Neutrosophic ℵ–Ideals (ℵ-Subalgebras) of Subtraction Algebra

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Abstract

The connection between neutrosophy and algebra has been of great interest with respect to many researchers. The objective of this paper is to provide a connection between neutrosophic ℵ−structures and subtraction algebras. In this regard, we introduce the concept of neutrosophic ℵ−ideals in subtraction algebra. Moreover, we study its properties and find a necessary and sufficient condition for a neutrosophic ℵ−structure to be a neutrosophic ℵ−ideal.

Keywords: Subtraction algebra, ℵ−structure, Neutrosophic ℵ−ideal, Level set.

1. Introduction

Neutrosophic sets were introduced by Florentin Smarandache [11] as a new mathematical tool for dealing with uncertainity. They can be viewed as a generalization of the fuzzy sets that were introduced in 1965 by Lotfi Zadeh [14]. Where Zadeh defined fuzzy sets as mathematical model of vagueness in which an element belongs to a given set to some degree that is a number between 0 and 1 (both inclusive). Neutrosophy is a base of neutrosophic logic which is an extension of fuzzy logic where indeterminacy is included [13]. In neutrosophic logic [10], each proposition is estimated to have the degree of truth in a subset T, the degree of indeterminacy in a subset I, and the degree of falsity in a subset F. The study of neutrosophic sets and their properties have a great importance in the sense of applications as well as in understanding the fundamentals of uncertainty. Some related work can be found in [1, 2, 3, 12].

A crisp set A in a universe X can be defined in the form of its membership function $\mu_A: X \to \{0,1\}$ where $\mu_A(x) = 1$ if $x \in A$ and $\mu_A(x) = 0$ if $x \notin A$. A single valued neutrosophic set is an example of neutrosophic set which has many applications [10]. A new function, which is called *negative-valued function*, was introduced by Jun et al. [5] and they used it to construct ℵ-*structures*. Some work related to neutrosophic ℵ-structures can be found in [6, 7]. Schein [9] considered systems of the form $(\Phi, \circ, \backslash)$, where Φ is a set of functions closed under the composition “$\circ$” of functions and the set theoretic subtraction “\” and hence $(\Phi, \backslash)$ is a *subtraction algebra*. Jun et al. [4] introduced the concept of ideals in subtraction algebras and discussed the properties of these ideals. Some researchers worked on combining the notions of neutrosophic sets and subtraction algebra. For example, Ibrahim et al. introduced neutrosophic subtraction algebra (semigroups) and presented some results about them. Moreover, Park [8] discussed neutrosophic ideals of subtraction algebras by using single valued neutrosophic sets.

In this paper, we apply the concept of neutrosophic ℵ-structures in subtraction algebras. And it is organized as follows: After an Introduction, in Section 2 and Section 3, we present some basic results about neutrosophic ℵ-strructures as well as about subtraction algebras that are used throughout the paper. In Section 4, we introduce neutrosophic ℵ-ideals (ℵ-subalgebras) of subtraction algebra and prove that the intersection, the product, the homomorphic preimage, and onto homomrphic image are neutrosophic ℵ-ideals. Finally, in Section 5, we prove a necessary and sufficient condition for ℵ-structures to be neutrosophic ℵ-ideals by introducing the $(\alpha, \beta, \gamma)-$ level sets.

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2. Neutrosophic ℵ-structures

In this section, we present some basic results about neutrosophic ℵ-structures. For more details about neutrosophy, we refer to [5, 6, 7].

Definition 2.1. [5] Let S be a non-empty set. A function from $S \to [-1,0]$ is called a negative-valued function (ℵ-function) from S to $[-1,0]$.

Definition 2.2. [7] Let S be a non-empty set. A neutrosophic ℵ-structure over S is defined as follows:

$$S_{\mathrm{N}} = \left\{\frac{x}{(T_{\mathrm{N}}, I_{\mathrm{N}}, F_{\mathrm{N}})} : x \in S\right\},$$

where $T_{\mathrm{N}}, I_{\mathrm{N}}, F_{\mathrm{N}}$ are ℵ-functions on S which are called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function, respectively, on S.
It is clear that for any ℵ-structure S_{N} over S, $-3 \leq T_{\mathrm{N}}(x) + I_{\mathrm{N}}(x) + F_{\mathrm{N}}(x) \leq 0$ for all $x \in S$.

Definition 2.3. [7] Let $S_{\mathrm{N}} = \left\{\frac{x}{(T_{\mathrm{N}}, I_{\mathrm{N}}, F_{\mathrm{N}})} : x \in S\right\}$ and $S_{\mathrm{M}} = \left\{\frac{x}{(T_{\mathrm{M}}, I_{\mathrm{M}}, F_{\mathrm{M}})} : x \in S\right\}$ be ℵ-structures over S.

(1) S_{N} is called a neutrosophic ℵ-substructure of S_{M}, denoted as $S_{\mathrm{N}} \subseteq S_{\mathrm{M}}$, if for all $x \in S$, $T_{\mathrm{N}}(x) \geq T_{\mathrm{M}}(x), I_N(x) \leq I_{\mathrm{M}}(x), F_{\mathrm{N}}(x) \geq F_{\mathrm{M}}(x)$.
If $S_{\mathrm{N}} \subseteq S_{\mathrm{M}}$ and $S_{\mathrm{M}} \subseteq S_{\mathrm{N}}$, we say that $S_{\mathrm{N}} = S_{\mathrm{M}}$.

