On Some W-Hosoya polynomials for Several Special Connected Graphs

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Abstract

Let u and v be any two distinct vertices in a connected graph *G*. A container *C*(*u*,*v*) is a set of internally disjoint *u* - *v* paths. The width of *C*(*u*,*v*) is denoted by *w* or *w*(*C*(*u*,*v*)), it is equal to $|C(u,v)|$, and the length of $\ell(C(u,v))$ is the length of the longest *u* – *v* path in *C*(*u*,*v*) . Then, for a given positive integer w, the width distance between any two distinct vertices u and v in a connected graph *G* is define by:

$d_w(u,v) = \min_{C(u,v)} \ell(C(u,v))$, where the minimum is taken over all containers C(u, v) of width w.

In this paper, we find the Hosoya polynomials and Wiener indices of the join of two special graphs such as bipartite complete graphs, paths, cycles, star graphs and wheel graphs with respect to the width distance.

Keywords: Connected graph; Path, Width; Hosoya polynomial

1. Introduction

Let u and v be any two different vertices in a connected statement G, the container between u and v is defined as the set of paths u -v internally separated, denoted by C(u, v). The width of the container C(u,v) is defined as the number of paths u-v in it, thus:

$$\boldsymbol{w = w(C(u,v)) = |C(u,v)|}$$

The length of the container c(u, v) is also defined as the length of the longest path u-v is denoted by the symbol $\ell(C(u,v))$.

Definition: *w*-width distance: for each positive integer and given w, w ≥2 the *w*-width distance between the two different vertices u and v in a connected statement G is defined as:

$$d_w(u,v) = \min_{C(u,v)} \ell(C(u,v))$$

The smallest is taken on all containers C(u,v) with width *w* [2,3].

We note that when *w* =1, the *w*-width distance becomes the usual distance between the vertices u and v. Therefore, we will assume that $w \geq 2$, and then $d_w(u,v) \geq 2$ is for any two different vertices in G. We also note that, *w* does not exceed the connection factor k_0 for the vertices of the statement G, so $2 \leq w \leq k_0$.

To define the diameter and radius of the connected statement G for the *w*-width distance, we will first define the central difference of the *w*-width distance for the vertex v AS [5-8]:

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$e_w(v) = \max_{u \in V(G)} \{d_w(u,v)\}$

The diameter of the w-width distance of the statement G is defined as:

$\delta_w(G) = diam_w G = \max_{v \in V(G)} e_w(v) = \max_{v,u \in V(G)} \{d_w(u,v)\}$

The radius of the w-width distance of the statement G is also defined as:

$r_w G = rad_w G = \min_{v \in V(G)} e_w(v)$

The Hosoya polynomial of the w-width distance is defined as follows:

$$H_w(G;x) = \sum_{k=m_k}^{\delta_w} C_w(G,k)x^k$$

Since:

$$m_w = m_w(G) = \min_{u,v \in V(G)} \{d_w(u,v)\}$$

Wiener's proof of the w-width distance is defined as the sum of the w-width distance in the statement G, thus:

$$W_w(G) = \sum_{u,v \in V(G)} d_w(u,v|G)$$

It is clear that:

$$W_w(G) = \frac{d}{dx} H_w(G;x)|_{x=1} = \sum_{k=m_w}^{\delta_w} kC_w(G,k)$$

Let v be a vertex in the connected statement, the Hosoya polynomial of w-width distance of the vertex v is defined as follows [1]:

$$H_w(v,G;x) = \sum_{k \geq m_w}^{\delta_w} C_w(v,G,k)x^k$$

Since $C_w(v,G,k)$represents the number of vertices of the statement G, each of which is a w-width distance equal to k from the vertex v, It is clear that:

$$\sum_{v \in V} H_w(v,G;x) = 2H_w(G;x)$$

Let G_1 and G_2 be two connected and separate statements for vertices, the connection of the statements G_1 and G_2, denoted by $G_1 + G_2$ is defined as the statement whose set of vertices is $V(G_1) \cup V(G_2)$ and its set of edges is:

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv: u \in V(G_1), v \in V(G_2)\}$$

For some of the concepts and terms contained in this research, you can refer to the source [4]. It is obvious that the connection of any two complete statements is a complete statement, so there is no need to find a hosoya-w his polynomial.

2. Hosoya-w polynomial for the connection of two complete binary hash graphs

Let K_{p_1,p_2} and K_{p_3,p_4} be two complete binary hash statements of rank $p_1 + p_2$and $p_3 + p_4$, respectively.

