On The Algebraic Properties of the Multiplication Operation of 4-Cyclic Refined Neutrosophic Real Matrices

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Abstract

The objective of this paper is to study some of the elementary algebraic properties of 4-cyclic refined neutrosophic matrices with real entries, where we study the algebraic structure of the multiplication operation and its properties such as associativity, commutativity, and the existence of algebraic multiplication inverse. Also, we illustrate many examples that explain the validity of our work.

Keywords: 4-cyclic refined neutrosophic matrix; eigenvalue; eigenvector; 4-cyclic refined neutrosophic ring

1. Introduction

The algebraic structures based on n-cyclic refined neutrosophic sets were studied by many different authors, where it can find some interesting generalizations such as n-cyclic refined neutrosophic rings, 3-cyclic refined neutrosophic units, 4-cyclic refined neutrosophic units, 4-cyclic refined neutrosophic Diophantine equations, and n-cyclic refined neutrosophic groups and complex numbers [1-13]. In [14], 2-cyclic refined neutrosophic matrices have been studied and formulated, where the computations of eigenvectors and eigenvalues were introduced, as well as the computation of determinants, inverses, and the diagonal zing properties.
This has motivated us to study the general case of 4-cyclic refined neutrosophic matrices for the first time, where some of the elementary algebraic properties of 4-cyclic refined neutrosophic matrices with real entries will be discussed, and we study the algebraic structure of the multiplication operation and its properties such as associativity, commutativity, and the existence of algebraic multiplication inverse. Also, we illustrate many examples that explain the validity of our work.

2. Main Discussion

Definition:
Let X_i ; $0 \leq i \leq 1$ be five $n \times n$ matrices with real entries, then the corresponding 4-cyclic refined neutrosophic matrix is defined as follows:

$$X = X_0 + \sum_{i=1}^{4} X_i I_i$$

Example:
Consider $X_0 = \begin{pmatrix} 1 & 2 \\ -1 & 7 \end{pmatrix}$, $X_1 = \begin{pmatrix} 3 & 3 \\ 0 & 1 \end{pmatrix}$,

$$X_2 = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, X_3 = \begin{pmatrix} -4 & 2 \\ 6 & -9 \end{pmatrix},$$

$X_4 = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$, then:

DOI: https://doi.org/10.54216/NIF.030102
Received: April 27, 2023 Accepted: January 05, 2024

$$X = X_0 + \sum_{i=1}^{4} X_i I_i = \begin{pmatrix} 1+3I_1+I_2-4I_3+I_4 & 2+3I_1-I_2+2I_3+3I_4 \\ -1+6I_3-I_4 & 7+I_1+2I_2-9I_3+2I_4 \end{pmatrix}$$

Remark:

We denote to the set of all $n \times n$ 4-cyclic refined neutrosophic matrices with real entries by $R_{n\times n}^{(4)}(I)$.

Remark:

Let $X = X_0 + \sum_{i=1}^{4} X_i I_i$, $Y = Y_0 + \sum_{i=1}^{4} Y_i I_i$, then:

$X + Y = (X_0 + Y_0) + \sum_{i=1}^{4} (X_i + Y_i) I_i$,

$X.Y = X_0Y_0 + I_1(X_0Y_1 + X_1Y_0 + X_1Y_4 + X_4Y_1 + X_2Y_3 + X_3Y_2) + I_2(X_0Y_2 + X_2Y_0 + X_1Y_1 + X_2Y_4 + X_4Y_2 + X_3Y_3) + I_3(X_0Y_3 + X_3Y_0 + X_1Y_2 + X_2Y_1 + X_3Y_4 + X_4Y_3) + I_4(X_0Y_4 + X_4Y_0 + X_1Y_3 + X_3Y_1 + X_4Y_4 + X_2Y_2)$.

Example:

Take $X = \begin{pmatrix} 1+I_1-I_3 & I_2+I_4 \\ 2I_1+I_3 & 1-I_2 \end{pmatrix}$,

$Y = \begin{pmatrix} 2+I_2 & I_4 \\ 1+I_2+I_3 & 1+I_1-I_4 \end{pmatrix}$, then:

$X_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $X_1 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$,

$X_2 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$, $X_3 = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$,

$X_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

$Y_0 = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$, $Y_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$,

$Y_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $Y_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$,

$Y_4 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$.

$X + Y = \begin{pmatrix} 3+I_1+I_2-I_3 & I_2+2I_4 \\ 1+2I_1+I_2+2I_3 & 2+I_1-I_2-I_4 \end{pmatrix}$,

$X_0Y_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$,

$X_0Y_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$,

$X_1Y_4 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$,

$X_1Y_0 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix}$,

$X_4Y_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

$X_2Y_3 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$,

$X_3Y_2 = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$

$\Longrightarrow X_0Y_1 + X_1Y_0 + X_1Y_4 + X_4Y_1 + X_2Y_3 + X_3Y_2 = \begin{pmatrix} 2 & 2 \\ 4 & 3 \end{pmatrix}$.