(2) The union of S_{N} and S_{M} is defined to be the ℵ-structure over S:

$$S_{\mathrm{N\cup M}} = \left\{\frac{x}{(T_{\mathrm{N\cup M}}, I_{\mathrm{N\cup M}}, F_{\mathrm{N\cup M}})} : x \in S\right\},$$

where $T_{\mathrm{N\cup M}}(x) = T_{\mathrm{N}}(x) \wedge T_{\mathrm{M}}(x)$, $I_{\mathrm{N\cup M}}(x) = I_{\mathrm{N}}(x) \vee I_{\mathrm{M}}(x)$, and $F_{\mathrm{N\cup M}}(x) = F_{\mathrm{N}}(x) \wedge F_{\mathrm{M}}(x)$ for all $x \in S$.

(3) The intersection of S_{N} and S_{M} is defined to be the ℵ-structure over S:

$$S_{\mathrm{N\cap M}} = \left\{\frac{x}{(T_{\mathrm{N\cap M}}, I_{\mathrm{N\cap M}}, F_{\mathrm{N\cap M}})} : x \in S\right\},$$

where $T_{\mathrm{N\cap M}}(x) = T_{\mathrm{N}}(x) \vee T_{\mathrm{M}}(x)$, $I_{\mathrm{N\cup M}}(x) = I_{\mathrm{N}}(x) \wedge I_{\mathrm{M}}(x)$, and $F_{\mathrm{N\cup M}}(x) = F_{\mathrm{N}}(x) \vee F_{\mathrm{M}}(x)$ for all $x \in S$.

(4) The complement of S_{N} is defined to be the ℵ-structure over S:

$$S_{\mathrm{N}^c} = \left\{\frac{x}{(T_{\mathrm{N}^c}, I_{\mathrm{N}^c}, F_{\mathrm{N}^c})} : x \in S\right\},$$

where $T_{\mathrm{N}^c} = -1 - T_{\mathrm{N}}(x)$, $I_{\mathrm{N}^c} = -1 - I_{\mathrm{N}}(x)$, and $F_{\mathrm{N}^c} = -1 - F_{\mathrm{N}}(x)$ for all $x \in S$.

Definition 2.4. [7] Let X, Y be non-empty sets, $f: X \to Y$ be any function, and $X_{\mathrm{N}} = \left\{\frac{x}{(T_{\mathrm{N}}, I_{\mathrm{N}}, F_{\mathrm{N}})} : x \in X\right\}$, $Y_{\mathrm{M}} = \left\{\frac{y}{(T_{\mathrm{M}}, I_{\mathrm{M}}, F_{\mathrm{M}})} : y \in Y\right\}$ be ℵ-structures over X, Y respectively. Then

(1) the ℵ-structure $X_{f^{-1}(M)} = \left\{\frac{x}{(T_{f^{-1}(M)}, I_{f^{-1}(M)}, F_{f^{-1}(M)})} : x \in X\right\}$ over X is defined as follows:

$$T_{f^{-1}(M)}(x) = T_{\mathrm{M}}(f(x)),\ I_{f^{-1}(M)}(x) = I_{\mathrm{M}}(f(x)), \text{ and } F_{f^{-1}(M)}(x) = F_{\mathrm{M}}(f(x));$$

(2) the ℵ-structure $Y_{\mathrm{f}(N)} = \left\{\frac{y}{(T_{\mathrm{f}(N)}, I_{\mathrm{f}(N)}, F_{\mathrm{f}(N)})} : y \in Y\right\}$ over Y is defined as follows:

$$T_{\mathrm{f}(N)}(y) = \begin{cases} \bigwedge_{f(x)=y} T_{\mathrm{M}}(x) & if\ f^{-1}(y) \neq \emptyset; \\ 0 & otherwise. \end{cases}, \quad I_{\mathrm{f}(N)}(y) = \begin{cases} \bigvee_{f(x)=y} I_{\mathrm{M}}(x) & if\ f^{-1}(y) \neq \emptyset; \\ -1 & otherwise. \end{cases},$$

and

$$F_{\mathrm{f}(N)}(y) = \begin{cases} \bigwedge_{f(x)=y} F_{\mathrm{M}}(x) & if\ f^{-1}(y) \neq \emptyset; \\ 0 & otherwise. \end{cases}.$$

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Remark 2.1. Let X, Y be non-empty sets, $f: X \to Y$ be any onto function, and $X_{\mathrm{N}} = \left\{\frac{x}{(T_{\mathrm{N}}, I_{\mathrm{N}}, F_{\mathrm{N}})} : x \in X\right\}$, $Y_{\mathrm{M}} = \left\{\frac{y}{(T_{\mathrm{M}}, I_{\mathrm{M}}, F_{\mathrm{M}})} : y \in Y\right\}$ be ℵ-structures over X, Y respectively. Then for all $y \in Y$

$$T_{\mathrm{f}(N)}(y) = \bigwedge_{f(x)=y} T_{\mathrm{M}}(x), \qquad I_{\mathrm{f}(N)}(y) = \bigvee_{f(x)=y} I_{\mathrm{M}}(x), \qquad F_{\mathrm{f}(N)}(y) = \bigwedge_{f(x)=y} F_{\mathrm{M}}(x).$$

3. Subtraction algebra

In this section, we present some results related to subtraction algebra that are used throughout the paper. For more details, we refer to [4, 9, 15].