Let $p_1 \leq p_2$and $p_3 \leq p_4$ then the Connection Factor [4] for $K_{p_1,p_2} + K_{p_3,p_4}$ is $k_0 = p_1 + p_3 + min\{p_2, p_4\}$, we assume that:

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$$V(K_{p_1,p_2}) = V_{P_1} \cup V_{P_2}$$

$$V(K_{p_3,p_4}) = V_{P_3} \cup V_{P_4}$$

Where V_{P_1} and V_{P_2} are hash sets of statement headers K_{p_1,p_2}, and V_{P_3} and V_{P_4} are hash sets of statement headers K_{p_3,p_4}. For convenience, we will encode the statement $K_{p_1,p_2} + K_{p_3,p_4}$ with the symbol G_b.

To find the Hosoya- w polynomial for the statement G_b we will take three cases to illustrate the proof. We will assume that $P_2 \leq P_3$, so the connection factor for this statement is $k_0(G_b) = P_1 + P_2 + P_3$.

The first case: if $u, v \in V_{P_i}$, for each *i=1,2,3,4*, then there are $\sum_{j=1 \ \ i \neq j}^{4} p_j$ of internally separated paths each of length 2. Therefore:

$$d_w(u, v) = 2 , \forall \ \ 2 \leq w \leq p_1 + p_2 + p_3$$

The second Case:

1. If $\boldsymbol{u} \in V_{P_1}$ and $\boldsymbol{v} \in V_{P_2}$, then there is a path of length 1 and $P_3 + P_4$ of internally separated paths each of length 2 and $p_1 - 1$ of internally separated paths each of length 3. Therefore:

$$d_w(u, v) = 2 , \forall \ \ 2 \leq w \leq P_3 + P_4 + 1$$

$$d_w(u, v) = 3 , \forall \ \ P_3 + P_4 + 2 \leq w \leq p_1 + P_3 + P_4$$

2.If $u \in V_{P_3}$ and $v \in V_{P_4}$, then there is a path of length 1 and $P_1 + P_2$ of internally separated paths each of length 2 and $p_3 - 1$ of internally separated paths each of length 3. Therefore:

$$d_w(u, v) = 2 , \forall \ \ 2 \leq w \leq P_1 + P_2 + 1$$

$$d_w(u, v) = 3 , \forall \ \ P_1 + P_2 + 2 \leq w \leq p_1 + P_2 + P_3$$

The third case:

1. If $u \in V_{P_1}$ and $v \in V_{P_3}$, then there is a path of length 1 and $P_2 + P_4$ of internally separated paths each of length 2 and $p_1 - 1$ of internally separated paths each of length 3. Therefore:

$$d_w(u, v) = 2 , \forall \ \ 2 \leq w \leq P_2 + P_4 + 1$$

$$d_w(u, v) = 3 , \forall \ \ P_2 + P_4 + 2 \leq w \leq p_1 + P_2 + P_4$$

2.If $u \in V_{P_1}$ and $v \in V_{P_4}$, then there is a path of length 1 and $P_2 + P_3$ of internally separated paths each of length 2 and $p_1 - 1$ of internally separated paths each of length 3. Therefore:

$$d_w(u, v) = 2 , \forall \ \ 2 \leq w \leq P_2 + P_3 + 1$$

$$d_w(u, v) = 3 , \forall \ \ P_2 + P_3 + 2 \leq w \leq p_1 + P_2 + P_3$$

3.If $u \in V_{P_2}$ and $v \in V_{P_3}$, then there is a path of length 1 and $P_1 + P_4$ of internally separated paths each of length 2 and $p_2 - 1$ of internally separated paths each of length 3. Therefore:

$$d_w(u, v) = 2 , \forall \ \ 2 \leq w \leq P_1 + P_4 + 1$$

$$d_w(u, v) = 3 , \forall \ \ P_1 + P_4 + 2 \leq w \leq p_1 + P_2 + P_4$$

4.If $u \in V_{P_2}$ and $v \in V_{P_4}$, then there is a path of length 1 and $P_1 + P_3$ of internally separated paths each of length 2 and $p_2 - 1$ of internally separated paths each of length 3. Therefore:

$$d_w(u, v) = 2 , \forall \ \ 2 \leq w \leq P_1 + P_3 + 1$$

$$d_w(u, v) = 3 , \forall \ \ P_1 + P_3 + 2 \leq w \leq p_1 + P_2 + P_3$$

From the previous three cases we can obtain the following theorem:

Theorem 1: let $p = P_1 + P_2 + P_3 + P_4$ and $N = \sum_{i=1}^{4} \binom{p_i}{2}$, then:

