On the Numerical Solutions Based On Exponential Finite Difference Method for Kuramoto-Sivashinsky Equation and Numerical Stability Analysis

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Abstract

In this paper, we solve the Kuramoto-Sivashinsky Equation numerically by finite-difference methods, using two different schemes which are the Fully Implicit scheme and Exponential finite difference scheme, because of the existence of the fourth derivative in the equation we suggested a treatment for the numerical solution of the two previous scheme by parting the mesh grid into five regions, the first region represents the first boundary condition, the second at the grid point x_1, while the third represents the grid points $x_2,x_3,\ldots x_{n-2}$, the fourth represents the grid point x_{n-1} and the fifth is the second boundary condition. We also, study the numerical stability by Fourier (Von-Neumann) method for the two scheme which used in the solution on all mesh points to ensure the stability of the point which had been treated in the suggested style, we using two interval with two initial condition and the numerical results obtained by using these schemes are compare with Exact Solution of Equation Excellent approximate is found between the Exact Solution and numerical Solutions of these methods.

Keywords: Kuramoto-Sivashinsky Equation; Numerical stability; Interval; Exact Solution; Numerical Solutions

1- Introduction

Partial differential equations have become a useful tool for describing natural phenomena for science and engineering models, therefore it became necessary to solve these partial differential equations using numerical and sophisticated methods, applying these methods [15].

Partial differential equations appear in some branches of Applied Mathematics, fluid dynamics of water, quantum mechanics and electromagnetic theory are examples of Partial Differential Equations, and the analytical processing of these equations is a very complex process, as it requires the application of advanced mathematical methods, on the other hand, the use of simple and generally efficient numerical methods has become easier to find sufficiently approximate solutions [12].

Many researchers have studied the Kuramoto-Sivashinsky equation due to its complexity and ability to describe a large variety of natural phenomena such as the stability of sloping membrane fronts, so this equation has attracted a lot of attention in scientific aggregates [3].

The researcher N.A. Kudryashov in [7] using the simplest equation method to search for exact solutions of the new method of nonlinear differential equations, offers to search for exact solutions of nonlinear differential equations, there are two basic ideas in the search, one of which is to use the general solutions of the easiest nonlinear differential equations, the other idea is to take into account the possible single equations of the lesson, and the method is used to adjusted for the K.S equation. The equation describes nonlinear waves in a heat-conducting fluid.

The researchers T. Mackenzie and A. J. Roberts in [11] analyzed K.S nonlinear equation. The analysis was based on the central manifold theory, and they were reassured that the model of finite differences regulates the motion

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accurately and has been arranged in an orderly manner, the theory is applied after dividing the natural field into small elements by introducing internal boundary elements that are later deleted, the equation K.S. used an example to prove that Universal finite differences can be applied to nonlinear fourth-order equations, and they proved that the innovative central manifold theory is universal in the sense that it treats the kinetic equations as a whole, not just as a sum of separate conditions.

As for the researchers Al-Rawi and Al-Baker in [1], they solved the Korteweg-de Vries-Burger's equation using two of the finite difference methods, the full implicit method and the exponential finite difference method, and studied the numerical stability by the Fourier(Von Neumann) method of the two methods and at all points of the clamp, compared the numerical results obtained using these methods with the results of the exact solution, and found an approximation between the analytical solution and the numerical solutions of the two methods.

2- Mathematical model

The formula of the Kuramoto-Sivashinsky equation is as follows [10]:

$$u_t = -uu_x - u_{xx} - u_{xxxx} \qquad , [0,32\pi] \qquad \ldots(1)$$

The boundary conditions are:

$$\left.\begin{aligned} u(0,t) = u(32\pi,t) = 1 \qquad &, t \geq 0 \\ u_{xx}(0,t) = u_{xx}(32\pi,t) = -\tfrac{1}{16^2} \quad &, t \geq 0 \end{aligned}\right\} \qquad \ldots(2)$$

Example (1): the period is $[0,32\pi]$ and the initial condition is [10]:

$$u(x,0) = cos\left(\frac{x}{16}\right)\left(1 + sin\left(\frac{x}{16}\right)\right) \qquad \ldots(3)$$

