(x)\beta f_2(y) + x\beta f_1 f_2(y) = a\alpha x\beta y + x\beta y\alpha b$$

$$\text{,for all } x \in M, \alpha, \beta \in \Gamma.$$

Using (1),we get

$$(a\alpha x + x\alpha b)\beta y + f_2(x)\beta f_1(y) + f_1(x)\beta f_2(y) + x\beta(a\alpha y + y\alpha b) = a\alpha x\beta y + x\beta y\alpha b$$

$$a\alpha x\beta y + x\alpha b\beta y + f_2(x)\beta f_1(y) + f_1(x)\beta f_2(y) + x\beta a\alpha y + x\beta y\alpha b = a\alpha x\beta y + x\beta y\alpha b$$

$$\text{, for all } x \in M, \alpha, \beta \in \Gamma$$

Comparing, then

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$$x\alpha b\beta y + f_2(x)\beta f_1(y) + f_1(x)\beta f_2(y) + x\beta a\alpha y = 0 \text{ for all } x \in M, \alpha, \beta \in \Gamma. \qquad (2)$$

Replace y by $y\gamma x$ where $\gamma \in \Gamma$in (2), we have

$$0 = x\alpha b\beta y\gamma x + x\beta a\alpha y\gamma x + f_2(x)\beta f_1(y\gamma x) + f_1(x)\beta f_2(y\gamma x)$$

$$\text{for all } x \in M, \alpha, \beta \in \Gamma.$$

Since$f_1$and $f_2$are semiderivations of M , we get

$$0 = x\alpha b\beta y\gamma x + x\beta a\alpha y\gamma x + f_2(x)\beta f_1(y)\gamma g_1(x) + f_2(x)\beta y\gamma f_1(x) + f_1(x)\beta f_2(y)\gamma g_2(x) + f_1(x)\beta y\gamma f_2(x)$$

$$,\text{for all } x \in M, \alpha, \beta, \gamma \in \Gamma.$$

Since g_1 and g_2 are surjective ,we get

$$0 = x\alpha b\beta y\gamma x + x\beta a\alpha y\gamma x + f_2(x)\beta f_1(y)\gamma x + f_2(x)\beta y\gamma f_1(x) + f_1(x)\beta f_2(y)\gamma x + f_1(x)\beta y\gamma f_2(x)$$

$$, \text{ for all } x \in M, \alpha, \beta, \gamma \in \Gamma.$$

Hence,

$$0 = (x\alpha b\beta y + x\beta a\alpha y + f_2(x)\beta f_1(y) + f_1(x)\beta f_2(y))\gamma x + f_2(x)\beta y\gamma f_1(x) + f_1(x)\beta y\gamma f_2(x)$$

$$\text{for all } x \in M, \alpha, \beta, \gamma \in \Gamma.$$

Using (2),we get

$$0 = f_2(x)\beta y\gamma f_1(x) + f_1(x)\beta y\gamma f_2(x) \text{ for all } x \in M, \alpha, \beta, \gamma \in \Gamma.$$

By Theorems 4.2 and 4.3 ,we have

$$f_2(x)\Gamma M\Gamma f_1(x) = 0 = f_1(x)\Gamma M\Gamma f_2(x) \text{for all } x, y \in M.$$

Hence ,

f_1 and f_2 are orthogonal.

Theorem 4.9

Let M be a 2-torsion free semi prime Γ-semiring and let f_1 and f_2 be semiderivations of M into M associated with functions $g_1: M \to M$ and $g_2: M \to M$ respectively such that $f_1 f_2$ is also a semiderivation associated with the function $g_1 g_2: M \to M$. Then either f_1 is zero or f_2 is zero.

Proof

Suppose that, $f_1 f_2$ is a semiderivation, we have

$$f_1 f_2(x\alpha y) = f_1 f_2(x)\alpha g_1 g_2(y) + x\alpha f_1 f_2(y) \text{ for all } x, y \in M, \alpha \in \Gamma. \qquad (1)$$

Again, since$f_1 f_2$ is a semiderivation, we get

$$f_1 f_2(x\alpha y) = f_1\big(f_2(x)\alpha g_2(y) + x\alpha f_2(y)\big)$$

$$= f_1(f_2(x))\alpha g_1(g_2(y)) + f_2(x)\alpha f_1(g_2(y)) + f_1(x)\alpha g_1(f_2(y)) + x\alpha f_1(f_2(y))$$

