A note on Category of SuperHyper BCI-Algebra

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Abstract

In this paper, we have put forward the SH-homo and $0-$preserving SH-homo to compare two SH-groupoids along with few examples. Some properties of SH-homo and $0-$preserving SH-homo are explored. Also, we proved the SH-homo between two SuperHyper BCI-Algebra is $0-$preserving SH-homo. Finally, the category of SuperHyper Groupoid, SuperHyper BCI-Algebra and Neutrosophic SuperHyper BCI-Algebra were investigated.

Keywords: Category; SuperHyper operation; SuperHyper homomorphism; Neutrosophic SuperHyper homomorphism

1 Introduction

S. Eilenberg and S. Mac Lane[3] initially studied categories in 1945 specifically to design the bridge between two quite different fields topology and algebra. Eilenberg and Mac Lane developed a framework applicable to topology, algebra, and other mathematical disciplines by precisely comparing links between these fields. Across the years, category theory[1,7] has developed into a global language that enables mathematicians working in apparently unrelated fields to exchange ideas and techniques and make significant advances. In modern times, category theory is a vital research tool in many fields of pure mathematics, including algebraic geometry, algebraic topology, homological algebra, and functional analysis. It additionally finds applications in theoretical computer science, electrical circuits, chemical interactions, biological systems, and medicine.[5]

Fuzzy set[19] is a powerful tool to handle uncertainty. Fuzzy set theory has been approached via the perspective of category theory. Objects in this category are fuzzy sets, and functions that preserve the fuzzy

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membership structure are called morphisms.[17]Category of fuzzy sets and fuzzy topological spaces were explored in.[15,18] Sugeno and Sasaki[16] established the notion of the functor and L-fuzzy category. Further various researches have been done in fuzzy category.[4,6,9] As a generalization of fuzzy set, Attanasov[2] initiated the notion of intuitionistic fuzzy set(IFS). Later, Lee et al.,[8] proposed the category theory in intuitionistic fuzzy topological spaces. Also, the properties of category theory have been applied in intuitionistic fuzzy graphs.[10] The categorical approach in the analysis of intuitionistic fuzzy modules over a commutative ring was explored by Poonam Kumar Sharma et al.,.[12]

The development of neutrosophic logic by Smarandache[13] is a crucial component in handling indeterminacy. In recent decades, the study of the algebraic aspects has emerged as one of developing topics. The most generalized algebra, known as SuperHyper(SH) Algebras, was introduced by Smarandache.[14] The comprehensive kinds of algebras were presented in.[11] Specifically categorical theory has become widespread in most mathematical domains; they rely on highly abstract definitions that explain the general idea underlying a given notion rather than basic properties. In accordance with this, we have introduced the category theory on SH groupoid , SH BCI-Algebra and Neutrosophic SuperHyper (NSH) BCI-Algebra. The framework of this study is as follows. The first section outlines the prerequisites needed for the study. In section 3 SH-homo and $0-$preserving SH-homo are introduced and compared with two SH-groupoids along with few examples. A few characteristics of $0-$preserving SH-homo and SH-homo are investigated. Furthermore, we established that the $0-$preserving SH-homo between two SH BCI-Algebras is the SH-homo. In following sections , we have examined the category of SH Groupoid, SH BCI-Algebra and NSH BCI- Algebra. The final section delivers the summary and implications for the future.

2 Preliminaries

Definition 2.1. [11] Suppose that Ɏ is a non-empty set with the binary operation '$\circledcirc$' and the constant 0. Then $(Ɏ, \circledcirc, 0)$ is known to be a BCI-algebra, if the following four axioms hold.
(i) $((\upsilon_1 \circledcirc \upsilon_2) \circledcirc (\upsilon_1 \circledcirc \upsilon_3)) \circledcirc (\upsilon_3 \circledcirc \upsilon_2) = 0$
(ii) $(\upsilon_1 \circledcirc (\upsilon_1 \circledcirc \upsilon_2)) \circledcirc \upsilon_2 = 0$
(iii) $\upsilon_1 \circledcirc \upsilon_1 = 0$
(iv) $\upsilon_1 \circledcirc \upsilon_2 = 0$ and $\upsilon_2 \circledcirc \upsilon_1 = 0 \implies \upsilon_1 = \upsilon_2$
for all $\upsilon_1, \upsilon_2, \upsilon_3 \in$ Ɏ

