Finding the complete List of Different K-Brackets for the Projective Plane PG (2, 8)

Khaled Moaz
University of Mosul, department of computer science and mathematics, Mosul, Iraq
Khaledmoaz_m13@gmail.com

Abstract

A k-arc in a plane PG (2, q) is a set of k point such that every line in the plane intersect it in at most two points and there is a line intersect it in exactly two points. A k-arc is complete if there is no k+1-arc containing it. This thesis is concerned with studies a k-arcs, k=4, 5,…., 10 and classification of protectively distinct k-arcs and distinct arcs under collineation. We prove by using computer program that the only complete k-arcs is for, k= 6, 10.

Keywords: k-bracket; Projection; Projective plane; k-arc

1. Introduction

The K-arc is called complete if it is not possible to have an k+1- containing arc, The Scientist Hirschfeid [6] has made a set of studies of K-arcs in the projective plane PG(2, q) defined on the Galois field GF(q) of values q<9. In [10] authors also conducted the first study of the K-arcs in the projective plane PG (2, 16). Recently there have also been several studies of the K-arcs, the researchers Coolsaet [3] and Sticker [4] classified the K-arcs in the projective plane PG (2,q) for q=23,25,27. As the researchers [2] Chao, Kaneta used to classify the k-arcs in the projective plane PG (2, q) for the values 23≤q≤29.

It is worth noting that the study of such a topic covered in this research requires a not short period of time to implement computer programs that were used to find k-different projective arcs and K- different arcs with the effect of straightness.

Example: let $F(x) = x^3 + x + 1$ be an indecomposable polynomial on Z_2, then:

$$\mathrm{GF}(8) = \{\sum_{i=0}^{h-1} a_i w^i \;/\; a_i \in Z_2, w^2 + w^{+1} = 0\}, \mathrm{h} = 3$$

Table 1: The elements of GF (8)

a_0	a_1	a_2	$a_0 + a_1 w + a_2 w^2$
0	0	0	0
0	1	0	w
0	1	1	$w + w^2$
0	0	1	w^2
1	0	0	1
1	1	0	$1 + w$
1	0	1	$1 + w^2$
1	1	1	$1 + w + w^2$

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The elements of GF (8) are:
w is a prime root $GF(8) = \{0,1, w, w^2, w^3, w^4, w^5, w^6 : w^7 = 1\}$

(1-2) [6] definition

If $F(x) = x^n - a_{n-1}x^{n-1} - \ldots - a_0$ is a monomial polynomial, the companion matrix C(F) is a matrix of dimension $n \times n$.

$$C(F) = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots. & 0 \\ 0 & 0 & 1 & 0 & \ldots. & 0 \\ . & . & . & . & \ldots. & . \\ 0 & 0 & 0 & 0 & \ldots. & 1 \\ a_0 & a_1 & a_2 & a_3 & \ldots. & a_{n-1} \end{bmatrix}$$

(1-3) [6] definition

If α' and α are projectively spaces, then the projection $S: \alpha \to \alpha'$is a relation corresponding to (Bijection) represented by the nonsingular Matrix T such that if $P(x') = P(x)S$, then $tx' = xT$, since x and x' are coordinate vectors of the points $P(x)$ and $P(x')$ and $t \in K_0$

(1-4) [6] principle of Duality

For any projectively space S = PG(n, K) there is a space S* called a binomial space in which the points are the primes are respectively the primes and the points in S.

(1-5) [6] Fundamental Theorem of Projective Geometry:

1- If it is $\{p_1, p_2, \ldots, p_{n+2}\}$and $\{p'_1, p'_2, \ldots, p'_{n+2}\}$ two sets of n+2 points in PG(n,k), so that no (n+1) of the selected points from the same set fall into prime, there is a single projection S since $p'_i = p_i S$ and $i \in \{0,1,2, \ldots \ldots . .\}$.

2- Let be A = PG (n, k), the function $S': A \to A$ is called Collination if $S' = \sigma S$, since σ is an auto morphism on A and S is a projection.

(1-6) [6] Cyclic protectively:

Let T be a projection in the plane PG (n,q), then T is called a cyclic projection if all the points of the space can be arranged in only one circle.

Example: the following projection:

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & \lambda^3 \end{bmatrix}$$

Where λ is the root of the field GF (8) of a cyclic projection.