$X_0Y_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$,

DOI: https://doi.org/10.54216/NIF.030102
Received: April 27, 2023 Accepted: January 05, 2024

$X_2Y_0 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$,

$X_1Y_1 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

$X_2Y_4 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$,

$X_4Y_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,

$X_3Y_3 = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

$\Longrightarrow X_0Y_2 + X_2Y_0 + X_1Y_1 + X_2Y_4 + X_4Y_2 + X_3Y_3 = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$.

$X_0Y_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$,

$X_3Y_0 = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 2 & 0 \end{pmatrix}$,

$X_1Y_2 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$,

$X_2Y_1 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$,

$X_3Y_4 = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$,

$X_4Y_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,

$\Longrightarrow X_0Y_3 + X_3Y_0 + X_1Y_2 + X_2Y_1 + X_3Y_4 + X_4Y_3 = \begin{pmatrix} 0 & 0 \\ 5 & 0 \end{pmatrix}$.

$X_0Y_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$,

$X_4Y_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$,

$X_1Y_3 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

$X_3Y_1 = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

$X_4Y_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$,

$X_2Y_2 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$,

$\Longrightarrow X_0Y_4 + X_4Y_0 + X_1Y_3 + X_3Y_1 + X_4Y_4 + X_2Y_2 = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$.

So that

$$X.Y = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} + I_1\begin{pmatrix} 2 & 2 \\ 4 & 3 \end{pmatrix} + I_2\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} + I_3\begin{pmatrix} 0 & 0 \\ 5 & 0 \end{pmatrix} + I_4\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 + 2I_1 + 3I_2 + 2I_4 & 2I_1 + I_4 \\ 1 + 4I_1 + 5I_3 - I_4 & 1 + 3I_1 - I_4 \end{pmatrix}.$$

Example:

Let us try to find the inverse of:

$$X = \begin{pmatrix} 1 + I_4 & 3 \\ I_4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}I_1 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}I_2 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}I_3 + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}I_4.$$

DOI: https://doi.org/10.54216/NIF.030102
Received: April 27, 2023 Accepted: January 05, 2024

Let $Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = X^{-1}$;

$$\begin{cases} A = a_0 + \sum_{i=1}^{4} a_i I_i \;, B = b_0 + \sum_{i=1}^{4} b_i I_i \\ C = c_0 + \sum_{i=1}^{4} c_i I_i \,, D = d_0 + \sum_{i=1}^{4} d_i I_i \end{cases}$$

$XY = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, thus

$$\begin{cases} A(1 + I_4) + BI_4 = 1 \\ 3A + B = 0 \\ C(1 + I_4) + DI_4 = 0 \\ 3C + D = 1 \end{cases}$$

$$A(1 + I_4) + BI_4 = a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3 + a_4 I_4 + a_0 I_4 + a_1 I_1 + a_2 I_2 + a_3 I_3 + a_4 I_4 + b_0 I_4 + b_1 I_1 + b_2 I_2 + b_3 I_3 + b_4 I_4 = 1,$$

Thus

$$a_0 + I_1(2a_1 + b_1) + I_2(2a_2 + b_2) + I_3(2a_3 + b_3) + I_4(a_0 + 2a_4 + b_0 + b_4) = 1,$$

so that: $\begin{cases} a_0 = 1 \,, 2a_1 + b_1 = 0 \\ 2a_2 + b_2 = 2a_3 + b_3 = 0 \\ 1 + 2a_4 + b_0 + b_4 = 0 \end{cases}$

From the equation $3A + B = 0$, we get:

$$\begin{cases} 3a_0 + b_0 = 0 \Longrightarrow b_0 = -3 \\ 3a_1 + b_1 = 0 \\ 3a_2 + b_2 = 0 \\ 3a_3 + b_3 = 0 \\ 3a_4 + b_4 = 0 \end{cases}$$

Thus: $\begin{cases} a_1 = 0 \,, b_1 = 0 \\ a_2 = 0 \,, b_2 = 0 \\ a_3 = 0 \,, b_3 = 0 \\ a_4 = -2 \,, b_4 = 6 \end{cases}$

Hence: $\begin{cases} A = 1 - 2I_4 \\ B = -3 + 6I_4 \end{cases}$

$C(1 + I_4) + DI_4 = 0$ implies that:

$$\begin{cases} c_0 = 0 \,, 2c_1 + d_1 = 0 \\ 2c_2 + d_2 = 2c_3 + d_3 = 0 \\ d_0 + d_4 + 2c_4 = 0 \end{cases}$$

$3C + D = 1$ implies

$$\begin{cases} 3c_0 + d_0 = 1 \Longrightarrow d_0 = 1 \\ 3c_1 + d_1 = 0 \\ 3c_2 + d_2 = 0 \\ 3c_3 + d_3 = 0 \\ 3c_4 + d_4 = 0 \end{cases}$$

Hence: $\begin{cases} c_1 = d_1 = 0 \\ c_2 = d_2 = 0 \\ c_3 = d_3 = 0 \\ c_4 = 1, d_4 = -3 \end{cases}$

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Received: April 27, 2023 Accepted: January 05, 2024

Which implies that:

$C = I_4, D = 1 - 3I_4$

and $Y = \begin{pmatrix} 1-2I_4 & -3+6I_4 \\ I_4 & 1-3I_4 \end{pmatrix}$.