Definition 2.2. [11] Consider $\odot : \mathcal{H} \times \mathcal{H} \to P^*(\mathcal{H})$ ($\mathcal{H}$ is a non-empty set and $P^*(\mathcal{H})$ represents the power set of $\mathcal{H} \setminus \emptyset$). Let Γ , $\Gamma' \subseteq \mathcal{H}$, then $\Gamma \odot \Gamma'$ is the collection $\bigcup\limits_{r\in\Gamma, s\in\Gamma'} r \odot s$ is termed as a hypergroupoid. $\odot$ is called as Hyperoperation. Also $r \ll s$ is stated as, for $0 \in r \odot s$ and $\forall\ \Gamma, \Gamma' \subseteq \mathcal{H}$, $\Gamma \ll \Gamma'$ means that $\forall$ $r \in \Gamma, \exists s \in \Gamma' \ni r \ll s$.

Definition 2.3. [11] If the hyper groupoid $(\mathcal{H}, \odot)$ satisfies the following axioms, it is known to be a hyper BCI-algebra.
(H1) $(\upsilon_1 \odot \upsilon_2) \odot (\upsilon_2 \odot \upsilon_3) \ll \upsilon_1 \odot \upsilon_2$
(H2) $(\upsilon_1 \odot \upsilon_2) \odot \upsilon_3 = (\upsilon_1 \odot \upsilon_3) \odot \upsilon_2$
(H3) $\upsilon_1 \ll \upsilon_1$
(H4) $\upsilon_1 \ll \upsilon_2$ and $\upsilon_2 \ll \upsilon_1$ implies $\upsilon_1 = \upsilon_2$
(H5) $0 \odot (0 \odot \upsilon_1) \ll \upsilon_1$
for all $\upsilon_1, \upsilon_2, \upsilon_3 \in \mathcal{H}$

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Notation 1. [14] *The* m^{th} *power set of the set* $\mathcal{H}$ *denoted by* $P_*^m(\mathcal{H})$ *, and the empty set* $\emptyset$ *is not contained in any of* $P^j(\mathcal{H}), j = 1, 2, ..., m$.

Definition 2.4. [14] The following defines a classical - type Binary SuperHyper Operation $\circledcirc^*_{(2,m)}$.
$\circledcirc^*_{(2,m)} : \mathcal{H}^2 \to P_*^m(\mathcal{H})$, $P_*^m(\mathcal{H})$ denotes the mth - power set of the set $\mathcal{H}$, without empty set $\emptyset$.

Notation 2. [11] *Given a set* $\mathcal{B}$ *and an element* ϱ*, the notation* $\varrho \prec \mathcal{B}$ *indicates that* $\varrho \in \mathcal{B}$*. Inductively, for* $\mathcal{F} \in P_*^m(\mathcal{B})$ *and an element* ϱ*,* $\varrho \prec \mathcal{F}$ *represents* $\varrho \prec \mathcal{X}$ *for some* $\mathcal{X} \in \mathcal{F}$.

Definition 2.5. [11] Let $\circledast$ be a SH operation on non-empty set S, defined as $\circledast : S \times S \to P_*^m(S)$. For the subsets Δ and Δ' of S, $\Delta \circledast \Delta'$ indicates the collection $\bigcup\limits_{\varsigma \in \Delta, \varsigma' \in \Delta'} \varsigma \circledast \varsigma'$ and for any $\mathcal{F}$ and $\mathcal{G}$ of $P_*^i(S)$, $\mathcal{F} \circledast \mathcal{G}$ represents the collection $\bigcup\limits_{\Delta \in \mathcal{F}, \Delta' \in \mathcal{G}} \Delta \circledast \Delta'$, $\forall i = 1, 2, ..., m$.