(1-7) [6] definition

An Arc ((k, n) - arc)-(k, n) in space PG (n, q) is a set consisting of k points such that at most n of them are on a line. And there is exactly n of them on the line. When n=2, the (k, 2) -arc is denoted by the k-arc.

(1-8) [10] theorem

Let t (p) represent the number of monomials through the point P, since P is a point of the k-arc with the symbol K. let T_i represent the number of cutters-i.

For a k-arc in the plane, the:

1- t(p) = q + 2 – k = t

2- a. $T_2 = k(k-1)/2$

b. $T_1 = kt$

c. $T_0 = q(q-1)/2 + t(t-1)/2$

(1-9) [6] definition

An (k, n)-arc is said to be complete if there is no (k+1,n)-arc containing it.

(1-10) [5] definition

Quadric is a priority of Rank 2 in the space PG (n, q), so if R = V (F), since F is a quadratic formula, then:

$F = \sum_{i \le j} a_{ij}x_i x_j = a_{00}x_0^2 + a_{01}x_0x_1 + a_{02}x_0x_2 + a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2$

(1-11) [6] definition

A non-singular quadratic in the plane PG (2, q) is called a conic. The general formula of the conic is$\boldsymbol{x_0^2 + x_1x_2}$, so the conic $C = V(x_0^2 + x_1x_2)$contains q + 1 of the points.

(2) Construction and classification of K-arcs in the protectively plane PG (2, 8)

In this chapter, we constructed the k-arcs from 4≤ k ≤10 and found the protectively different arcs and the different arcs with the effect of straightness and also found the effect of the PGL (3, 8) clique on each of these arcs.

(2-2) PG (2, 8) Plane

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The plane PG (2, 8) contains 73 points and 73 lines, each line contains nine points and each point lies on nine lines. If GF (8) is a kalois field of rank (8), then the elements of this field are
$0,1,\lambda,\lambda^2,\lambda^3,\ldots\ldots,\lambda^6:\lambda^7=1$, and the following equation is achieved$\lambda^2+\lambda^3+1=0$, so that:

$$C(f)=\begin{bmatrix}0&1&0\\0&0&1\\1&0&\lambda^3\end{bmatrix}$$

Is the incidence matrix that rotates the 73 points one turn.
Let $P_0=(1,0,0)$, then:

$$P_i=P_{i-1}=\begin{bmatrix}0&1&0\\0&0&1\\1&0&\lambda^3\end{bmatrix}^i, i=1,2,\ldots,72$$

The 73 points that appear are shown in Table 1.

Table 2: level points PG (2, 8)

I	points	I	points	I	Points
0.	(1,0,0)	25.	(1,1, λ)	50.	(1, λ, λ^4)
1.	(0,1,0)	26.	(1, λ^6, λ^5)	51.	(1, λ^3, λ)
2.	(0,0,1)	27.	(1, λ^2, λ^4)	52.	(1, λ^6,1)
3.	(1,0, λ^3)	28.	(1, λ^3, λ^6)	53.	(1,1, λ^5)
4.	(1, λ^4, λ^3)	29.	(1, λ, λ)	54.	(1, λ^2,1)
5.	(1, λ^4, λ^4)	30.	(1, λ^6, λ^2)	55.	(1,1,1)
6.	(1, λ^3, λ^2)	31.	(1, λ^5, λ)	56.	(1,1, λ^2)
7.	(1, λ^5, λ^4)	32.	(1, λ^6, λ)	57.	(1, λ^5, λ^6)
8.	(1, λ^3, λ^4)	33.	(1, λ^6, λ^6)	58.	(1, λ, λ^5)
9.	(1, λ^3, λ^5)	34.	(1, λ, λ^2)	59.	(1, λ^2,0)
10.	(1, λ^2, λ^6)	35.	(1, λ^5, λ^5)	60.	(0,1, λ^2)
11.	(1, λ,0)	36.	(1, λ^2, λ^2)	61.	(1,0, λ^6)
12.	(0,1, λ)	37.	(1, λ^5, λ^2)	62.	(1, λ, λ^3)
13.	(1,0, λ^5)	38.	(1, λ^5,0)	63.	(1, λ^4, λ^6)
14.	(1, λ^2, λ^3)	39.	(0,1, λ^5)	64.	(1, λ, λ^6)
15.	(1, λ^4, λ^5)	40.	(1,0,1)	65.	(1, λ,1)
16.	(1, λ^2, λ^5)	41.	(1,1, λ^3)	66.	(1,1, λ^4)
17.	(1, λ^2, λ)	42.	(1, λ^4, λ)	67.	(1, λ^3,0)
18.	(1, λ^6, λ^4)	43.	(1, λ^6,0)	68.	(0,1, λ^3)
19.	(1, λ^3,1)	44.	(0,1, λ^6)	69.	(1,0, λ)
20.	(1,1,0)	45.	(1,0, λ^4)	70.	(1, λ^6, λ^3)
21.	(0,1,1)	46.	(1, λ^3, λ^3)	71.	(1, λ^4,0)
22.	(1,0, λ^2)	47.	(1, λ^4, λ^2)	72.	(0,1, λ4)
23.	(1, λ^5, λ^3)	48.	(1, λ^5, 1)		
24.	(1, λ^4, 1)	49.	(1,1, λ^6)		