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$$\text{for all } x, y \in M, \alpha \in \Gamma.$$

since g_1 and g_2 are surjective ,we get

$$f_1 f_2(x\alpha y) = f_1 f_2(x)\alpha g_1 g_2(y) + f_2(x)\alpha f_1(y) + f_1(x)\alpha f_2(y) + x\alpha f_1 f_2(y)$$
$$\text{for all } x, y \in M, \alpha \in \Gamma. \qquad (2)$$

Comparing (1) and (2), we get

$$f_2(x)\alpha f_1(y) + f_1(x)\alpha f_2(y) = 0 \text{ for all } x, y \in M, \alpha \in \Gamma. \qquad (3)$$

Replace x by $x\beta f_1(z)$, where $\beta \in \Gamma$, we get

$$f_1(x\beta f_1(z))\alpha f_2(y) + f_2(x\beta f_1(z))\alpha f_1(y) = 0 \text{ for all } x, y, z \in M.$$

Since f_1 and f_2 be semiderivations of M, we get

$$0 = f_1(x)\beta g_1(f_1(z))\alpha f_2(y) + x\beta f_1(f_1(z))\alpha f_2(y) + f_2(x)\beta g_2(f_1(z))\alpha f_1(y) + x\beta f_2(f_1(z))\alpha f_1(y)$$

Using (3)by , we get

$$0 = f_1(x)\beta g_1(f_1(z))\alpha f_2(y) + f_2(x)\beta g_2(f_1(z))\alpha f_1(y) \text{ for all } x, y, z \in M$$

Since g_1 and g_2 are surjective , we get

$$0 = f_1(x)\beta f_1(z)\alpha f_2(y) + f_2(x)\beta f_1(z)\alpha f_1(y) \text{ for all } x, y \in M \qquad (4)$$

Replace x by z where $z \in M$ in (3) , we get

$$f_2(z)\alpha f_1(y) + f_1(z)\alpha f_2(y) = 0 \text{ for all } z, y \in M.$$

$$f_1(z)\alpha f_2(y) = -f_2(z)\alpha f_1(y) \text{ for all } y, z \in M. \qquad (5)$$

Using (5) in (4),we get

$$- f_1(x)\beta f_2(z)\alpha f_1(y) + f_2(x)\beta f_1(z)\alpha f_1(y) = 0 \text{ forall } x, y, z \in M$$
$$(f_2(x)\beta f_1(z) - f_1(x)\beta f_2(z))\alpha f_1(y) = 0 \text{ for all } x, y, z \in M$$

By Theorem (4.1), we get

$$\text{Either } f_2(x)\beta f_1(z) - f_1(x)\beta f_2(z) = 0 \text{ or } f_1(y) = 0$$

Let

$$f_2(x)\beta f_1(z) - f_1(x)\beta f_2(z) = 0 \text{ for all } x, z \in M, \beta \in \Gamma. \qquad (6)$$

In (3) replace y by z and β by α we get

$$f_2(x)\beta f_1(z) + f_1(x)\beta f_2(z) = 0 \text{ for all } x, z \in M, \beta \in \Gamma. \qquad (7)$$

Adding (6) and (7), we get

$$2f_2(x)\beta f_1(z) = 0 \text{ for all } x, z \in M, \beta \in \Gamma.$$

Since M is 2-torsion free,we get

$$f_2(x)\beta f_1(z) = 0 \text{ for all } x, z \in M, \beta \in \Gamma.$$

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Again By Theorem (4.1), we get

$$f_2(x) = 0 \text{ or } f_1(z) = 0 \text{ for all } x, z \in M$$

Hence,

$$f_1 = 0 \text{ or } f_2 = 0.$$

Theorem 4.10

Let M be a 2-torsion free semiprime Γ-semiring and let f be a semiderivations of M into M associated with functions $g: M \to M$ such thatf^2 is also a semiderivation associated with the function $g^2: M \to M$. Then f is zero.

Proof

Suppose f^2 is a semiderivations 2 -torsion free semiprime Γ-semiring M.

Let $x, y \in M, \alpha \in \Gamma$, we have

$$f^2(x\alpha y) = f(f(x\alpha y))$$

Since f andf^2 be semiderivations of M, weget

$$f^2(x)\alpha g^2(y) + x\alpha f^2(y) \; = f^2(x)\alpha g(g(y)) + f(x)\alpha f(g(y)) + f(x)\alpha g(f(y)) + x\alpha f^2(y)$$

$$\text{for all } x, y \in M, \alpha \in \Gamma.$$

Since gis surjective , we get

$$2f(x)\alpha f(y) = 0 \text{ for all } x, y \in M, \alpha \in \Gamma.$$

Since M is a 2-torsion free , we get

$$f(x)\alpha f(y) = 0 \text{ for all } x, y \in M, \alpha \in \Gamma. \qquad (1)$$