Definition 3.1. [15] An algebra $(X, -)$ is called a subtraction algebra if the single binary operation "$-$" satisfies the following identities: for any $x, y, z \in X$,

(1) $x - (y - x) = x$;
(2) $x - (x - y) = y - (y - x)$;
(3) $(x - y) - z = (x - z) - y$.

We introduce an order relation "≤" on subtraction algebras: $a \leq b$ if and only if $a - b = 0$; where $0 = a - a$ is an element that does not depend on the choice of $a \in X$.

It is clear that $a - 0 = a$ and $0 - a = 0$ for all $a \in X$.

Example 3.1. Let $A_1 = \{0, 1\}$. Then $(A_1, -_1)$ is a subtraction algebra defined in Table 1.

Table 1. The subtraction algebra $(A_1, -_1)$

$-_1$	0	1
0	0	0
1	1	0

Example 3.2. Let $A_2 = \{0, a, b, c\}$. Then $(A_2, -_2)$ is a subtraction algebra defined in Table 2.

Table 2. The subtraction algebra $(A_2, -_2)$

$-_2$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Definition 3.3. [4] A non-empty subset A of a subtraction algebra X is called a subalgebra of X if for all $a, b \in A$, $a - b \in A$.

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Definition 3.4. [4] A non-empty subset A of a subtraction algebra X is called an ideal of X if it satisfies the following conditions.
(1) $a - x \in A$ for all $a \in A$ and $x \in X$;
(2) for all $a, b \in A$, whenever $a \vee b$ exists in X then $a \vee b \in A$.

Remark 3.1. Every ideal of a subtraction algebra is a subalgebra. But the converse may not hold.

We illustrate Remark 3.1 by Example 3.3.

Example 3.3. Let $(A_2, -_2)$ be the subtraction algebra in Example 3.2. Then $\{0, c\}$ is a subalgebra of A_2 that is not an ideal of A_2. This is clear as $c - a = b \notin \{0, c\}$.

Example 3.4. Let $A_3 = \{0, d, e\}$. Then $(A_3, -_3)$ is a subtraction algebra defined in Table 3.

Table 3. The subtraction algebra $(A_3, -_3)$

$-_3$	0	d	e
0	0	0	0
d	d	0	d
e	e	e	0

Example 3.5. Let $(A_3, -_3)$ be the subtraction algebra in Example 3.3. Then $\{0\}, \{0, d\}, \{0, e\}, A_3$ are the only subalgebras of A_3. Moreover, every subalgebra of A_3 is an ideal of A_3.

Definition 3.5. Let $(X, -_1)$, $(Y, -_2)$ be subtraction algebras and $f: X \to Y$ be a function. Then

(1) f is a homomrphism if $f(x -_1 y) = f(x) -_2 f(y)$ and $f(x \vee y) = f(x) \vee f(y)$.
(2) f is an isomorphism if f is a bijective homomrphism. In this case, we say that X and Y are isomorphic subtraction algebras and we write $X \cong Y$.

Example 3.6. Let $(X, -)$ be a subtraction algebra and Let S be any subalgebra of Let X. Then $f: S \to X$ defined as $f(x) = x$ is a homomrphism.

Example 3.7. Let $(X, -_1)$, $(Y, -_2)$ be subtraction algebras and $f: X \to Y$ be defined as $f(x) = 0$. Then f is a homomrphism.

4. Operations on neutrosophic ℵ-ideals (ℵ-subalgebra) of subtraction algebra

In this section, we introduce neutrosophic ℵ-ideals (ℵ-subalgebras) of subtraction algebra, present some examples, and study different operations on them.

Definition 4.1. Let $(X, -)$ be a subtraction algebra. An ℵ-structure X_N over X is called a neutrosophic ℵ-subalgebra of X if the following conditions hold for all $x, y \in X$.

$$T_N(x - y) \leq T_N(x) \vee T_N(y),\ I_N(x - y) \geq I_N(x) \wedge I_N(y), \text{ and } F_N(x - y) \leq F_N(x) \vee F_N(y).$$

Definition 4.2. Let $(X, -)$ be a subtraction algebra. An ℵ-structure X_N over X is called a neutrosophic ℵ-ideal of X if the following conditions hold.
(1) $T_N(x - y) \leq T_N(x)$, $I_N(x - y) \geq I_N(x)$, and $F_N(x - y) \leq F_N(x)$ for all $x, y \in X$,

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(2) if $x \vee y$ exists in X then

$$T_N(x \vee y) \leq T_N(x) \vee T_N(y),\ I_N(x \vee y) \geq I_N(x) \wedge I_N(y), \text{ and } F_N(x \vee y) \leq F_N(x) \vee F_N(y).$$

Example 4.1. Let $(X,-)$ be any subtraction algebra and $t,i,f \in [-1,0]$. Then $X_{\mathrm{N}} = \left\{\frac{x}{(t,i,f)} : x \in S\right\}$ is a neutrosophic $\aleph$-ideal of X. We call this neutrosophic $\aleph$-ideal as **constant neutrosophic $\aleph$-ideal**.

Example 4.2. Let $(X,-)$ be any non-trivial subtraction algebra and $t,i,f \in [-1,0]$ with $(t,i,f) \neq (0,-1,0)$. Then $X_{\mathrm{N}} = \left\{\frac{0}{(t,i,f)}, \frac{x}{(0,-1,0)} : x \in X - \{0\}\right\}$ is a neutrosophic $\aleph$-ideal of X.