A SH groupoid, symbolized with $(S, \circledast)$, is a set S that has a SH operation $\circledast$ and all the aforementioned notations.

Notation 3. [11] *We represent* $\varpi \ll \varpi'$ *if* $0 \prec \varpi \circledast \varpi' \in P_*^{n-1}(S)$*. For all* Ω*,* $\Omega' \subseteq S$*,* $\Omega \ll \Omega'$ *represents that* $\forall\ \kappa \in \Omega$*,* $\exists\ \kappa' \in \Omega'$ *such that* $\kappa \ll \kappa'$ *and for each* $\mathcal{F}$ *and* $\mathcal{G} \in P_*^i(S)$*,* $\mathcal{F} \ll \mathcal{G}$ *refers that for each* $\Omega \in \mathcal{F}$*,* $\exists\ \Omega' \in \mathcal{G}$ *such that* $\Omega \ll \Omega'$*,* $\forall i = 1, 2, ..., n$.

Definition 2.6. [11] The following criteria characterizes a SH groupoid $(S, \circledast)$ with a constant 0 as a SH BCI-algebra.
(SH1) $(\wp \circledast ç) \circledast (ą \circledast ç) \ll \wp \circledast ą$
(SH2) $(\wp \circledast ą) \circledast ç = (\wp \circledast ç) \circledast ą$
(SH3) $\wp \ll \wp$
(SH4) $\wp \ll ą$ and $ą \ll \wp$ implies $\wp = ą$
(SH5) $0 \circledast (0 \circledast \wp) \ll \wp$
for every $\wp, ą, ç \in S$

Definition 2.7. [11] Given a SH BCI-Algebra $(S, \circledast)$, let $S' \subset S$ such that $0 \in S'$. S' is referred to as a SH subalgebra of S if it is a SH BCI-Algebra with SH operation $\circledast$ on S.

Definition 2.8. [18] A category $C = (\mathcal{C}, M)$ is an ordered pair comprises

(i) a collection $\mathcal{C}$ of objects $S_1, S_2, ...$

(ii) a collection $M = \text{Mor}(\mathcal{C}) = \{\text{Mor}(S_1, S_2) | S_1\ \&\ S_2 \in \mathcal{C}\}$ of morphisms corresponding to each pair of objects S_1 and S_2 of $\mathcal{C}$ (An element ζ of $\text{Mor}(S_1, S_2)$ is denoted by $\zeta : S_1 \to S_2$) and

(iii) a map $\text{Mor}(S_1, S_2) \times \text{Mor}(S_2, S_3) \to \text{Mor}(S_1, S_3)$ for every three objects S_1, S_2 and S_3 of $\mathcal{C}$ (known as the composition $\eta \circ \zeta$ of the morphisms ζ and η) such that
(R1) if $\zeta : S_1 \to S_2$, $\eta : S_2 \to S_3$, $\vartheta : S_3 \to S_4$ then $\vartheta \circ (\eta \circ \zeta) = (\vartheta \circ \eta) \circ \zeta$.
(R2) to each S_1 of $\mathcal{C}$, there exist $i_{S_1} : S_1 \to S_1$ such that $\zeta \circ i_{S_1} = \zeta$, for every $\zeta : S_1 \to S_2$ and $i_{C_1} \circ \eta = \eta$, for every $\eta : S_2 \to S_1$.

Definition 2.9. [18] Given a category $C = (\mathcal{C}, M)$, a subcategory $C' = (\mathcal{C}', M')$ comprises a some objects and morphisms of C, such that

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(i) if $S_1 \in \mathcal{C}'$, then $i_{S_1} \in M'$,

(ii) if $\zeta : S_1 \to S_2 \in M'$, then $S_1, S_2 \in \mathcal{C}'$ and

(iii) if $\zeta : S_1 \to S_2, \eta : S_2 \to S_3$ are in M', then $\eta \circ \zeta \in M'$.