Example (2): the period is [-30,30] and the initial condition is [5]:

$$u(x,0) = c + \tfrac{15}{19}\sqrt{\tfrac{11}{19}}\left(-9tanh(k(x - x_0)) + 11tanh^3(k(x - x_0))\right) \qquad \ldots(40)$$

Since $x_0 = -12$ ، $c = 1.2$ ، $k = \frac{1}{2}\sqrt{\frac{11}{19}}$

3- Numerical solution of the Kuramoto-Sivashinsky equation:

The finite difference method is one of the most classical methods in the numerical analysis of differential equations and is one of the most important applications, and finite differences form the basis of numerical analysis as applied in other numerical methods such as numerical differential and Numerical Integration. [13,14].

3-1 Derivation of the formula of the full implicit scheme for equation K.S.:

The method is called implicit, since the solution at each level we progress in it needs to solve a system of linear equations with an amplitude (nxm) and the points to find the solution at which in level j+1 depend on one point in level j [4] as follows:

To find u_1^{j+1} at the point x_1 we use approximations of progressive differentials as shown below:

$$\frac{u_i^{j+1} - u_i^j}{k} = -u_i^j\left[\frac{-3u_i^{j+1} + 4u_{i+1}^{j+1} - u_{i+2}^{j+1}}{2h}\right] - \left[\frac{2u_i^{j+1} - 5u_{i+1}^{j+1} + 4u_{i+2}^{j+1} - u_{i+3}^{j+1}}{h^2}\right]$$
$$-\left[\frac{-3u_i^{j+1} - 14u_{i+1}^{j+1} + 26u_{i+2}^{j+1} - 24u_{i+3}^{j+1} + 11u_{i+4}^{j+1} - 2u_{i+5}^{j+1}}{h^4}\right]$$

Multiplying the equation by K and transferring the elements of the plane (j+1) to the right side and the elements of the plane (j) To the left side assuming that $r = \frac{k}{h^2}$ and simplifying the equation we get:

$$u_i^j = \left(1 + \frac{3hr}{2}u_i^j + 2r + \frac{3r}{h^2}\right)u_i^{j+1} + \left(-2rhu_i^j - 5r - \frac{14r}{h^2}\right)u_{i+1}^{j+1}$$
$$+\left(-\tfrac{rh}{2}u_i^j + 4r + \tfrac{26r}{h^2}\right)u_{i+2}^{j+1} + (-r - \tfrac{24r}{h^2})u_{i+3}^{j+1} + \tfrac{11r}{h^2}u_{i+4}^{j+1} - \tfrac{2r}{h^2}u_{i+5}^{j+1} \qquad ..(5)$$

The rest of the solutions u_i^{j+1} at points x_i, where i =2,…,n-2, are found from approximations of the central differences:

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$$\frac{u_i^{j+1}-u_i^j}{k} = -u_i^j\left[\frac{u_{i+1}^{j+1}-u_{i-1}^{j+1}}{2h}\right]-\left[\frac{u_{i+1}^{j+1}-2u_i^{j+1}+2u_{i-1}^{j+1}}{h^2}\right]$$

$$-\left[\frac{u_{i+2}^{j+1}-4u_{i+1}^{j+1}+6u_i^{j+1}-4u_{i-1}^{j+1}+u_{i+2}^{j+1}}{h^4}\right]$$

After simplification, the above equation becomes as follows:

$$u_i^j = \left(1-2r+\frac{6r}{h^2}\right)u_i^{j+1}+\left(\frac{rh}{2}+r-\frac{4r}{h^2}\right)u_{i+1}^{j+1}$$

$$+\left(-\tfrac{rh}{2}u_i^j+r-\tfrac{4r}{h^2}\right)u_{i-1}^{j+1}+\tfrac{r}{h^2}u_{i+2}^{j+1}+\tfrac{r}{h^2}u_{i-2}^{j+1} \quad ..(6)$$