Corollary 4.1. Let $(X,-)$ be any non-trivial subtraction algebra (i.e., $X \neq \emptyset$). Then X has at least two neutrosophic $\aleph$-ideals.

Proof. The proof follows from Example 4.1 and Example 4.2.

Remark 4.1. Let $(X,-)$ be a subtraction algebra. Then every neutrosophic $\aleph$-ideal of X is a neutrosophic $\aleph$-subalgebra of X. But the converse may not hold.

We illustrate Remark 4.1 by Example 4.3.

Example 4.3. Let $(A_2,-_2)$ be the subtraction algebra defined in Example 3.2 and define $A_{2_{\mathrm{N}}}$ as follows:

$$A_{2_{\mathrm{N}}} = < \frac{0}{(-0.8,-0.1,-0.7)}, \frac{a}{(-0.4,-0.5,-0.6)}, \frac{b}{(-0.4,-0.5,-0.6)}, \frac{c}{(-0.8,-0.1,-0.7)} >.$$

Then $A_{2_{\mathrm{N}}}$ is a neutrosophic $\aleph$-subalgebra of A_2 that is not neutrosophic $\aleph$-ideal of A_2.

Remark 4.2. The results in this section are also valid for neutrosophic $\aleph$-subalgebras. But we restrict our proof to neutrosophic $\aleph$-ideals.

Proposition 4.1. Let $(X,-)$ be a subtraction algebra and X_{N} be a neutrosophic $\aleph$-ideal ($\aleph$-subalgebra) of X. Then for all $x \in X$, $T_N(0) \leq T_N(x)$, $I_N(0) \geq I_N(x)$, and $F_N(0) \leq F_N(x)$.

Proof. Since $0 = x - x$ for all all $x \in X$, it follows that $T_N(0) = T_N(x-x) \leq T_N(x)$, $I_N(0) = I_N(x-x) \geq I_N(x)$, and $F_N(0) = F_N(x-x) \leq F_N(x)$.

Theorem 4.1. Let $(X,-)$ be a subtraction algebra. Then X_{N} and X_{N^c} are neutrosophic $\aleph$-ideals ($\aleph$-subalgebras) of X if and only if X_{N} is the constant neutrosophic $\aleph$-ideal of X.

Proof. It is clear that if X_{N} is the constant neutrosophic $\aleph$-ideal of X then X_{N} and X_{N^c} are neutrosophic $\aleph$-ideals ($\aleph$-subalgebras) of X.

Let X_{N} and X_{N^c} be neutrosophic $\aleph$-ideals ($\aleph$-subalgebras) of X. Proposition 4.1 asserts that $T_N(0) \leq T_N(x)$ and $T_{N^c}(0) \leq T_{N^c}(x)$, $I_N(0) \geq I_N(x)$ and $I_{N^c}(0) \geq I_{N^c}(x)$, and $F_N(0) \leq F_N(x)$ and $F_{N^c}(0) \leq F_{N^c}(x)$. The latter implies that

$$T_N(0) \leq T_N(x) \text{ and } -1 - T_N(0) \leq -1 - T_N(x),$$
$$I_N(0) \geq I_N(x) \text{ and } -1 - I_N(0) \geq -1 - I_N(x),$$
$$F_N(0) \leq F_N(x) \text{ and } -1 - F_N(0) \leq -1 - F_N(x).$$

We get now that $T_N(x) = T_N(0)$, $I_N(x) = I_N(0)$, and $F_N(x) = F_N(0)$. Thus, X_{N} is the constant neutrosophic $\aleph$-ideal of X.

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Proposition 4.2. Let $(X,-)$ be a subtraction algebra and $X_{\mathrm{N}}, X_{\mathrm{M}}$ be neutrosophic ℵ-ideals (ℵ-subalgebras) of X. Then $X_{\mathrm{N}\cap\mathrm{M}}$ is a neutrosophic ℵ-ideal (ℵ-subalgebra) of X.

Proof. Let $x,y\in X$. Then

$$T_{\mathrm{N}\cap\mathrm{M}}(x-y)=T_{\mathrm{N}}(x-y)\vee T_{\mathrm{M}}(x-y)\leq T_{\mathrm{N}}(x)\vee T_{\mathrm{M}}(x)=T_{\mathrm{N}\cap\mathrm{M}}(x);$$
$$I_{\mathrm{N}\cap\mathrm{M}}(x-y)=I_{\mathrm{N}}(x-y)\wedge I_{\mathrm{M}}(x-y)\geq I_{\mathrm{N}}(x)\wedge I_{\mathrm{M}}(x)=I_{\mathrm{N}\cap\mathrm{M}}(x);$$
$$F_{\mathrm{N}\cap\mathrm{M}}(x-y)=F_{\mathrm{N}}(x-y)\vee F_{\mathrm{M}}(x-y)\leq F_{\mathrm{N}}(x)\vee F_{\mathrm{M}}(x)=F_{\mathrm{N}\cap\mathrm{M}}(x).$$