Moreover, if Mor(S_1, S_2)in C is equal to Mor(S_1, S_2) in C' for all $S_1, S_2 \in C'$, then C' is called full subcategory of C.

Definition 2.10. [11] Let S be the SH BCI-algebras, and $\mathcal{A} =< T_{\mathcal{A}}, I_{\mathcal{A}}, F_{\mathcal{A}} >$ be a neutrosophic set over S. If

$$T_{\mathcal{A}}(\omega \circledast \omega') \geq min(T_{\mathcal{A}}(\omega), T_{\mathcal{A}}(\omega')),$$

$$I_{\mathcal{A}}(\omega \circledast \omega') \geq min(I_{\mathcal{A}}(\omega), I_{\mathcal{A}}(\omega')) \quad \text{and}$$

$$F_{\mathcal{A}}(\omega \circledast \omega') \leq max(F_{\mathcal{A}}(\omega), F_{\mathcal{A}}(\omega'))$$

for all elements ω & $\omega' \in S$ then $\mathcal{A}$ is called as NSH BCI-algebra.

3 SuperHyper Homomorphism

In this section we have introduced the notion SuperHyper homomorphism (SH-homo). The SH-homo is a map between two SuperHyper groupoid which preserve the SuperHyper operations. Also, we have introduced the notion $0-$preserving SH-homo. Then we have explored some basic proberties of SH-homo and $0-$preserving SH-homo. Also, we proved that SH-homo between two SuperHyper BCI-algebras is $0-$preserving SH-homo.

Definition 3.1. Consider the two SuperHyper groupoid $(S, \circledast_1)$ and $(S, \circledast_2)$ on the sets S_1 and S_2 with the SuperHyper operations $\circledast_1$ and $\circledast_2$ respectively. Then the function $\zeta : S_1 \to S_2$ is said to be a SuperHyper homomorphism (SH-homo) if $\zeta(s \circledast_1 s') = \zeta(s) \circledast_2 \zeta(s')$ for all $s, s' \in S_1$

The following two examples are examples for SH-homo. First one is trivial example and second example is a non-trivial example.

Example 3.2. The identity map $i : S \to S$ is always a SH-homo on the SuperHyper groupoid $(S, \circledast)$ into $(S, \circledast)$.

Example 3.3. Let $S_1 = \{0, 1\}$, $S_2 = \{0, 1\}$ and $\circledast_1$, $\circledast_2$ be defined as in the table.

$\circledast_1$	0	1
0	{{0}, {0,1}}	{{0},{1}}
1	{{1}}	{{0},{1},{0,1}}

$\circledast_2$	0	1
0	{{1}, {0,1}}	{{0},{1}}
1	{{0}}	{{0},{1},{0,1}}

Here the function $\zeta : S_1 \to S_2$ defined by $\zeta(0) = 1$ and $\zeta(1) = 0$ is a SH-homo.

Definition 3.4. Let $(S_1, \circledast_1)$ and $(S_2, \circledast_2)$ be two SuperHyper groupoid with zeros 0_1 and 0_2 then the SH-homo $\zeta : S_1 \to S_2$ is said to be $0-$preserving SH-homo if $\zeta(0_1) = 0_2$.

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In the following two examples, the first is the example for $0-$preserving SH-homo and the second is the example for non $0-$preserving SH-homo.

Example 3.5. Let $(S, \circledast)$ be a SuperHyper groupoid with zero 0.Then the identity map $i : S \to S$ is always $0-$preserving SH-homo.

Example 3.6. Consider the SuperHyper groupoids $S_1 = S_2 = \{0, 1\}$ with zeroes $0_1 = 0_2 = 0$ and the SuperHyper operators $\circledast_1$ and $\circledast_2$ defined as in the Example 3.3. Then the function $\zeta : S_1 \to S_2$ defined by $\zeta(0) = 1$ and $\zeta(1) = 0$ is a SH-homo but not $0-$preserving.