As for finding u_{n-1}^{j+1}, we use the regression difference approximations as shown below:

$$\frac{u_i^{j+1}-u_i^j}{k} = -u_i^j\left[\frac{-3u_i^{j+1}-4u_{i-1}^{j+1}+u_{i-2}^{j+1}}{2h}\right]-\left[\frac{2u_i^{j+1}-5u_{i-1}^{j+1}+4u_{i-2}^{j+1}-u_{i-3}^{j+1}}{h^2}\right]$$

$$-\left[\frac{3u_i^{j+1}-14u_{i-1}^{j+1}+26u_{i-2}^{j+1}-24u_{i-3}^{j+1}+11u_{i-4}^{j+1}-2u_{i-5}^{j+1}}{h^4}\right]$$

So, we get the equation:

$$u_i^j = \left(1+\frac{3hr}{2}u_i^j+2r+\frac{3r}{h^2}\right)u_i^{j+1}+\left(-2rhu_i^j-5r-\frac{14r}{h^2}\right)u_{i-1}^{j+1}$$

$$+\left(\tfrac{rh}{2}u_i^j+4r+\tfrac{26r}{h^2}\right)u_{i-2}^{j+1}+(-r-\tfrac{24r}{h^2})u_{i-3}^{j+1}+\tfrac{11r}{h^2}u_{i-4}^{j+1}-\tfrac{2r}{h^2}u_{i-5}^{j+1} \quad ..(7)$$

Boundary conditions:

We mentioned the mathematical model of the K.S. equation. And the boundary conditions of the equation, represented by the equation *x*=*a*, *x*=*b* and to approximate the boundary conditions using the explicit method, we need to define the points $x_{-2}, x_{-1}, x_{n+1}, x_{n+2}$ as follows:

$x_{-2} = x_0 - 2h$

$x_{-1} = x_0 - h$

$x_{n+1} = x_n + h$

$x_{n+2} = x_n + 2h$

Since these points are located outside the grid, as in Figure (1)

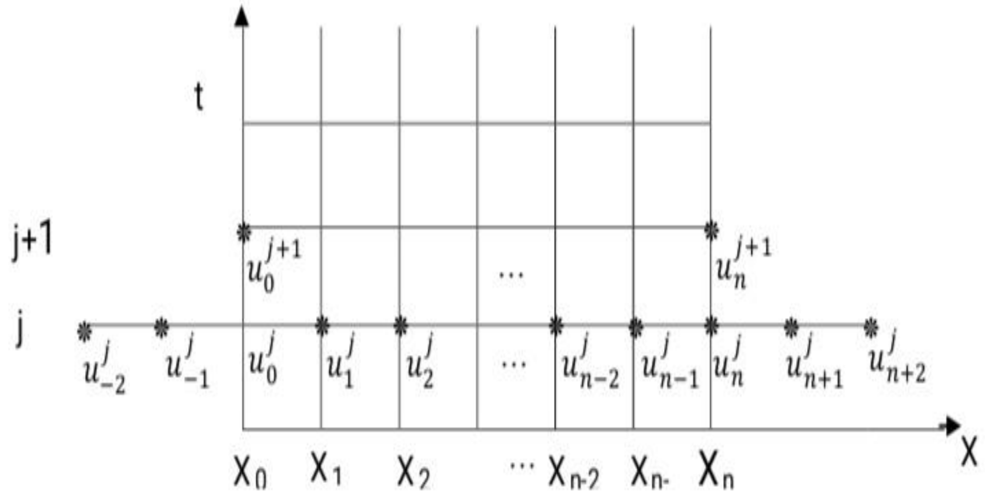

Figure 1. shows the points at which the synapse abscess is located in plane j

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Now using the central differentials of the second boundary condition of the second rank:

$$\frac{\partial^2 u(a,t)}{\partial x^2} = -\frac{1}{16^2}$$

$$\frac{\partial^2 u(b,t)}{\partial x^2} = -\frac{1}{16^2}$$

And using the central difference approximations of the second derivative when *i* = 0

$$\frac{u_1^j - 2u_0^j + u_{-1}^j}{h^2} = -= -\frac{1}{16^2}$$

$$u_{-1}^j = -u_1^j + 2u_0^j - \frac{h^2}{16^2} \qquad \ldots(8)$$

And when *i* = *n*

$$u_{n+1}^j = -u_{n-1}^j + 2u_n^j - \frac{h^2}{16^2} \qquad \ldots(9)$$

Using the central differences of the second boundary condition of the fourth rank at *i* = 0, we obtain:

$$\frac{-u_{-2}^j + 16u_1^j - 30u_0^j + 16u_{-1}^j - u_{-2}^j}{12h^2} = -\frac{1}{16^2}$$

$$u_{-2}^j = -u_2^j + 16u_1^j - 30u_0^j + 16u_{-1}^j + \frac{12h^2}{16^2} \qquad (10)$$