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1. $H_w(G_b;x) = \frac{1}{2}p(p-1)x^2$, $2 \le w \le p_1 + p_2 + 1$
2. $H_w(G_b;x) = \{\frac{1}{2}p(p-1) - P_3P_4\}x^2 + P_3P_4x^3, p_1 + p_2 + 2 \le w \le p_1 + p_3 + 1$
3. $H_w(G_b;x) = \{\frac{1}{2}p(p-1) - (P_2 + P_3)\}x^2 + (P_2 + P_3)P_4x^3$

If $p_1 + p_4 < p_2 + p_3$ and $p_1 + p_3 + 2 \le w \le p_1 + p_4 + 1$

Or $p_1 + p_4 > p_2 + p_3$ and $p_1 + p_3 + 2 \le w \le p_2 + p_3 + 1$

4. $H_w(G_b;x) = \{\frac{1}{2}p(p-1) - P_2P_3 - (P_2 + P_3)P_4\}x^2 + \{P_2P_3 + (P_2 + P_3)P_4\}x^3$,

If $p_1 + p_4 < p_2 + p_3$ and $p_1 + p_4 + 2 \le w \le p_2 + p_3 + 1$

$$p_1 + p_4 + 2 \le w \le p_2 + p_3 + 1$$

$$H_w(G_b;x) = \{\frac{1}{2}p(p-1) - (P_1 + P_2 + P_3)P_4\}x^2 + (P_1 + P_2 + P_3)P_4x^3$$

If $p_1 + p_4 > p_2 + p_3$ and $p_2 + p_3 + 2 \le w \le p_1 + p_4 + 1$

5. $H_w(G_b;x) = \{N + P_1(P_2 + P_3)\}x^2 + \{p_4(P_1 + P_2 + P_3) + P_2P_3\}x^3$

If $p_1 + p_4 < p_2 + p_3$ and $p_2 + p_3 + 2 \le w \le p_2 + p_4 + 1$

Or $p_1 + p_4 > p_2 + p_3$ and $p_1 + p_4 + 2 \le w \le p_2 + p_4 + 1$

6. $H_w(G_b;x) = \{N + P_1P_2\}x^2 + \{(p_4 + P_3)(P_1 + P_2) + P_3P_4)\}x^3$

$$p_2 + p_4 + 2 \le w \le p_3 + p_4 + 1$$

7. $H_w(G_b;x) = NX^2 + \{\sum_{i=1}^{3}\sum_{j=i+1}^{4} P_iP_j\}x^3$.

   $p_3 + p_4 + 2 \le w \le p_1 + p_2 + P_3$ #

Result 1.1: let $p = p_1 + p_2 + p_3 + p_4$, then:

1. $H_w(G_b) = p(p-1)$, $2 \le w \le p_1 + p_2 + 1$
2. $H_w(G_b) = p(p-1) + P_3P_4$, $p_1 + p_2 + 2 \le w \le p_1 + p_3 + 1$
3. $H_w(G_b) = p\,(p-1) + (P_2 + P_3)P_4\}$

If $p_1 + p_4 < p_2 + p_3$ and $p_1 + p_3 + 2 \le w \le p_1 + p_4 + 1$

Or $p_1 + p_4 > p_2 + p_3$ and $p_1 + p_3 + 2 \le w \le p_2 + p_3 + 1$

4. $H_w(G_b) = p\,(p-1) + P_2P_3 + (P_2 + P_3)P_4$

If $p_1 + p_4 < p_2 + p_3$ and $p_1 + p_4 + 2 \le w \le p_2 + p_3 + 1$

$$H_w(G_b) = p\,(p-1) + (p_1 + P_2 + P_3)P_4$$

If $p_1 + p_4 > p_2 + p_3$ and $p_2 + p_3 + 2 \le w \le p_1 + p_4 + 1$

5. $H_w(G_b) = p\,(p-1) + (P_1 + P_2 + P_3)P_4 + P_2P_3$

If $p_1 + p_4 < p_2 + p_3$ and $p_2 + p_3 + 2 \le w \le p_2 + p_4 + 1$

Or $p_1 + p_4 > p_2 + p_3$ and $p_1 + p_4 + 2 \le w \le p_2 + p_4 + 1$

6. $H_w(G_b) = p\,(p-1) + (P_4 + P_3) + (P_1 + P_2) + P_3P_4$

$$p_2 + p_4 + 2 \le w \le p_3 + p_4 + 1$$

7. $H_w(G_b) = p\,(p-1) + \sum_{i=1}^{3}\sum_{j=i+1}^{4} P_iP_j$.