Proposition 3.7. *Let $(S_1, \circledast_1)$, $(S_2, \circledast_2)$ and $(S_3, \circledast_3)$ be SuperHyper groupoids, if $\zeta : S_1 \to S_2$ and $\eta : S_2 \to S_3$ are SH-homo then $\eta \circ \zeta$ is also SH-homo.*
Proof: *Since ζ is SH-homo, for every $s, s' \in S_1$ we have $\zeta(s \circledast_1 s') = \zeta(s) \circledast_2 \zeta(s')$.*

$$\begin{aligned} \text{Now } \eta \circ \zeta(s \circledast_1 s') &= \eta[\zeta(s \circledast_1 s')] \\ &= \eta[\zeta(s) \circledast_2 \zeta(s')] \\ &= \eta(\zeta(s)) \circledast_3 \eta(\zeta(s'))(\text{since } \eta \text{ is SH-homo}) \\ &= \eta \circ \zeta(s) \circledast_3 \eta \circ \zeta(s') \end{aligned}$$

In the above theorem, if the SH-homo are $0-$preserving then the composition also preserve the 0. Thus, the subsequent corollary is obtained.

Corollary 3.8. *Let $(S_1, \circledast_1)$, $(S_2, \circledast_2)$ and $(S_3, \circledast_3)$ be SuperHyper groupoids, if $\zeta : S_1 \to S_2$ and $\eta : S_2 \to S_3$ are SH-homo with $0-$preserving then $\eta \circ \zeta$ is also SH-homo with $0-$preserving.*

The following definition represents the equivalence of two SH-groupoid through the SH-homo.

Definition 3.9. If the SH-homo $\zeta : S_1 \to S_2$ is bijection then ζ is said to be SH-isomorphism(SH-iso). Also, S_1 and S_2 are said to be SH-isomorphic.

The following proposition states that a map and its inverse are both SH-iso.

Proposition 3.10. *Consider the SH groupoids $(S_1, \circledast_1)$ and $(S_2, \circledast_2)$. If $\zeta : S_1 \to S_2$ is SH-iso iff its inverse $\zeta^{-1} : S_2 \to S_1$ is SH-iso.*
Proof: *Since ζ is bijective ,ζ^{-1} exist and it is bijective. Let $s_2, s_2' \in S_2$ then there exist $s_1, s_1' \in S_1$ such that $\zeta(s_1) = s_2$ and $\zeta(s_1') = s_2'$. Now $\zeta(s_1 \circledast_1 s_1') = \zeta(s_1) \circledast_2 \zeta(s_1') \implies \zeta^{-1}(s_2 \circledast_2 s_2') = s_1 \circledast_1 s_1' = \zeta^{-1}(s_2) \circledast_1 \zeta^{-1}(s_2')$.*

Corollary 3.11. *Let $(S_1, \circledast_1)$ and $(S_2, \circledast_2)$ be SH groupoids. If $\zeta : S_1 \to S_2$ is $0-$preserving SH-iso iff its inverse $\zeta^{-1} : S_2 \to S_1$ is $0-$preserving SH-iso.*

The key outcome of this section is the following theorem, which demonstrates that an SH-homo between two SuperHyper BCI-algebras is a $0-$preserving SH-homo.