Example:

Consider $A = \begin{pmatrix} 1+I_3 & I_2 \\ I_4 & 1-I_3 \end{pmatrix}$, we will try to find its eigenvalues.

The equation: $AX = a.X$, where $X = x_0 + \sum_{i=1}^{4} x_i I_i$; $x_i \in \mathbb{R}^2, a = a_0 + \sum_{i=1}^{4} a_i I_i$; $X_i \in R_4(I)$,

$A.X = \begin{pmatrix} 1+I_3 & I_2 \\ I_4 & 1-I_3 \end{pmatrix}\begin{pmatrix} X' \\ X'' \end{pmatrix}$;

$$\begin{cases} X' = x_0' + \sum_{i=1}^{4} x_i' I_i \\ X'' = x_0'' + \sum_{i=1}^{4} x_i'' I_i \end{cases}, x_i', x_i'' \in \mathbb{R}$$

$A.X = \begin{pmatrix} (1+I_3)X' + I_2 X'' \\ I_4 X' + (1-I_3)X'' \end{pmatrix} = \begin{pmatrix} T \\ L \end{pmatrix}$.

$T = (1+I_3)(x_0' + x_1' I_1 + x_2' I_2 + x_3' I_3 + x_4' I_4) + I_2(x_0'' + x_1'' I_1 + x_2'' I_2 + x_3'' I_3 + x_4'' I_4) =$
$x_0' + I_1(x_1' + x_2') + I_2(x_2' + x_3') + I_3(x_3' + x_4') + I_4(x_4' + x_1') + x_0'' I_2 + x_1'' I_3 + x_2'' I_4 + x_3'' I_1 + x_4'' I_2,$
$= x_0' + I_1(x_1' + x_2' + x_3'') + I_2(x_2' + x_3' + x_0'' + x_4'') + I_3(x_3' + x_4' + x_1'') + I_4(x_4' + x_1' + x_2'')$.

$L = I_4(x_0' + x_1' I_1 + x_2' I_2 + x_3' I_3 + x_4' I_4) + (1-I_3)(x_0'' + x_1'' I_1 + x_2'' I_2 + x_3'' I_3 + x_4'' I_4) =$
$x_0'' + I_1(x_1' + x_1'' - x_2'') + I_2(x_2' + x_2'' - x_3'') + I_3(x_3' + x_3'' - x_0'' - x_4'') + I_4(x_0' + x_4' + x_4'' - x_1'')$.

By a similar argument, we can compute $a.X$.

The equation $AX = aX$ is equivalent to:

$$\begin{cases} T = a.X' \\ L = a.X'' \end{cases}$$

Hence:
$$\begin{cases} x_0' = a_0 x_0' \\ x_1' + x_2' + x_3'' = a_0 x_1' + a_1 x_0' + a_2 x_3' + a_3 x_2' + a_1 x_4' + a_4 x_1' \\ x_2' + x_3' + x_0'' + x_4'' = a_0 x_2' + a_2 x_0' + a_1 x_1' + a_2 x_4' + a_4 x_2' \\ x_3' + x_4' + x_1'' = a_0 x_3' + a_3 x_0' + a_1 x_2' + a_2 x_1' + a_3 x_4' + a_4 x_3' \\ x_4' + x_1' + x_2'' = a_0 x_4' + a_4 x_0' + a_2 x_2' + a_3 x_1' + a_1 x_3' + a_4 x_4' \end{cases}$$

And :
$$\begin{cases} x_0'' = a_0 x_0'' \\ x_1' + x_1'' - x_2'' = a_0 x_1'' + a_1 x_0'' + a_2 x_3'' + a_3 x_2'' + a_1 x_4'' + a_4 x_1'' \\ x_2' + x_2'' - x_3'' = a_0 x_2'' + a_2 x_0'' + a_1 x_1'' + a_2 x_4'' + a_4 x_2'' \\ x_3' + x_3'' - x_0'' - x_4'' = a_0 x_3'' + a_3 x_0'' + a_1 x_2'' + a_2 x_1'' + a_3 x_4'' + a_4 x_3'' \\ x_0' + x_4' + x_4'' - x_1'' = a_0 x_4'' + a_4 x_0'' + a_1 x_3'' + a_3 x_1'' + a_2 x_2'' + a_4 x_4'' \end{cases}$$

It is clear that: $x_0' = x_0'' = 0, a_0 = 1$.
Solving the other equations is still a hard problem. It can be solved by using computers.

3. Conclusion

In this paper we studied some of the elementary algebraic properties of 4-cyclic refined neutrosophic matrices with real entries, where we studied the algebraic structure of the multiplication operation and its properties such as associativity, commutativity, and the existence of algebraic multiplication inverse. Also, we illustrated many examples that explain the validity of our work.

DOI: https://doi.org/10.54216/NIF.030102
Received: April 27, 2023 Accepted: January 05, 2024

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DOI: https://doi.org/10.54216/NIF.030102
Received: April 27, 2023 Accepted: January 05, 2024