Suppose that $x\vee y$ exists in X. Then

$$T_{\mathrm{N}\cap\mathrm{M}}(x\vee y)=T_{\mathrm{N}}(x\vee y)\vee T_{\mathrm{M}}(x\vee y)\leq T_{\mathrm{N}}(x)\vee T_{\mathrm{N}}(y)\vee T_{\mathrm{M}}(x)\vee T_{\mathrm{M}}(y)=T_{\mathrm{N}\cap\mathrm{M}}(x)\vee T_{\mathrm{N}\cap\mathrm{M}}(y);$$
$$I_{\mathrm{N}\cap\mathrm{M}}(x\vee y)=I_{\mathrm{N}}(x\vee y)\wedge I_{\mathrm{M}}(x\vee y)\geq I_{\mathrm{N}}(x)\wedge I_{\mathrm{N}}(y)\wedge I_{\mathrm{M}}(x)\wedge I_{\mathrm{M}}(y)=I_{\mathrm{N}\cap\mathrm{M}}(x)\wedge I_{\mathrm{N}\cap\mathrm{M}}(y);$$
$$F_{\mathrm{N}\cap\mathrm{M}}(x\vee y)=F_{\mathrm{N}}(x\vee y)\vee F_{\mathrm{M}}(x\vee y)\leq F_{\mathrm{N}}(x)\vee F_{\mathrm{N}}(y)\vee F_{\mathrm{M}}(x)\vee F_{\mathrm{M}}(y)=F_{\mathrm{N}\cap\mathrm{M}}(x)\vee F_{\mathrm{N}\cap\mathrm{M}}(y).$$

Therefore, $X_{\mathrm{N}\cap\mathrm{M}}$ is a neutrosophic ℵ-ideal (ℵ-subalgebra) of X.

Corollary 4.2. Let $(X,-)$ be a subtraction algebra and X_{N_i} be a neutrosophic ℵ-ideal (ℵ-subalgebra) of X for $i=1,2,\dots,n$. Then $X_{\cap_{i=1}^{n}\mathrm{N}_i}$ is a neutrosophic ℵ-ideal (ℵ-subalgebra) of X.

Remark 4.3. Let $(X,-)$ be a subtraction algebra and $X_{\mathrm{N}}, X_{\mathrm{M}}$ be neutrosophic ℵ-ideals (ℵ-subalgebras) of X. Then $X_{\mathrm{N}\cup\mathrm{M}}$ may not be a neutrosophic ℵ-ideal (ℵ-subalgebra) of X.

We illustrate Remark 4.3 by Example 4.4.

Example 4.4. Let $(A_2,-_2)$ be the subtraction algebra defined in Example 3.2 and define the neutrosophic ℵ-ideals of X $A_{2_{\mathrm{N}}}, A_{2_{\mathrm{M}}}$ as follows:

$$A_{2_{\mathrm{N}}}=<\frac{0}{(-0.8,-0.1,-0.7)},\frac{a}{(-0.8,-0.1,-0.7)},\frac{b}{(-0.4,-0.5,-0.6)},\frac{c}{(-0.4,-0.5,-0.6)}>,$$
$$A_{2_{\mathrm{M}}}=<\frac{0}{(-0.8,-0.2,-0.7)},\frac{a}{(-0.4,-0.3,-0.6)},\frac{b}{(-0.8,-0.2,-0.7)},\frac{c}{(-0.4,-0.3,-0.6)}>.$$

Then

$$A_{2_{\mathrm{N}\cup\mathrm{M}}}=<\frac{0}{(-0.8,-0.1,-0.7)},\frac{a}{(-0.8,-0.1,-0.7)},\frac{b}{(-0.8,-0.2,-0.7)},\frac{c}{(-0.4,-0.3,-0.6)}>$$

is not a neutrosophic ℵ-ideal of X as $-0.3=I_{N\cup M}(c)=I_{\mathrm{N}\cup\mathrm{M}}(a\vee b)\ngeq I_{\mathrm{N}\cup\mathrm{M}}(a)\wedge I_{\mathrm{N}\cup\mathrm{M}}(b)=-0.2$.

Proposition 4.3. Let $(X,-_1)$, $(Y,-_2)$be subtraction algebras and $X_{\mathrm{N}}, Y_{\mathrm{M}}$ be neutrosophic ℵ-ideals (ℵ-subalgebras) of X,Y respectively. Then $(X\times Y)_{\mathrm{N}\times\mathrm{M}}$ is a neutrosophic ℵ-ideal (ℵ-subalgebra) of $X\times Y$. Here, for all $(x,y)\in X\times Y$, $T_{\mathrm{N}\times\mathrm{M}}((x,y))=T_{\mathrm{N}}(x)\vee T_{\mathrm{M}}(y)$, $I_{\mathrm{N}\times\mathrm{M}}((x,y))=I_{\mathrm{N}}(x)\wedge I_{\mathrm{M}}(y)$, and $F_{\mathrm{N}\times\mathrm{M}}((x,y))=F_{\mathrm{N}}(x)\vee F_{\mathrm{M}}(y)$.

Proof. The proof is straightforward.

Example 4.5. Let $(A_1,-_1)$ be the subtraction algebra defined in Example 3.1 and define $A_{1_{\mathrm{N}}}$ as follows:

$$A_{1_{\mathrm{N}}}=<\frac{0}{(-0.8,-0.1,-0.7)},\frac{1}{(-0.4,-0.5,-0.6)}>.$$

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Then $(A_1 \times A_1)_{\mathrm{N}\times\mathrm{N}} = < \frac{(0,0)}{(-0.8,-0.1,-0.7)}, \frac{(0,1)}{(-0.4,-0.5,-0.6)}, \frac{(1,0)}{(-0.4,-0.5,-0.6)}, \frac{(1,1)}{(-0.4,-0.5,-0.6)} >$ is a neutrosophic ℵ-ideal for the subtraction algebra $(A_1 \times A_1, -)$ presented in Table 4.