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Theorem 3.12. *Every SH-homo $\zeta : S_1 \to S_2$ between two SuperHyper BCI-algebras $(S_1, \circledast_1)$ and $(S_2, \circledast_2)$ is $0-$preserving.*

Proof : *We need to prove $\zeta(0_1) = 0_2$. Since ζ is SH-homo, we have $\zeta(0_1 \circledast_1 0_1) = \zeta(0_1) \circledast_2 \zeta(0_1)$. We know that $\zeta(0_1) \prec \zeta(0_1) \implies 0_2 \prec \zeta(0_1) \circledast_2 \zeta(0_1) = \zeta(0_1 \circledast_1 0_1) \implies 0_2 \prec \zeta(0_1 \circledast_1 0_1)$ and $\zeta(0_1) \prec \zeta(0_1 \circledast_1 0_1)$ (since $0_1 \prec 0_1 \circledast_1 0_1$). But $\zeta(0_1 \circledast_1 0_1) \prec P_*^m(S_2)$ and $0_2 \prec \zeta(0_1 \circledast_1 0_1)$ implies $0_2 \prec P_*^{m-1}(\{0\})$. $0_2 \prec 0_2 \circledast_2 0_2 \prec (\zeta(0_1) \circledast_2 \zeta(0_1)) \circledast_2 P_*^{m-1}(\{0\}) \implies \zeta(0_1) \circledast_2 \zeta(0_1) \prec P_*^{m-1}(\{0\})$(By theorem 3.7 (vii) in[11]) $\implies \zeta(0_1) \circledast_2 \zeta(0_1) = P_*^{m-1}(\{0\}) \implies \zeta(0_1) \prec P_*^{m-1}(\{0\}) \implies \zeta(0_1) = 0_2$.*

From the above theorem, we show that SH-homo between two SuperHyper BCI-algebras preserve the zero from the domain to codomain. Using this property we prove the following corollary.

Corollary 3.13. *The image of SH-homo between two SuperHyper BCI algebras is a SuperHyper BCI subalgebra.*

Proof : *Let $\zeta : S_1 \to S_2$ be a SH-homo between two SuperHyper BCI-algebras $(S_1, \circledast_1)$ and $(S_2, \circledast_2)$. By the above theorem, zero is there in the image set $\zeta(S_1)$. Let $\zeta(s_1), \zeta(s_2') \in \zeta(S_1)$, $\zeta(s_1) \circledast_2 \zeta(s_2') = \zeta(s_1 \circledast_1 s_2') \in \zeta(S_1)$. Therefore, $\circledast_2|_{\zeta(S_1)}$ is binary. [By Theorem 4.2[11]] , $\circledast_2|_{\zeta(S_1)}$ is binary implies the result.*

Corollary 3.14. *The image of SuperHyper BCI subalgebra under SH-homo between two SuperHyper BCI algebras is a SuperHyper BCI subalgebra.*

4 Category of SuperHyper BCI-algebras

In this section we have explored the category of SH-groupoids and the category of SuperHyper BCI subalgebras and Neutrosophic SuperHyper BCI algebra.

Notation 4.

- *Let C_1 be the ordered pair $(\mathcal{C}_1, M_1)$, where $\mathcal{C}_1$ is collection of all SH-groupoid and*

$$M_1 = \{\zeta : S_1 \to S_2 : \zeta \text{ is SH-homo for all } S_1, S_2 \in \mathcal{C}_1\}$$

.

- *Let C_2 be the ordered pair $(\mathcal{C}_2, M_2)$, where $\mathcal{C}_2$ is collection of all SH-groupoid and*

$$M_2 = \{\zeta : S_1 \to S_2 : \zeta \text{ is } 0- \text{ preserving SH-homo for all } S_1, S_2 \in \mathcal{C}_2\}$$

.

- *Let C_3 be the ordered pair $(\mathcal{C}_3, M_3)$, where $\mathcal{C}_3$ is collection of all SH-BCI-algebra and*

$$M_3 = \{\zeta : S_1 \to S_2 : \zeta \text{ is SH-homo for all } S_1, S_2 \in \mathcal{C}_3\}$$

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The next theorem states that the set of all SH-groupoids and all SH-homo sets constitutes a category.