And when *i*=*n*

$$u_{n+2}^j = -u_{2-2}^j + 16u_{n+1}^j - 30u_n^j + 16u_{n-1}^j + \frac{12h^2}{16^2} \qquad (11)$$

Substituting equation (10) into equation (6) when *i*=0, we obtain equation (12) after simplification:

$$u_0^{j+1} = \left(1 + 2r + \frac{24r}{h^2}\right)u_0^j + \left(-\frac{rh}{2}u_0^j - r - \frac{12r}{h^2}\right)u_1^j + \left(\frac{rh}{2}u_0^j - r - \frac{12r}{h^2}\right)u_{-1}^j - \frac{12r}{h^2}$$
(12)

We note that there is another point located outside the clamp, we substitute equation (8) into equation (12) to get:

$$u_0^{j+1} = \left(1 + rhu_0^j - \frac{rh^3}{2(16^2)}\right)u_0^j - \left(rhu_0^j\right)u_1^j + \frac{h^2}{16^2} \qquad (13)$$

We substitute equation (11) into equation (6) when *i* = *n*, we get:

$$u_n^{j+1} = (1 + 2r + \frac{24r}{h^2})u_n^j + (-\frac{rh}{2}u_n^j - r - \frac{12r}{h^2})u_{n+1}^j + (\frac{rh}{2}u_n^j - r - \frac{12r}{h^2})u_{n-1}^j - \frac{12r}{h^2}$$

(14)

We also notice that there is a point located outside the boundaries of the clamp, we substitute equation (9) into equation (14) to get:

$$u_n^{j+1} = (1 - rhu_n^j + \frac{rh^3}{2(16^2)})u_n^j + (rhu_n^j)u_{n-1}^j + \frac{rh^2}{16^2} \qquad (15)$$

Having found the solution at the boundary conditions, we obtain the following five-diagonal linear system:

$$\begin{bmatrix}
A1 & B1 & 0 & 0 & \cdots & 0 & & 0 & 0 & & & & 0 & \cdots & 0 \\
0 & A2 & B2 & C1 & D1 & E1 & F1 & 0 & .. & & & & 0 & \cdots & 0 \\
T & H & A & B & C & 0 & & \cdots & 0 & & & & 0 & \cdots & 0 \\
0 & T & H & A & B & C & 0 & & & & .. & & 0 & \cdots & 0 \\
0 & 0 & T & H & A & B & C & 0 & . & & & & 0 & \cdots & 0 \\
 & & & & & & \vdots & & & & & & & & \\
0 & \cdots & 0 & 0 & .. & 0 & & & & & T & H & A & B & C \\
0 & \cdots & 0 & & .. & 0 & & & F2 & E2 & D2 & T1 & H1 & A3 & 0 \\
0 & \cdots & 0 & & & & & & 0 & 0 & 0 & \cdots & 0 & H2 & A4
\end{bmatrix}
\begin{bmatrix}
u_0^{j+1} \\ u_1^{j+1} \\ u_2^{j+1} \\ . \\ . \\ . \\ u_{n-2}^{j+1} \\ u_{n-1}^{j+1} \\ u_n^{j+1}
\end{bmatrix}$$