Table 4. The subtraction algebra $(A_1 \times A_1, -)$

−	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
(0,1)	(0,1)	(0,0)	(0,1)	(0,0)
(1,0)	(1,0)	(1,0)	(0,0)	(0,0)
(1,1)	(1,1)	(1,0)	(0,1)	(0,0)

Theorem 4.2. Let $(X, -_1)$, $(Y, -_2)$ be subtraction algebras, X_{N} , Y_{M} be neutrosophic ℵ-ideals (ℵ-subalgebras) of X, Y respectively, and $f: X \to Y$ be a homomorphism. Then $X_{f^{-1}(\mathrm{M})}$ is a neutrosophic ℵ-ideal (ℵ-subalgebra) of X.

Proof. Let $x, a \in X$. Then

$$T_{f^{-1}(\mathrm{M})}(x-a) = T_{\mathrm{M}}(f(x-a)) = T_{\mathrm{M}}(f(x)-f(a)) \leq T_{\mathrm{M}}\big(f(x)\big) = T_{f^{-1}(\mathrm{M})}(x),$$
$$I_{f^{-1}(\mathrm{M})}(x-a) = I_{\mathrm{M}}(f(x-a)) = I_{\mathrm{M}}(f(x)-f(a)) \geq I_{\mathrm{M}}\big(f(x)\big) = I_{f^{-1}(\mathrm{M})}(x),$$
$$F_{f^{-1}(\mathrm{M})}(x-a) = F_{\mathrm{M}}(f(x-a)) = F_{\mathrm{M}}(f(x)-f(a)) \leq F_{\mathrm{M}}\big(f(x)\big) = F_{f^{-1}(\mathrm{M})}(x).$$

Suppose that $x \vee a$ exists in X. Then

$$T_{f^{-1}(\mathrm{M})}(x \vee a) = T_{\mathrm{M}}(f(x \vee a)) = T_{\mathrm{M}}(f(x) \vee f(a)) \leq T_{\mathrm{M}}\big(f(x)\big) \vee T_{\mathrm{M}}\big(f(a)\big) = T_{f^{-1}(\mathrm{M})}(x) \vee T_{f^{-1}(\mathrm{M})}(a),$$
$$I_{f^{-1}(\mathrm{M})}(x \vee a) = I_{\mathrm{M}}(f(x \vee a)) = I_{\mathrm{M}}(f(x) \vee f(a)) \geq I_{\mathrm{M}}\big(f(x)\big) \wedge I_{\mathrm{M}}\big(f(a)\big) = I_{f^{-1}(\mathrm{M})}(x) \wedge I_{f^{-1}(\mathrm{M})}(a),$$
$$F_{f^{-1}(\mathrm{M})}(x \vee a) = F_{\mathrm{M}}(f(x \vee a)) = F_{\mathrm{M}}(f(x) \vee f(a)) \leq F_{\mathrm{M}}\big(f(x)\big) \vee F_{\mathrm{M}}\big(f(a)\big) = F_{f^{-1}(\mathrm{M})}(x) \vee F_{f^{-1}(\mathrm{M})}(a).$$

Therefore, $X_{f^{-1}(\mathrm{M})}$ is a neutrosophic ℵ-ideal of X.

Example 4.6. Let $(X, -)$ be a subtraction algebra, S a subalgebra of X, and $X_{\mathrm{N}} = \left\{\frac{x}{(T_{\mathrm{N}}, I_{\mathrm{N}}, F_{\mathrm{N}})} : x \in X\right\}$ a neutrosophic ℵ-ideal of X. Then by Theorem 4.1 and by taking $f: S \to X$ as $f(x) = x$ for all $x \in S$ we get that S_{N} is a neutrosophic ℵ-ideal of S. Where $S_{\mathrm{N}} = \left\{\frac{x}{(T_{\mathrm{N}}, I_{\mathrm{N}}, F_{\mathrm{N}})} : x \in S\right\}$

Theorem 4.3. Let $(X, -_1)$, $(Y, -_2)$ be subtraction algebras, X_{N} , Y_{M} be neutrosophic ℵ-ideals (ℵ-subalgebras) of X, Y respectively, and $f: X \to Y$ be a surjective homomorphism. Then $Y_{\mathrm{f(N)}}$ is a neutrosophic ℵ-ideal (ℵ-subalgebra) of Y.

Proof. Let $y, b \in Y$. Since f is surjective, it follows that $T_{\mathrm{f(N)}}(y-b) = \bigwedge_{y-b=f(x)} T_{\mathrm{N}}(x)$. Moreover, there exist $a \in X$ such that $b = f(a)$. We have that $T_{\mathrm{f(N)}}(y) = \bigwedge_{y=f(x)} T_{\mathrm{N}}(x) = T_{\mathrm{N}}(x')$ for some $x' \in X$ with $f(x') = y$. We get now that $y - b = f(x') - f(a) = f(x' - a)$. The latter implies that that $T_{\mathrm{f(N)}}(y-b) \leq T_{\mathrm{N}}(x'-a) \leq T_{\mathrm{N}}(x') = T_{\mathrm{f(N)}}(y)$. Similarly, we get that $F_{\mathrm{f(N)}}(y-b) \leq F_{\mathrm{f(N)}}(y)$.