Let L be a line in the plane and we take L is the line at Infinity, that is, if $X_2=0$, the points of this line are:
0 1 11 20 38 43 59 67 71
Since the accompanying matrix C (f) rotates all the points of the plane in one cycle, and since the points and lines are bent one to the other, according to the principle of duality, C (f) rotates the lines in one cycle containing 73 lines
The last line is:
72 0 10 19 37 42 58 66 70

2. Construction

of k-arcs k=4, 5,..., 10, (Construction k-arcs for, k=4,5,..., 10)

Let P_0,P_1,P_2,P_3 be: $P_0=(1,0,0)$ $P_1=(0,1,0),P_2=(0,0,1),P_3=(1,1,1)$the source, which forms the 4-arc according to the basic theorem in projection (1-8), the 4-arc is equivalent to the arc $\{P_0,P_1,P_2,P_3\}$, so the arcs can be constructed from k=5 to k=10 based on the 4-Arc which includes the points $\{P_0,P_1,P_2,P_3\}$ and according to the following algorithm used by the computer program:

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(3-2) The working algorithm used to classify parentheses -K, $4 \le k \le 10$
First: lines with a 2-segment are assigned to the arc $K = \{P_0, P_1, P_2, P_3\}$
Secondly: we find the points that do not lie on the binary lines.
Third: we add these points individually to the bracket to get all the 5-arcs
Fourth: we find the different arcs projection as well as the different arcs by the effect of straightness.
Fifth: we find the cliques that prove these arcs.
Sixth: we repeat these steps from the second to the fifth to find the 6-arcs and 7-arcs and 8-arcs and 9-arcs and 10-arcs.
Seventh: we find the perfect brackets for all steps, if any.
(3-3) K-arc Projectively Distinct and Collnieation Distinct:
(3-3-1) 5-arcs Projectively Distinct and Collnieation Distinct:
To build the 5-arcs we point the four Source points on the plane lines and delete the points of the lines with a two-segment from the points of the plane PG(2.8) and then add the points individually to the source points and we get the 5-arcs.
Using the computer program to find the protectively different 5-arcs, we found that the number of protectively different arcs is one arc whose points are:
(1,0,0) , (0,1,0) , (0,0,1) , (1,1,1) , (1, , λ^4, λ^3)
The clique of this bracket is $C_2 \times C_2$ and generated by:

$$\emptyset_1: (X_0, X_1, X_2) \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \lambda^4 \\ 0 & \lambda^3 & 0 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}$$

$$\emptyset_2: (X_0, X_1, X_2) \rightarrow \begin{bmatrix} 1 & \lambda & \lambda^5 \\ 0 & \lambda^5 & \lambda^5 \\ 0 & \lambda^4 & \lambda^5 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}$$