Theorem 4.1. *$C_1 = \{\mathcal{C}_1, M_1\}$ is a category with $\mathcal{C}_1$ as the collection of all objects and M_1 as the collection of all morphisms.*
Proof: *By the Proposition 3.7 and Example 3.2, it is clear that composition $\eta \circ \zeta$ of two SH-homo ζ and η and identity map i are SH-homo. Since SH-homo are basically functions so those will satisfy the associative law. Hence C_1 is a category.*

The following theorem shows that the collection of all SH-groupoid along with the collection of all $0-$ preserving SH-homo is a category.

Theorem 4.2. *$C_2 = \{\mathcal{C}_2, M_2\}$ is a category with collection of all objects $\mathcal{C}_2$ and collection of all morphisms M_2.*
Proof: *By the Corollary 3.8 and Example 3.5, it is clear that composition $\eta \circ \zeta$ of two $0-$ preserving SH-homo ζ and η and identity map i are $0-$ preserving SH-homo. Hence C_2 is a category.*

The following theorem indicates that the collection of all SH-BCI-algebras along with the collection of all SH-homo is a category.

Theorem 4.3. *$C_3 = \{\mathcal{C}_3, M_3\}$ is a category with collection of all objects $\mathcal{C}_3$ and collection of all morphisms M_3.*
Proof: *This proof is analogous to 4.1.*

The following proposition deals with the relation between the categories C_1 and C_2.

Proposition 4.4. *C_2 is a subcategory of C_1.*
Proof: *Clearly $\mathcal{C}_2 = \mathcal{C}_1$, $M_2 \subseteq M_1$. By Example 3.5, identity is $0-$preserving and by Corollary 3.8, composition of two $0-$preserving SH-homo is again $0-$preserving SH-homo. Hence, C_2 is a subcategory of C_1.*

The following proposition delivers the relation of the category C_3 with C_1 and C_2.

Proposition 4.5. *C_3 is a full subcategory of C_1 and C_2.*
Proof: *Clearly $\mathcal{C}_3 \subseteq \mathcal{C}_1$, $\mathcal{C}_3 \subseteq \mathcal{C}_1$, $M_3 \subseteq M_1$ and $M_3 \subseteq M_2$. By Theorem 3.12, any SH-homo between two SH-BCI-algebras is $0-$preserving SH-homo. Hence, Mor(S_1, S_2) with respect to C_3 is equal to Mor(S_1, S_2) with respect to C_1 and C_2 for all $S_1, S_2 \in \mathcal{C}_3$.*

5 Category of Neutrosophic SH-BCI-Algebra

Definition 5.1. Let $\mathcal{A}_1 =< T_{\mathcal{A}_1}, I_{\mathcal{A}_1}, F_{\mathcal{A}_1} >$ and $\mathcal{A}_2 =< T_{\mathcal{A}_2}, I_{\mathcal{A}_2}, F_{\mathcal{A}_2} >$ be two NSH-BCI algebras on SH-BCI algebras $(S_1, \circledast_1)$ and $(S_2, \circledast_2)$ respectively, and $\zeta : S_1 \to S_2$ be a SH-homo. Then ζ is said to be Neutrosophic SuperHyper homomorphism (NSH-homo) if
(i) $T_{\mathcal{A}_1}(s_1 \circledast_1 s_2) \leq T_{\mathcal{A}_2}(\zeta(s_1) \circledast_2 \zeta(s_2))$.
(ii) $I_{\mathcal{A}_1}(s_1 \circledast_1 s_2) \leq I_{\mathcal{A}_2}(\zeta(s_1) \circledast_2 \zeta(s_2))$.
(iii) $F_{\mathcal{A}_1}(s_1 \circledast_1 s_2) \geq F_{\mathcal{A}_2}(\zeta(s_1) \circledast_2 \zeta(s_2))$.

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Example 5.2. Let $\mathcal{A} =< T_{\mathcal{A}}, I_{\mathcal{A}}, F_{\mathcal{A}} >$ be a NSH-BCI on a SH-BCI algebra S, then the identity $i : S \to S$ is a NSH-homo.