   $p_3 + p_4 + 2 \le w \le p_1 + p_2 + P_3$ #

Result 1.2: if $p_1 = p_2 = p_3 = p_4 = m$, then:

$$H_w(G_b;x) = \begin{cases} 2m(4m-1)x^2 & , \quad 2 \le w \le 2m+1 \\ 2m(m-1)x^2 + 6m^2x^3 & , \quad 2(m+1) \le w \le 2m \end{cases} \quad \ldots(3.2.1)$$

Proof: to illustrate the proof, we take only two cases:

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The first case: if $u, v \in V_{P_i}$ for each *i=1,2,3,4*, then there are $3m$ internally separated paths each of length 2. Therefore:

$$d_w(u, v) = 2 , \forall \; 2 \leq w \leq 3m$$

The second case: if $u \in V_{P_i}$ and $v \in V_{P_j}$, for each $i \neq j$ where *i=1,2,3* and *j=i+1*, there is a path of length 1 and 2m of internally separated paths each of length 2 and m-1 of internally separated paths each of length 3. Therefore:

$$d_w(u, v) = 2 , \forall \; 2 \leq w \leq 2m + 1$$

$$d_w(u, v) = 3 , \forall \; 2(m + 1) \leq w \leq 3m$$

From the above two cases it is possible to obtain the formula (3.2.1).

Result 1.3: if $p_1 = p_2 = p_3 = p_4 = m$, then:

$$H_w(G_b) = \begin{cases} 4m(4m - 1) & , \; 2 \leq w \leq 2m + 1 \\ 2m(11m - 2) & , \; 2(m + 1) \leq w \leq 3m \end{cases} \qquad \#$$

3. Hosoya-w polynomial for two-Path connection

Let p_{p_1} and p_{p_2} be two separate paths of rank p_1 and p_2, respectively, since $p_1 \leq p_2$. Let $G_p = p_{p_1} + p_{p_2}$ then the connection operator of the statement G_p is $k_0(G_p) = p_1 + 1$, assuming that $V(G_p) = V(p_{p_1}) \cup V(p_{p_2})$ since $V(p_{p_1})$ and $V(p_{p_2})$ are the sets of vertices p_{p_1} and p_{p_2} respectively which are:

$$V(p_{p_1}) = \{u_1, u_2, u_3, \ldots., u_{p_1}\}$$

$$V(p_{p_2}) = \{v_1, v_2, v_3, \ldots., v_{p_2}\}$$

To find the Hosoya polynomial we take the following cases:

The first case:

1. If $u, u' \in V(p_{p_1})$, then there are p_2 internally separated paths each with length 2. In addition to one path with length k, where $1 \leq k \leq p_1 - 1$ ،$d(u, u') = k$. Therefore:

$$d_W(u, u') = 2 \;, \qquad \forall \; 2 \leq w \leq P_1 + 1 \,, P_1 < P_2$$

$$d_{P_1+1}(u, u') = 2 \;, k = 1{,}2 \,, \qquad P_1 = P_2$$

$$d_{P_1+1}(u, u') = 2 \;, 3 \leq k \leq P_1 - 1 \,, P_1 = P_2$$

2. If $v, v' \in V(p_{p_2})$, then there are p_1 internally separated paths each with length 2. In addition to a path with length k where $1 \leq k \leq p_2 - 1$ ،$d(v, v') = k$. Therefore:

$$d_W(v, v') = 2 \;, \qquad \forall \; 2 \leq w \leq P_1$$

$$d_{P_1+1}(v, v') = 2 \;, k = 1{,}2$$

$$d_{P_1+1}(v, v') = k \;, 3 \leq k \leq P_2 - 1$$

The second case:

1. If $u_i \in V(p_{p_1})$ ،$v_j \in V(p_{p_2})$،, since $i = 2,3, \ldots, p_1 - 1 \,, j = 2,3, \ldots, p_2 - 1$, there is a trail of length 1, four internally separated trails of length 2 each plus $p_1 - 3$ internally separated trails of length 3 each. Then:

$$d_W(u_i, v_j) = 2 \;, \qquad \forall \; 2 \leq w \leq 5$$

$$d_W(u_i, v_j) = 3 \;, \qquad \forall \; 6 \leq w \leq P_1 + 1$$

2. If $u_i \in V(p_{p_1})$ ،$v_j \in V(p_{p_2})$, since $i = 1, p_1$ and $j = 1, p_2$, then there is a path of length 1, Two Paths of length 2 and $p_1 - 2$ of the paths of length 3, so that:

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$$d_W(u_i, v_j) = 2 \ , \qquad w = 2,3$$

$$d_W(u_i, v_j) = 3 \ , \qquad \forall\, 4 \le w \le P_1 + 1$$

3. If $u_i \in V(p_{p_1})$ ،$v_j \in V(p_{p_2})$, since $i = 1, p_1$ ، $j = 2,3, \dots, p_2 - 1$, then there is a path of length 1 and three paths of length 2 and $p_1 - 2$ of paths of length 3 if $p_1 < p_2$ (or $p_1 - 3$ of paths of length 3 if $p_1 = p_2$). Therefore:

$$d_W(u_i, v_j) = 2 \ , \qquad \forall\, 2 \le w \le 4$$

$$d_W(u_i, v_j) = 3 \ , \qquad \forall\, 5 \le w \le P_1 + 1$$

4. If $u_i \in V(p_{p_1})$ ،$v_j \in V(p_{p_2})$, since $i = 2,3, \dots, p_1 - 1$ and $j = 1, p_2$, there is a path of length 1, three paths of length 2 and $p_1 - 3$ of the paths of length 3. Therefore:

$$d_W(u_i, v_j) = 2 \ , \qquad \forall\, 2 \le w \le 4$$

$$d_W(u_i, v_j) = 3 \ , \qquad \forall\, 5 \le w \le P_1 + 1$$

From the above cases we obtain the following theorem:

Theorem 2: for every $p_1 < p_2$ where $p_1 \ge 6$ and $w \le p_1 + 1$, then:

$$H_w(G_P; x) = \left(\binom{p_1}{2} + \binom{p_2}{2} + p_1 p_2\right) x^2, w = 2,3$$

$$H_4(G_P; x) = \left(\binom{p_1}{2} + \binom{p_2}{2} + p_1 p_2 - 2\right) x^2 + 2x^3$$

$$H_5(G_P; x) = \left(\binom{p_1}{2} + \binom{p_2}{2} + p_1(p_2 - 2)\right) x^2 + 2p_1 x^3$$

$$H_w(G_P; x) = \left(\binom{p_1}{2} + \binom{p_2}{2}\right) x^2 + p_1 p_2 x^3, 6 \le w \le p_1$$

$$H_{p_1+1}(G_P; x) = \left(\binom{p_1}{2} + 2p_2 - 3\right) x^2 + p_1 p_2 x^3 + \sum_{k=3}^{p_2-1} (p_2 - k) x^k \quad \#$$

It is easy to obtain a formula similar to theorem 1.3 when $p_1 = p_2 = m.$

Result 2.1: for every $p_1 < p_2$ and $w \le p_1 + 1$, then:

$$H_w(G_P) = 2\left(\binom{p_1}{2} + \binom{p_2}{2} + p_1 p_2\right), w = 2,3$$

$$H_4(G_P) = 2\left(\binom{p_1}{2} + \binom{p_2}{2} + p_1 p_2 + 1\right)$$

$$H_5(G_P) = 2\left(\binom{p_1}{2} + \binom{p_2}{2} + p_1(p_2 + 1)\right)$$

$$H_w(G_P) = 2\left(\binom{p_1}{2} + \binom{p_2}{2}\right) + 3p_1 p_2, 6 \le w \le p_1$$

$$H_{p_1+1}(G_P) = 2\binom{p_1}{2} + \binom{p_2 + 1}{3} + 3p_1 p_2 + p_2 + 1 \qquad \#$$

4. Polynomials Hosoya-w for two-circuit connection

Let $C_{p_1}$and C_{p_2} be two separate paths of rank p_1 and p_2, respectively, since $p_1 \le p_2$.. Let $G_C = C_{p_1} + C_{p_2}$ then the connection operator of the G_C statement is $k_0(G_p) = p_1 + 2$, assuming that $V(G_C) = V(C_{p_1}) \cup V(C_{p_2})$. In order to find the Hosoya-w polynomial for $G_C$we take the following cases:

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The first case:

1. If $u, v \in V(C_{p_1})$ and $d(u, v|C_{p_1}) = k$, then there are two internally separated paths of length k and $p_1 - k$, since $1 \leq k \leq \left\lfloor\frac{p_1}{2}\right\rfloor$ and p_2 are internally separated paths of length 2 each. Therefore:

$$d_w(u, v) = 2, \forall 2 \leq w \leq p_1 + 2, \qquad if\ p_1 + 2 \leq p_2$$

$$d_{p_1+2}(u, v) = 2, k = 1,2, \qquad if\ p_1 + 1 = p_2\ or\ p_1 = p_2$$

$$d_{p_1+2}(u, v) = 3 \leq k \leq \left\lfloor\frac{p_1}{2}\right\rfloor\ \ if\ p_1 + 1 = p_2$$

$$d_{p_1+2}(u, v) = p_1 - k\ , 3 \leq k \leq \left\lfloor\frac{p_1}{2}\right\rfloor\ \ if\ p_1 = p_2$$