The transformation $\emptyset_1$ fixes the point (1, 0, 0) and divides the rest of the arc points into two orbits:
First orbit {(0, 1, 0), (0, 0, 1)}
And the second source {(1, 1, 1),$(1, \lambda^4, \lambda^3)$}
The transformation $\emptyset_2$ fixes the point (1, 0, 0) and divides the rest of the arc points into two orbits:
The first orbit {(0, 1, 0), $(1, \lambda^4, \lambda^3)$}
And the second source {(0, 0, 1), (1, 1, 1)}
5-Arc content in the conic:
$V(X_0X_1 + \lambda^4 X_0X_2 + \lambda^6 X_1X_2)$
Let T_0, T_1, T_2be the number of i-incisors of the 5-arc.
t (p) are the single incisors of a point of 5-arc.
According to the theorem (1-12):
T (p)=5 , $T_0 = 38$, $T_1 = 25$, , $T_2 = 10$
Since the number of protectively different 5-arcs is one.
The number of 5-different arcs with the effect of straightening is also one.
(3-3-2) 6-arcs Projectively Distinct and Collnieation Distinct In the same way in (3-3-1) we get five 6- arcs protectively different as in Table number (2).

Table 3: Projectively different 6-arcs

	P_0	P_1	P_2	P_3	P_4	P_5	G	$\lvert G \rvert$
X_0	0	1	2	55	4	8	$C_2 \times C_2$	4
X_2	0	1	2	55	4	9	S_3	6
X_3	0	1	2	55	4	28	S_4	24
X_4	0	1	2	55	4	30	S_4	24
X_5	0	1	2	55	4	50	S_4	24

We found that the arcs X_3, X_4, X_5 are perfect arcs. The invariant group of the X_1 arc is $C_2 \times C_2$ generated from the two transformations:

$$\emptyset_1: (X_0, X_1, X_2) \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}$$

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$$\emptyset_2: (X_0, X_1, X_2) \rightarrow \begin{bmatrix} 1 & \lambda^6 & \lambda^6 \\ 0 & \lambda^3 & \lambda^2 \\ 0 & \lambda^2 & \lambda^3 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}$$

The transformation $\emptyset_1$ fixes the point (1, 0, 0), (1, 1, 1) and divides the rest of the arc points into two orbits:
First orbit {(0, 0, 1), (0, 1,0)}
And the second orbit $\{(1, \lambda^4, \lambda^3), (1, \lambda^3, \lambda^4)\}$
The transformation $\emptyset_2$ fixes the point (1, 0, 0), (1,1,1) and divides the remaining points of the arc into two orbits:
The first orbit $\{(0, 1, 0), (1, \lambda^4, \lambda^3)\}$
And the second orbit $\{(0, 0,1), (1, \lambda^3, \lambda^4)\}$
The conic $V(X_0X_1 + \lambda^4 X_0X_2 + \lambda^6 X_1X_2)$ has five ARC points X_1 which are:
4 55 2 1 0
The Affine clique of the arc X_2 is S_3 generated from:

$$\emptyset_1: (X_0, X_1, X_2) \rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}$$

$$\emptyset_2: (X_0, X_1, X_2) \rightarrow \begin{bmatrix} 1 & \lambda^5 & 1 \\ \lambda^3 & 0 & 1 \\ \lambda^5 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}$$

The transformation $\emptyset_1$ rotates the points in three orbits are:
First orbit {(0,0,1), (1,0,0)}
The second orbit {(0,1,0), (1,1,1)}
And the third orbit $\{(1, \lambda^3, \lambda^5), (1, \lambda^4, \lambda^3)\}$
The transformation $\emptyset_2$ rotates the points in two orbits:
First orbit: $\{(1,0,0), (1, \lambda^3, \lambda^5), (0,1,0)\}$
The second orbit: $\{(0,0,1), (1,1,1), (1, \lambda^4, \lambda^3)\}$
The conic $V(X_0X_1 + \lambda^4 X_0X_2 + \lambda^6 X_1X_2)$ contains the six points of X_2.
We note that the installed clique of arcs X_3 , X_4 , X_5 is S_4, and none of these arcs is completely contained in a single conic.
We used the collineation program to find the different arcs with the straightening effect, so we got only three arcs as shown in the following table:

Table 4: 6-different arcs with the effect of straightening

	P_0	P_1	P_2	P_3	P_4	P_5
Y_1	0	1	2	55	4	8
Y_2	0	1	2	55	4	9
Y_3	0	1	2	55	4	28

The Y_3 arc is a perfect arc.
(3-3-3) 7-arcs Projectively Distinct and Collnieation Distinct:
In the same way in (3-3-1) we obtained two protectively different arcs as in the following table:

Table 5: Projectively different -7 arcs

	P_0	**P_1**	**P_2**	**P_3**	**P_4**	**P_5**	**P_6**	**G**	$\|G\|$
Z_1	0	1	2	55	4	8	16	S_3	6
Z_2	0	1	2	55	4	9	10	D_7	14

The arcs Z_1 ، Z_2 are not complete and from the table above we get the following result:
Result: there is no perfect 7-arc in the plane PG (2, 8)
The arc Z_1 has the clique S_3 generated from the two transformations:

$$\emptyset_1: (X_0, X_1, X_2) \rightarrow \begin{bmatrix} 0 & \lambda^5 & \lambda \\ 0 & \lambda^5 & \lambda^4 \\ 1 & \lambda^5 & \lambda^5 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}$$

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$$\emptyset_2: (X_0, X_1, X_2) \rightarrow \begin{bmatrix} 1 & 0 & \lambda^4 \\ 0 & 0 & \lambda \\ 0 & \lambda^6 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}$$

The transformation $\emptyset_1$ fixes the point (1, 0, 0) and rotates the remaining points in two orbits:
And the first orbit: {(1, 1, 1), $(1, \lambda^2, \lambda^5)$, $(1, \lambda^3, \lambda^4)$}
And the second orbit: {(0,1,0) , (0,0,1) , $(1, \lambda^4, \lambda^3)$}
And the transformation $\emptyset_2$ fixes the point (1, 0, 0) as well and rotates the remaining points in three orbits are:
The first Orbit: {$(1, \lambda^4, \lambda^3)$, $(1, \lambda^2, \lambda^5)$}
The second orbit: {(0,0,1), $(1, \lambda^3, \lambda^4)$}
The third orbit: {(0,1,0), (1,1,1)}
The conic $V(X_0X_1 + \lambda^4 X_0X_2 + \lambda^6 X_1X_2)$ contains five points of Z_1 and two external points.
The Z_2 arc has the D_7 clique generated from the two transformations:

$$\emptyset_1: (X_0, X_1, X_2) \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \lambda^4 \\ 0 & \lambda^3 & 0 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}$$

$$\emptyset_2: (X_0, X_1, X_2) \rightarrow \begin{bmatrix} 0 & \lambda^6 & \lambda^4 \\ \lambda^4 & \lambda & 0 \\ 0 & \lambda^5 & 0 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}$$

The ranks are 2, 7 respectively.
The transformation $\emptyset_1$ fixes the point (1,0,0) and rotates the remaining points in three orbits are:
The first Orbit: {$(1, \lambda^3, \lambda^5)$, $(1, \lambda^2, \lambda^6)$}
The second orbit: {(1,1,1), $(1, \lambda^4, \lambda^3)$}
The third orbit: {(0,1,0), (0,0,1)}
And that the transformation $\emptyset_2$rotates all the points of the 7-Arc by one orbit.
And that the conic $V(X_0X_1 + \lambda^4 X_0X_2 + \lambda^6 X_1X_2)$ contains the seven points of the arc Z_2.
To find the 7-different arcs with the straightening effect, we used the Collineation computer program and got only two arcs, the same arcs Z_2, Z_1

(3-3-4) 8-arcs Projectively Distinct and Collnieation Distinct:
In the same way in (3-3-1) we obtained two protectively different arcs as in the following table:

Table 6: Projectively different 8-arcs

	P0	**P1**	**P2**	**P3**	**P4**	**P5**	**P6**	**P7**	**G**	$\|G\|$
F1	0	1	2	55	4	8	16	32	D4	14
F2	0	1	2	55	4	9	10	18	PGL(2,16)	56

The arcs F_1 ، F_2 are not complete and from the table above we get the following result:
Result: there is no perfect 8-arc in the plane PG (2, 8)
The F_1 bracket has the D_4 clique generated by the two transformations:

$$\emptyset_1: (X_0, X_1, X_2) \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \lambda^6 \\ 0 & \lambda & 0 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}$$

$$\emptyset_2: (X_0, X_1, X_2) \rightarrow \begin{bmatrix} 1 & 1 & \lambda^5 \\ 0 & \lambda^3 & \lambda^2 \\ 0 & \lambda^4 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}$$