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$$\begin{bmatrix} (1-\frac{rh^3}{2(16^2)})u_n^j+\frac{rh^2}{16^2} \\ u_1^j \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ u_{n-1}^j \\ (1+\frac{rh^3}{2(16^2)})u_n^j+\frac{rh^2}{16^2} \end{bmatrix}$$

$$\begin{bmatrix} A1 & B1 & 0 & 0 & \cdots & 0 & & 0 & 0 & & & & 0 & \cdots & 0 \\ 0 & A2 & B2 & C1 & D1 & E1 & F1 & 0 & .. & & & & 0 & \cdots & 0 \\ T & H & A & B & C & 0 & & \cdots & 0 & & & & 0 & \cdots & 0 \\ 0 & T & H & A & B & C & 0 & & & & .. & & 0 & \cdots & 0 \\ 0 & 0 & T & H & A & B & C & 0 & . & & & & 0 & \cdots & 0 \\ & & & & & & \vdots & & & & & & & & \\ 0 & \cdots & 0 & 0 & .. & 0 & & & & & T & H & A & B & C \\ 0 & \cdots & 0 & & .. & 0 & & & F2 & E2 & D2 & T1 & H1 & A3 & 0 \\ 0 & \cdots & 0 & & & & & & 0 & 0 & 0 & \cdots & 0 & H2 & A4 \end{bmatrix} . \begin{bmatrix} u_0^{j+1} \\ u_1^{j+1} \\ u_2^{j+1} \\ . \\ . \\ . \\ u_{n-2}^{j+1} \\ u_{n-1}^{j+1} \\ u_n^{j+1} \end{bmatrix}$$

$$= \begin{bmatrix} (1-\frac{rh^3}{2(16^2)})u_n^j+\frac{rh^2}{16^2} \\ u_1^j \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ u_{n-1}^j \\ (1+\frac{rh^3}{2(16^2)})u_n^j+\frac{rh^2}{16^2} \end{bmatrix}$$

$$\forall j = 1,2,3,\dots,m$$

Then:

$$A=(1-2r+\frac{6r}{h^2}), A1=(1-rhu_n^j), A2=(1-\frac{3hr}{2}u_i^j+2r+\frac{3r}{h^2})$$

$A3=(1+\frac{3hr}{2}u_i^j+2r+\frac{3r}{h^2}), A4=(1+rhu_n^j), B=(\frac{hr}{2}u_i^j+r-\frac{4r}{h^2}), B1=(rhu_n^j),$

$B2=(-2rhu_i^j-5r-\frac{14r}{h^2}), C=\frac{r}{h^2}$

And by arranging the equations, we get a hexadecimal system:

$C1=(-\frac{rh}{2}u_i^j+4r+\frac{26r}{h^2}), D1=(-r-\frac{24r}{h^2}), D2=(-r-\frac{26r}{h^2}), E1=\frac{11r}{h^2}, E2=\frac{11r}{h^2},$

$F1=-\frac{2r}{h^2}, F2=-\frac{2r}{h^2}, H=(-\frac{rh}{2}u_i^j+r-\frac{4r}{h^2}), H1=(-2rhu_i^j-5r-\frac{14r}{h^2}), H2=(-rhu_n^j),$

$T=\frac{r}{h^2}, T1=(\frac{rh}{2}u_i^j+4r+\frac{26r}{h^2})$

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The above linear system can be solved using direct methods or iterative methods used one of the direct methods, which is the Gaussian Elimination Method.

2-3 Derive the formula of the exponential finite difference Scheme for equation K.S:

This method was first introduced by the scientist Bhattacharya in 1985 for the unstable one-dimensional case of thermal conductivity in Cartesian coordinates, after which the algorithm of the method was developed to solve the one-dimensional diffusion equation in cylinder coordinates and applied to two-dimensional and three-dimensional problems. The method is used to solve nonlinear partial differential equations in one and two dimensions of Cartesian coordinates [6].