$I_{\mathrm{f(N)}}(y-b) = \bigvee_{f(x)=y-b} I_N(x)$. Moreover, there exists $a \in X$ such that $b = f(a)$. We have that $I_{\mathrm{f(N)}}(y) = \bigvee_{f(x)=y} I_N(x) = I_{\mathrm{N}}(x')$ for some $x' \in X$ with $f(x') = y$. We get now that $y - b = f(x') - f(a) = f(x' - a)$. The latter implies that $I_{\mathrm{f(N)}}(y-b) \geq I_{\mathrm{N}}(x'-a) \geq I_{\mathrm{N}}(x') = I_{\mathrm{f(N)}}(y)$.

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Suppose that $y \vee b \in Y$. We prove that $T_{f(N)}(y \vee b) \leq T_{f(N)}(y) \vee T_{f(N)}(b)$ and $I_{f(N)}(y \vee b) \geq I_{f(N)}(y) \wedge I_{f(N)}(b)$, $F_{f(N)}(y \vee b) \leq F_{f(N)}(y) \vee F_{f(N)}(b)$ are done similarly.
We have $T_{f(N)}(y \vee b) = \wedge_{y \vee b = f(x)} T_N(x)$. Moreover, there exists $a \in X$ such that $b = f(a)$. We have that $T_{f(N)}(y) = \wedge_{y=f(x)} T_N(x) = T_N(x')$ and $T_{f(N)}(b) = \wedge_{b=f(x)} T_N(x) = T_N(a)$for some $x', a \in X$ with $f(x') = y$, $f(a) = b$ .We get now that $y \vee b = f(x') \vee f(a) = f(x' \vee a)$. The latter implies that $T_{f(N)}(y \vee b) \leq T_N(x' \vee a) \leq T_N(x') \vee T_N(a) = T_{f(N)}(y) \vee T_{f(N)}(b)$.

5. Level sets and neutrosophic ℵ-ideals (ℵ-subalgebra) of subtraction algebra

In this section, we define $(\alpha, \beta, \gamma)-$ level sets of X_N and study their relation with ℵ-ideals of X.

Let X_N be a neutrosophic ℵ- structure over X and let $\alpha, \beta, \gamma \in [-1,0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Then $(\alpha, \beta, \gamma)-$ level set of X_N is defined as follows:

$\{x \in X: T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\}$.

Remark 5.1. The results in this section are also valid for neutrosophic ℵ-subalgebras (instead of ideal we have subalgebra). But we restrict our proof to neutrosophic ℵ-ideals.

Proposition 5.1. Let $(X, -)$ be a subtraction algebra, $\alpha \in [-1,0]$, and X_N a neutrosophic ℵ-ideal of X. Then ${T_N}^{\alpha}$is either an empty set or an ideal of X. Here, ${T_N}^{\alpha} = \{x \in X: T_N(x) \leq \alpha\}$ is either an empty set or an ideal of X.

Proof. Let $x, y \in {T_N}^{\alpha} \neq \emptyset$. Since $T_N(x - y) \leq T_N(x) \leq \alpha$, it follows that $x - y \in {T_N}^{\alpha}$. Suppose that $x \vee y$ exists in X. Then $T_N(x \vee y) \leq T_N(x) \vee T_N(y) \leq \alpha$. Thus, $x \vee y \in {T_N}^{\alpha}$. Therefore, ${T_N}^{\alpha}$is an ideal of X.

Proposition 5.2. Let $(X, -)$ be a subtraction algebra, $\beta \in [-1,0]$, and X_N a neutrosophic ℵ-ideal of X. Then ${I_N}^{\beta}$ is either an empty set or an ideal of X. Here, ${I_N}^{\beta} = \{x \in X: I_N(x) \geq \beta\}$.

Proof. The proof is similar to that of Proposition 5.1.

Proposition 5.3. Let $(X, -)$ be a subtraction algebra, $\gamma \in [-1,0]$, and X_N a neutrosophic ℵ-ideal of X. Then ${F_N}^{\gamma}$is either an empty set or an ideal of X. Here, ${F_N}^{\gamma} = \{x \in X: F_N(x) \leq \gamma\}$.

Proof. The proof is similar to that of Proposition 5.1.

Corollary 5.1. Let $(X, -)$ be a subtraction algebra, $\alpha, \beta, \gamma \in [-1,0]$, and X_N a neutrosophic ℵ-ideal of X. Then the $(\alpha, \beta, \gamma)-$ level set of X_N is either an empty set or an ideal of X.

Proof. We have the $(\alpha, \beta, \gamma)-$ level set of X_N is ${T_N}^{\alpha} \cap I_{N_\beta} \cap {F_N}^{\gamma}$. And by Propositions 5.1, 5.2, and 5.3, we have ${T_N}^{\alpha}$, I_{N_β}, and ${F_N}^{\gamma}$ are either empty sets or ideals of X. Thus, the $(\alpha, \beta, \gamma)-$ level set of X_N is either empty or an intersection of ideals of X and hence, it is an ideal.

Lemma 5.1. Let $(X, -)$ be a subtraction algebra, $\alpha, \beta, \gamma \in [-1,0]$, and X_N an ℵ-structure over X. If every non-empty $(\alpha, \beta, \gamma)-$ level set of X_N is an ideal of X then X_N a neutrosophic ℵ-ideal of X.