Proposition 5.3. *Let $\mathcal{A}_1 =< T_{\mathcal{A}_1}, I_{\mathcal{A}_1}, F_{\mathcal{A}_1} >$, $\mathcal{A}_2 =< T_{\mathcal{A}_2}, I_{\mathcal{A}_2}, F_{\mathcal{A}_2} >$ and $\mathcal{A}_3 =< T_{\mathcal{A}_3}, I_{\mathcal{A}_3}, F_{\mathcal{A}_3} >$ be NSH-BCI algebras on SH-BCI algebras $(S_1, \circledast_1)$,$(S_2, \circledast_2)$ and $(S_3, \circledast_3)$ respectively, and $\zeta : S_1 \to S_2$, $\eta : S_2 \to S_3$ then the composition of two NSH-homo is again NSH-homo.*
Proof: *By Proposition 3.7 $\zeta \circ \eta$ is SH-homo. Consider $s_1, s_2 \in S_1$,*

$$T_{\mathcal{A}_1}(s_1 \circledast_1 s_2) \leq T_{\mathcal{A}_2}(\zeta(s_1) \circledast_2 \zeta(s_2)) \leq T_{\mathcal{A}_3}(\eta(\zeta(s_1)) \circledast_3 \eta(\zeta(s_2)))$$

$$\implies T_{\mathcal{A}_1}(s_1 \circledast_1 s_2) \leq T_{\mathcal{A}_3}(\eta \circ \zeta(s_1) \circledast_3 \eta \circ \zeta(s_2)))$$

Similarly,

$$I_{\mathcal{A}_1}(s_1 \circledast_1 s_2) \leq I_{\mathcal{A}_3}(\eta \circ \zeta(s_1) \circledast_3 \eta \circ \zeta(s_2)))$$

and

$$F_{\mathcal{A}_1}(s_1 \circledast_1 s_2) \geq F_{\mathcal{A}_2}(\zeta(s_1) \circledast_2 \zeta(s_2)) \geq F_{\mathcal{A}_3}(\eta(\zeta(s_1)) \circledast_3 \eta(\zeta(s_2)))$$
$$\implies F_{\mathcal{A}_1}(s_1 \circledast_1 s_2) \geq F_{\mathcal{A}_3}(\eta \circ \zeta(s_1) \circledast_3 \eta \circ \zeta(s_2))).$$

Theorem 5.4. *Let S be a SH-BCI algebra. The collection of all NSH-BCI algebra $\mathcal{C}_S$ (say) on S and the collection of all NSH-homo between any two elements of $\mathcal{C}_S$, M_S (say) form a category (denoted by $C_S = \{\mathcal{C}_S, M_S\}$).*
Proof: *By Proposition 5.3 and Example 5.2 the proof follows.*

Theorem 5.5. *Let the collection of all NSH-BCI algebra $\mathcal{C}_4$ (say) on $S \in \mathcal{C}_3$ and the collection of all NSH-homo between any two elements of $\mathcal{C}_4$, M_4(say) form a category (denoted by $C_4 = \{\mathcal{C}_4, M_4\}$).*
Proof: *By Proposition 5.3 and Example 5.2 the proof follows.*

Remark 5.6. The category C_4 is the union of C_S over the BCI algebras $S \in \mathcal{C}_3$.

6 Conclusion

This study presents SH-homo, and $0-$preserving SH-homo, to compare two different SH-groupoids. Several properties of $0-$preserving SH-homo and SH-homo were studied. The relation between the category of groupoid and category of SH-homo was explored. Additionally, we proved that the SH-homo is the $0-$preserving SH-homo between two SuperHyper BCI-Algebras. Finally the Category theory in SuperHyper BCI-Algebra and Neutrosophic SuperHyper BCI algebra were studied. One of the important aspect of algebra is category theory, which offers a unifying framework for different algebraic structures. Category theory has the ability to develop many areas within algebra, such as representation theory, homotopical algebra, monoidal and tensor categories, homological algebra, topos theory, etc.

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