The ranks are 2, 7 respectively.
The transformation $\emptyset_1$ fixes the point (1,0,0) , $(1, \lambda^3, \lambda^4)$and rotates the remaining points in three orbits are:
First orbit: {(0,1,0), (0,0,1)}
The second orbit: {(1,1,1), $(1, \lambda^6, \lambda)$}
The third orbit: {$(1, \lambda^4, \lambda^3)$, $(1, \lambda^2, \lambda^5)$}
And that the transformation $\emptyset_2$ fixes the point (1,0,0) as well and rotates the remaining points of the arc by one orbit is: {(0,1,0), $(1, \lambda^3, \lambda^4)$, (0,0,1), $(1, \lambda^4, \lambda^5)$, $(1, \lambda^6, \lambda)$, (1,1,1), $(1, \lambda^2, \lambda^5)$}
The conic $V(X_0X_1 + \lambda^4 X_0X_2 + \lambda^6 X_1X_2)$ has five points of F_1 and three exterior points.
The F_2 arc is composed of Clique PG(2,16).

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And that the conic $V(X_0X_1 + \lambda^4X_0X_2 + \lambda^6X_1X_2)$ contains all the points of the arc F_2
To find the 8- different arcs by the effect of straightness, we used the Collineation computer program, and we got only two arcs, which are the same arcs F_1 , F_2.
(3-3-5) 9-arcs Projectively Distinct and Collnieation Distinct:
In the same way in (3-3-1) we got two protectively different arcs as in the following table.

Table 7: Projectively different 9- arcs

	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	G	$\|G\|$
H_1	0	1	2	55	4	8	16	32	37	PGL(2,16)	56
H_2	0	1	2	55	4	9	10	18	34	PGO(3,16)	504

The arcs H_1 ، H_2 are incomplete and from the table above we get the following result:
Result: there is no perfect 9-arc in the plane PG (2, 8).
And that the conic $V(x_0^2 + x_0x_1 + x_0x_2 + \lambda^5x_1x_2)$ contains five points of the arc H_1 and four points located outside it.
And the conic $V(x_0x_1 + \lambda^4x_0x_2 + \lambda^6x_1x_2)$contains all nine points of the arc H_2.
To find the 9-different arcs with the straightness effect, we used the Collineation computer program and got only two arcs, H_2 ، H_1
(3-3-6) 10-arcs Projectively Distinct and Collnieation Distinct:
The number of protectively different 9-arcs is only two arcs, H_2 ، H_1
We add the points remaining from the deletion of the points that lie on the lines with a biconvex of H2 ، H_1 individually to H_2 ، H_1 and we get only two arcs.
Using the computer program A, we obtained one complete arc, as in the following table:

Table 8: Projectively different 9-arcs

	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	G	$\|G\|$
J	0	1	2	55	4	8	16	32	37	64	PGO(3,16)	504

4. The Results

(4-1) k- arc protectively different

Table No. (3) represents the complete classification of different arcs protectively (k=5,...10) since N_k represents the number of different arcs protectively. Gs represents the k-arc clique. The computer work to obtain these results took (150) computer hours.

Table 9: Obtained results A

K=5 N_k=1		K=6 N_k=3		K=7 N_k =2		K=8 N_k =2		K=9 N_k =2		K=10 N_k =1	
Gs	#	Gs	#	Gs	#	Gs	#	Gs	#	Gs	#
$C_2 \times C_2$	1	$C_2 \times C_2$ S_3 S_4	1 1 3	S_3 D_7	1 1	D_4 PGL (2,16)	1 1	PGL (2,16) PGO(3,16)	1 1	PGL (2,16)	1

(4-2) K-different arcs with straightening effect

Table No. (4) includes the classification of K - different arcs with the effect of straightness, since Nk*represents the number of different arcs with the effect of straightness. Gs represents the k-arc clique.

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Table 10: Obtained results B

K=5 N_k*=1		K=6 N_k*=3		K=7 N_k*=2		K=8 N_k*=2		K=9 N_k*=2		K=10 N_k*=1	
Gs	#	Gs	#	Gs	#	Gs	#	Gs	#	Gs	#
$C_2 \times C_2$	1	$C_2 \times C_2$ S_3 S_4	1 1 1	S_3 D_7	1 1	D_4 PGL (2,16)	1 1	PGL (2,16) PGO(3,16)	1 1	PGL (2,16)	1

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