Proof. Let $x, a \in X$. Then there exist $\alpha', \beta', \gamma' \in [-1,0]$ such that $T_N(a) = \alpha'$, $I_N(a) = \beta', F_N(a) = \gamma'$. Then a is in the $(\alpha', \beta', \gamma')-$ level set of X_N which is an ideal of X. The latter implies that $a - x$ is in the $(\alpha', \beta', \gamma')-$ level set of X_N. We get now that $T_N(a - x) \leq \alpha' = T_N(a)$, $I_N(a - x) \geq \beta' = I_N(a), F_N(a - x) \leq \gamma' = F_N(a)$.

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Suppose that $x \vee a \in X$. Then there exist $\alpha', \beta', \gamma', \alpha'', \beta'', \gamma'' \in [-1,0]$ such that $T_N(a) = \alpha'$, $I_N(a) = \beta'$, $F_N(a) = \gamma'$, $T_N(x) = \alpha''$, $I_N(x) = \beta''$, $F_N(x) = \gamma''$. Let $\alpha = \alpha' \vee \alpha''$, $\beta = \beta' \wedge \beta''$, $\gamma = \gamma' \vee \gamma''$. Then a, x are in the $(\alpha, \beta, \gamma)-$ level set of X_N which is an ideal of X. The latter implies that $a \vee x$ is in the $(\alpha, \beta, \gamma)-$ level set of X_N. Thus, $T_N(a \vee x) \leq \alpha = T_N(a) \vee T_N(x)$, $I_N(a \vee x) \geq \beta = I_N(a) \wedge I_N(x)$, $F_N(a \vee x) \leq \gamma = F_N(a) \vee F_N(x)$.

Therefore, X_N is a neutrosophic $\aleph$-ideal of X.

Theorem 5.1. Let $(X, -)$ be a subtraction algebra, $\alpha, \beta, \gamma \in [-1,0]$, and X_N an $\aleph$-structure over X. Then X_N is a neutrosophic $\aleph$-ideal of X if and only if every non-empty $(\alpha, \beta, \gamma)-$ level set of X_N is an ideal of X.

Proof. The proof follows from Corollary 5.1 and Lemma 5.1.

Theorem 5.2. Let $(X, -)$ be a subtraction algebra, $\alpha, \beta, \gamma \in [-1,0]$, and X_N an $\aleph$-structure over X. Then the following statements are equivalent.

(1) X_N is a neutrosophic $\aleph$-ideal of X;
(2) ${T_N}^{\alpha}$, I_{N_β}, and ${F_N}^{\gamma}$ are either empty sets or ideals of X;
(3) Every non-empty $(\alpha, \beta, \gamma)-$ level set of X_N is an ideal of X.

Proof. **(1) ⇒ (2):** If X_N is a neutrosophic $\aleph$-ideal of X then by Propositions 5.1, 5.2, and 5.3, we have ${T_N}^{\alpha}$, I_{N_β}, and ${F_N}^{\gamma}$ are ideals of X.

(2) ⇒ (3): If ${T_N}^{\alpha}$, I_{N_β}, and ${F_N}^{\gamma}$ are ideals of X then the $(\alpha, \beta, \gamma)-$ level set of X_N is intersection of ideals of X (intersection of ${T_N}^{\alpha}$, I_{N_β}, and ${F_N}^{\gamma}$) and hence, it is an ideal of X.

(3) ⇒ (1): By Lemma 5.1.

Theorem 5.3. Let $(X, -)$ be a subtraction algebra and $\alpha, \beta, \gamma \in [-1,0]$ with $(\alpha, \beta, \gamma) \neq (0, -1, 0)$. Then every ideal of X is an $(\alpha, \beta, \gamma)-$ level set of a neutrosophic $\aleph$-ideal of X.

Proof. Let I be an ideal of X and $X_N = \left\{\frac{x}{(T_N, I_N, F_N)} : x \in X\right\}$ be $\aleph$-structure of X defined as follows.

$$(T_N(x), I_N(x), F_N(x)) = \begin{cases} (\alpha, \beta, \gamma) & if\ x \in I; \\ (0, -1, 0) & otherwise. \end{cases}$$

Let $\alpha', \beta', \gamma' \in [-1,0]$. Then the $(\alpha', \beta', \gamma')-$ level set of X_N is given as follows:

$$\begin{cases} I & if\ \alpha \leq \alpha' < 0, -1 < \beta' \leq \beta, \gamma \leq \gamma' < 0; \\ X & (\alpha', \beta', \gamma') = (0, -1, 0) \\ \emptyset & otherwise. \end{cases}$$

And it is either an empty set or an ideal of X. Therefore, by Lemma 5.1, X_N is a neutrosophic $\aleph$-ideal of X.

6. Conclusion

In this paper, we combined the notions of $\aleph-$structures, neutrosophy, and subtraction algebra to introduce $\aleph-$ideals ($\aleph$-subalgebras) of subtraction algebras. Some operations on the defined notions were discussed. Moreover, the $(\alpha, \beta, \gamma)-$ level sets were introduced and used to find a necessary and sufficient condition for $\aleph$-structures to be neutrosophic $\aleph$-ideals ($\aleph$-subalgebras).

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For future work, it would be interesting to check whether there is a relation between our results about ℵ−ideals (ℵ-subalgebras) of subtraction algebras and the results related to single valued neutrosophic subtraction algebras discussed by Chul Hwan Park in [8].

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