2. If $u, v \in V(C_{p_2})$ and $d(u, v|C_{p_2}) = k$, then there are two paths, one with length k and the other $p_2 - k$, since $1 \leq k \leq \left\lfloor\frac{p_2}{2}\right\rfloor$ and p_1 are internally separated paths, each with length 2. Therefore:

$$d_w(u, v) = 2, \forall 2 \leq w \leq p_1$$

$$d_{p_1+1}(u, v) = 2, k = 1,2, d_{p_1+1}(u, v) = k, 3 \leq k \leq \left\lfloor\frac{p_2}{2}\right\rfloor$$

$$d_{p_1+2}(u, v) = 2, k = 1,2, d_{p_1+2}(u, v) = p_2 - k, 1 \leq k \leq \left\lfloor\frac{p_2}{2}\right\rfloor$$

The second case: if $u \in V(C_{p_1})$ ،$v \in V(C_{p_2})$, then there is a path of length 1 and four internally separated paths of length 2 Plus $p_1 - 3$ internally separated paths each of length 3. Therefore:

$$d_w(u, v) = 2\ , \qquad 2 \leq w \leq 5$$

$$d_w(u, v) = 3\ , \qquad \forall 6 \leq w \leq p_1 + 2$$

From the first and Second cases we get:

Theorem 3: for every $p_1 + 2 < p_2$ where $p_1 \geq 6$, then:

$$H_w(G_C; x) = \left(\binom{p_1}{2} + \binom{p_2}{2} + p_1 p_2\right) x^2, \forall 2 \leq\ w \leq 5$$

$$H_w(G_C; x) = \left(\binom{p_1}{2} + \binom{p_2}{2}\right) x^2 + p_1 p_2 x^3, \forall 6 \leq\ w \leq p_1$$

$$H_{p_1+1}(G_C; x) = \left(\binom{p_1}{2} + \binom{p_2}{2} + 2p_2\right) x^2 + p_1 p_2 x^3 + p_2 \sum_{k=3}^{\left\lfloor\frac{p_2}{2}\right\rfloor - 1} x^k + \begin{cases} p_2 x^{\frac{p_2-1}{2}}, p_2\ odd \\ \frac{p_2}{2} x^{\frac{p_2}{2}}, p_2\ even \end{cases}$$

$$H_{p_1+2}(G_C; x) = \left(\binom{p_1}{2} + \binom{p_2}{2} + 2p_2\right) x^2 + p_1 p_2 x^3 + p_2 \sum_{k=3}^{\left\lfloor\frac{p_2}{2}\right\rfloor - 1} x^{p_2-k} + \begin{cases} p_2 x^{\frac{p_2+1}{2}}, p_2\ odd \\ \frac{p_2}{2} x^{\frac{p_2}{2}}, p_2\ even \end{cases} \qquad \#$$

Approximate formulas of the 1.4 theorem can be obtained when $p_1 + 1 = p_2$ and $p_1 = p_2$from the previous cases.

Result 3.1: for every $p_1 + 1 \leq p_2$ where $p_1 \geq 6$, then:

$$H_w(G_C) = 2\left(\binom{p_1}{2} + \binom{p_2}{2} + p_1 p_2\right), \forall 2 \leq\ w \leq 5$$

$$H_w(G_C) = 2\left(\binom{p_1}{2} + \binom{p_2}{2}\right) + 3 p_1 p_2\ , \forall 6 \leq\ w \leq p_1$$

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$$H_{p_1+1}(G_C) = 2\left(\binom{p_1}{2} + \binom{p_2}{2} + \binom{\left\lceil \frac{p_2}{2} \right\rceil}{2}\right) + p_2(3p_1 + 1) + \begin{cases} \frac{p_2(p_2-1)}{2} & , p_2\ odd \\ \binom{p_2}{2} & , p_2\ even \end{cases}$$

$$H_{p_1+2}(G_C) = 2\left(\binom{p_1}{2} + \binom{p_2}{2} + 2p_2\right) + p_2\left(\binom{\left\lceil \frac{p_2}{2} \right\rceil}{2}\left(p_2 - \frac{2}{3}\left(\left\lceil \frac{p_2}{2} \right\rceil - 1\right)\right) + (5 - 3p_2)\right) + 3p_1p_2 + \begin{cases} \frac{p_2(p_2+1)}{2} & , p_2\ odd0 \\ \binom{p_2}{2}^2 & , p_2\ even \end{cases}$$