Replace x by $x\beta z$ in $(1), z \in M, \beta \in \Gamma$, we get

$$f(\,x\beta z)\alpha f(y) = 0 \text{ for all } x, y \in M, \alpha \in \Gamma.$$

Since f is semiderivations of M, we get

$$0 = [f(x)\beta g(z) + x\beta f(z)]\alpha f(y) \; \text{ for all } x, y, z \in M, \alpha, \beta \in \Gamma.$$

Hence,

$$0 = f(x)\beta z\alpha f(y) \; + x\beta f(z)\alpha f(y) \text{ for all } x, y, z \in M, \alpha, \beta \in \Gamma.$$

Using (1) , we get

$$0 = f(x)\beta z\alpha f(y) \text{ for all } x, y, z \in M, \alpha, \beta \in \Gamma. \qquad (2)$$

Replace y by $x + y$ in (2), we get

$$0 = f(x)\beta z\alpha f(\,x + y) \text{ for all } x, y, z \in M, \alpha, \beta \in \Gamma.$$

Since f is additive mapping of M, we get

$$0 = f(x)\beta z\alpha f(\,x) + f(x)\beta z\alpha f(y) \text{ for all } x, y, z \in M, \alpha, \beta \in \Gamma.$$

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Using (2) , we get

$$0 = f(x)\Gamma M\Gamma f(x) \text{ for all } x, y, z \in M, \alpha, \beta \in \Gamma.$$

Since M is a semiprime, we get

$$f(x) = 0 \text{ for all } x \in M .$$

Hence,

$$f = 0.$$

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References

[1] Nobusawa N. On a generalization of the ring theory 1964.
[2] Sen MK. Proceeding of International Conference on Algebra and its Applications 1981.
[3] Lehmer DH. A ternary analogue of abelian groups. Am J Math 1932;54:329–38.
[4] Lister WG. Ternary rings. Trans Am Math Soc 1971;154:37–55.
[5] Dutta TK, Kar S. On regular ternary semirings. Adv. Algebr. Proc. ICM Satell. Conf. Algebr. Relat. Top. World Sci., 2003, p. 343–55.
[6] Rao MMK. Γ-semirings-I. Southeast Asian Bull Math 1995;19:49–54.
[7] Murali Krishna Rao M. Prime bi-interior ideals of Γ-semirings. J Hyperstructures 2023;11:20–30.
[8] Bergen J. Derivations in prime rings. Can Math Bull 1983;26:267–70.
[9] Brešar M, Vukman J. On some additive mappings in rings with involution. Aequationes Math 1989;38:178–85.
[10] Bresar M. Orthogonal derivations and extension of a theorem of Posner. Rad Math 1989;5:237–46.
[11] Posner EC. Derivations in prime rings. Proc Am Math Soc 1957;8:1093–100.
[12] Chang J-C. On semi-derivations of prime rings. Chinese J Math 1984:255–62.
[13] Chuang C-L. On the structure of semiderivations in prime rings. Proc Am Math Soc 1990;108:867–9.
[14] Firat A. Some results for semi derivations of prime rings. Int J Pure Appl Math 2006;28:363.
[15] Ali S, Khan MS. On orthogonal (σ, τ)-derivations in semiprime Γ-rings. Int Electron J Algebr 2013;13:23–39.
[16] Hamil SA. Commutativity of Prime Γ-Semirings with Derivations and Generalized Derivations. J. Phys. Conf. Ser., vol. 1804, IOP Publishing; 2021, p. 12091.
[17] Rasheed MK, Hameed FA, Majeed AH. On Generalized (α, β) Derivation on Prime Semirings. J. Phys. Conf. Ser., vol. 1591, IOP Publishing; 2020, p. 12080.
[18] Rasheed MK, Majeed AH. Developing an improved grey prediction model for application to electricity consumption prediction: toward enhanced model accuracy. J. Phys. Conf. Ser., vol. 1362, IOP Publishing; 2019, p. 12137.
[19] Ibraheem RKH, Majeed AH. On Lie Structure in Semiprime Inverse Semirings. Iraqi J Sci 2019:2711–8.
[20] Dey KK, Paul AC, Rakhimov IS. Semiprime gamma rings with orthogonal reverse derivations. Int J Pure Appl Math 2013;83:233–45.
[21] Sindhu KK, Murugesan R, Namasivayam P. Orthogonl Semiderivtions on Semiprime Semirings. Int Organ Sci Res J Math 2015;11:18–24.
[22] Venkateswarlu B, Rao MMK, Narayana YA. ORTHOGONAL DERIVATIONS ON Γ− SEMIRINGS n.d.

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