#

5. Hosoya-w polynomial for two-star connection:

Let S_{p_1} and S_{p_2}be two separate stars of rank p_1 and p_2 , respectively, where, $p_1 \leq p_2$ let: $G_S = S_{p_1} + S_{p_2}$ then the connection operator of the G_S statement is $k_0(G_S) = p_1 + 1$, assuming that $V(G_S) = V(S_{p_1}) \cup V(S_{p_2})$. In order to find the Hosoya-w polynomial for G_S we assume that u' is the vertex adjacent to all vertices in the star $S_{p_1}$and that v' is the vertex adjacent to all vertices in the star S_{p_2}then we have the following cases:

The first case:

1. If $u, v \in V(S_{p_1})$and neither Vertex represents u', then we notice that there are $p_2 + 1$ internally separated paths connecting u and v, each with a length of 2.when one of the two vertices represents u', we find that there is a path of length one and p_2 of internally separated paths each of length 2. then we get:

$$d_w(u, v) = 2, \forall 2 \leq w \leq p_1 + 1$$

2. If $u, v \in V(S_{p_2})$and neither Vertex represents v', then we observe that there are $p_1 + 1$ internally separated paths connecting U and v and each of them with a length 2.When one of the two vertices represents v', we find that there is a path of length one and p_1 of internally separated paths each of length 2. then we get:

$$d_w(u, v) = 2, \forall 2 \leq w \leq p_1 + 1$$

The second case:

1. If $u \in V(S_{p_1}) - \{u'\}$ and $v \in V(S_{p_2}) - \{v'\}$, then there is a path of length 1, Two Paths of length 2, and $p_1 - 2$ of internally separated paths, each of length 3, so that:

$$d_w(u, v) = 2 \quad , \qquad w = 2,3$$

$$d_w(u, v) = 3, \forall 4 \leq w \leq p_1 + 1$$

2. If $v \in V(S_{p_2}) - \{v'\}$ and u=u', then there is a path of length 1 and p_1 of internally separated paths of length 2, so:

$$d_w(u', v) = 2, \qquad 2 \leq w \leq p_1 + 1$$

3. If $u \in V(S_{p_1})$and v=v', then there is a path of length 1 and p_2 of internally separated paths each of length 2, so:

$$d_w(u', v) = 2, \qquad 2 \leq w \leq p_1 + 1$$

4. If u=u' and v=v', then there is a path of length 1 and $p_1 + p_2 - 2$ internally separated paths each of length 2, then:

$$d_w(u', v') = 2, \qquad \forall\ 2 \leq w \leq p_1 + 1$$

From the two previous cases we obtain the following theorem:

Theorem 4: for every $p_1 \leq p_2$ then:

$$H_w(G_S; x) = \left(\binom{p_1}{2} + \binom{p_2}{2} + p_1p_2\right)x^2, w = 2,3$$

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$H_w(G_S;x) = \left(\binom{p_1}{2} + \binom{p_2}{2} + (p_1+p_2-1)\right)x^2 + (p_1-1)(p_2-1)x^3, \forall\, 4 \le w \le p_1+1$
#

Result 4.1: for every $p_1 \le p_2$, then:

$$H_w(G_S) = 2\left(\binom{p_1}{2} + \binom{p_2}{2} + p_1p_2\right), w = 2,3$$

$H_w(G_S) = 2\left(\binom{p_1}{2} + \binom{p_2}{2} + p_1p_2\right) + (p_1-1)(p_2-1), \forall\, 4 \le w \le p_1+1$ #

6. Hosoya-w polynomial for two-wheel connection:

Let W_{p_1} and W_{p_2} be two wheels of rank p_1 and p_2, respectively, and that:

$$V(W_{p_1}) = \{w_1, w_2, \dots, w_{p_1-1}, c_1\}, deg\; c_1 = p_1 - 1$$

$$V(W_{p_2}) = \{y_1, y_2, \dots, y_{p_2-1}, c_2\}, deg\; c_2 = p_2 - 1$$

Obviously, the connection operator of the statement $G_W = W_{p_1} + W_{p_2}$ is $k_0(G_W) = \min\{p_1, p_2\} + 3$

First:

If $u, v \in W_{p_i} - \{c_i\}$ for each *i=1,2*, then there is a path of length k, and another path of length $p_i - 1 - k$, since $1 \le k \le \left\lfloor \frac{p_i-1}{2} \right\rfloor$, and $p_{3-i} + 1$ internally separated paths each of length 2. Therefore:

$$d_w(u,v) = 2, \forall\; 2 \le w \le p_{3-i} + 1$$

$$d_{p_{3-i}+2}(u,v) = 2, k = 1,2, d_{p_{3-i}+2}(u,v) = k, 3 \le k \le \left\lfloor \frac{p_i-1}{2} \right\rfloor$$

$$d_{p_{3-i}+3}(u,v) = 2, k = 1,2, d_{p_{3-i}+3}(u,v) = p_i - 1 - k, 3 \le k \le \left\lfloor \frac{p_i-1}{2} \right\rfloor$$

If $u \in W_{p_i} - \{c_i\}$ and $v = c_i$, for every *i=1,2*, there is a path of length one and $p_{3-i} + 2$ internally separated paths of length 2 each. Therefore:

$$d_w(u,v) = 2, \forall\; 2 \le w \le p_{3-i} + 3, i = 1,2$$

Second:

1. If $u \in V(W_{p_1}) - \{c_1\}$ and $v \in V(W_{p_2}) - \{c_2\}$, then there is a path of length one and 6 internally separated paths of length 2 each, then:

$$d_w(u,v) = 2, \forall\; 2 \le w \le 7$$

2. If $u = c_i$ and $v \in V(W_{p_{3-i}}) - \{c_{3-i}\}$ for each *i=1,2*, then there is a path of length one and $p_i + 2$ internally separated paths each of length 2, then:

$$d_w(c_i,v) = 2, \forall\; 2 \le w \le p_i + 3, i = 1,2$$

3. If $u = c_1$ and $v = c_2$, then there is a path of length one and $p_1 + p_2 - 2$ internally separated paths each of length 2, then:

$$d_w(c_1,c_2) = 2, \forall\; 2 \le w \le p_1 + p_2 - 2$$

From the two previous cases we can obtain the following theorem:

Theorem 5: for every $p_1 \le p_2$ and for every $p_1 \ge 6$, then:

- $H_w(G_W;x) = \frac{1}{2}\{p_1(p_1-1) + p_2(p_2-1) + 2p_1p_2\}x^2, if\; 2 \le w \le 7$

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- $H_w(G_W;x) = \frac{1}{2}\{p_1(p_1-1) + p_2(p_2-1) + 2(p_1+p_2-1)\}x^2 \text{ , } if\ 8 \le w \le p_1+1$
- $H_{p_1+2}(G_W;x) = \binom{p_1-1}{2}\sum_{k=2}^{\left\lfloor\frac{p_1-1}{2}\right\rfloor} x^k + \binom{p_2-1}{2}\sum_{k=2}^{\left\lfloor\frac{p_2-1}{2}\right\rfloor} x^k + \{p_1+p_2-1\}x^2$
- $H_{p_1+3}(G_W;x) = \binom{p_1-1}{2}\sum_{k=2}^{\left\lfloor\frac{p_1-1}{2}\right\rfloor} x^{p_1-1-k} + \binom{p_2-1}{2}\sum_{k=2}^{\left\lfloor\frac{p_2-1}{2}\right\rfloor} x^{p_2-1-k} + \{p_1+p_2-1\}x^2$

Result 5.1: for every $p_1 \le p_2$ and for every $p_1 \ge 6$, then:

- $H_w(G_W) = \{p_1(p_1-1) + p_2(p_2-1) + 2p_1p_2\} \text{ , } if\ 2 \le w \le 7$
- $H_w(G_W) = \{p_1(p_1-1) + p_2(p_2-1) + 2(p_1+p_2-1)\} \text{ , } if\ 8 \le w \le p_1+1$
- $H_{p_1+2}(G_W) = \frac{1}{2}\binom{p_1-1}{2}\left\lfloor\frac{p_1-1}{2}\right\rfloor\left\lfloor\frac{p_1+1}{2}\right\rfloor + \binom{p_2-1}{2}\left\lfloor\frac{p_2-1}{2}\right\rfloor\left\lfloor\frac{p_2+1}{2}\right\rfloor + 2\{p_1+p_2-1\}$
- $H_{p_1+3}(G_W) = \frac{1}{2}\binom{p_1-1}{2}\left\{2(p_1-1)\left\lfloor\frac{p_1-3}{2}\right\rfloor - \left\lfloor\frac{p_1-1}{2}\right\rfloor\left\lfloor\frac{p_1+1}{2}\right\rfloor + 2\right\} + \frac{1}{2}\binom{p_2-1}{2}\left\{2(p_2-1)\left\lfloor\frac{p_2-3}{2}\right\rfloor - \left\lfloor\frac{p_2-1}{2}\right\rfloor\left\lfloor\frac{p_2+1}{2}\right\rfloor + 2\right\} + 2\{p_1+p_2-1\}$

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