The method of deriving the formula for exponential finite differences is by assuming that F (u) stands for any continuous and derivable function [2], and multiplying Equation (1) by the derivative of F yields:

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial t} = F'(u)(-u\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^2}) \qquad \ldots(16)$$

Thus:

$$\frac{\partial F}{\partial t} = F'(u)(-u\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^2}) \qquad \ldots(17)$$

Using the usual Progressive differentials and compensating them for the $\frac{\partial F}{\partial t}$,we get:

$$\frac{F(u_i^{j+1}) - F(u_i^j)}{k} = F'(u_i^j)\left[-u_i^j\left(\frac{\partial u}{\partial x}\right)_i^j - \left(\frac{\partial^2 u}{\partial x^2}\right)_i^j - \left(\frac{\partial^4 u}{\partial x^4}\right)_i^j\right] \qquad \ldots(18)$$

Suppose that $F(u) = ln\, u$, we obtain the formula for exponential differences as follows:

$$u_i^{j+1} = u_i^j\ exp\left[\frac{k}{u_i^j}(-u_i^j\left(\frac{\partial u}{\partial x}\right)_i^j - \left(\frac{\partial^2 u}{\partial x^2}\right)_i^j - \left(\frac{\partial^4 u}{\partial x^4}\right)_i^j)\right] \qquad \ldots(19)$$

To find the solution in this way, we follow the proposed method, which is to calculate the solution u_i^{j+1} using the arithmetic mean of progressive differential equations as follows:

$$\left(\frac{\partial u}{\partial x}\right)_i^j = \frac{-3u_i^j + 4u_{i+1}^j - u_{i+2}^j}{2h}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i^j = \frac{2u_i^j - 5u_{i+1}^j + 4u_{i+2}^j - u_{i+3}^j}{h^2}$$

$$\left(\frac{\partial^4 u}{\partial x^4}\right)_i^j = \frac{3u_i^j - 14u_{i+1}^j + 26u_{i+2}^j - 24u_{i+3}^j + 11u_{i+4}^j - 2u_{i+5}^j}{h^4}$$

Substituting it into equation (19), we get:

$$u_i^{j+1} = u_i^j\ exp\left[\frac{k}{u_i^j}(-u_i^j(\frac{-3u_i^j+4u_{i+1}^j-u_{i+2}^j}{2h}) - (\frac{2u_i^j-5u_{i+1}^j+4u_{i+2}^j-u_{i+3}^j}{h^2}) - (\frac{3u_i^j-14u_{i+1}^j+26u_{i+2}^j-24u_{i+3}^j+11u_{i+4}^j-2u_{i+5}^j}{h^4}))\right]$$

…(20)

To calculate the approximate solutions $u(x_i, t_j)$, since i=2,3,…,n-2, we substitute the central differential equations in equation (19), we get:

$$u_i^{j+1} = u_i^j\ exp\left[\frac{k}{u_i^j}(-u_i^j(\frac{u_{i+1}^j - u_{i-1}^j}{2h}) - (\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2}) - (\frac{u_{i+2}^j - 4u_{i+1}^j + 6u_i^j - 4u_{i-1}^j + u_{i-2}^j}{h^4}))\right]$$

…(21)

At i=n-1, we substitute the posterior differential equations in equation (19) to obtain:

$$u_i^{j+1} = u_i^j\ exp\left[\frac{k}{u_i^j}(-u_i^j(\frac{3u_i^j-4u_{i-1}^j+u_{i-2}^j}{2h}) - (\frac{2u_i^j-5u_{i-1}^j+4u_{i-2}^j-u_{i-3}^j}{h^2}) - (\frac{3u_i^j-14u_{i-1}^j+26u_{i-2}^j-24u_{i-3}^j+11u_{i-4}^j-2u_{i-5}^j}{h^4}))\right]$$

…(22)

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Boundary Conditions:

As shown in the implicit method, when approximating the boundary conditions of the equation, the same equations will be obtained for the exponential method, that is, equation (10) will be substituted into equation (12) when i=0, we get:

$$u_0^{j+1} = u_0^j \, exp\left[\frac{k}{u_0^j}(-u_0^j[\frac{u_1^j - u_{-1}^j}{2h}] - [\frac{u_1^j - 2u_0^j + 4u_{-1}^j}{h^2}] - [\frac{12u_1^j - 24u_0^j + 12u_{-1}^j}{h^4}] - \frac{12}{16^2h^2})\right] \quad \ldots(23)$$

And we substitute equation (8) into equation (23) to get:

$$u_0^{j+1} = u_0^j \, exp\left[\frac{k}{u_0^j}\left(\frac{u_0^j}{2h}\left(2u_1^j + 2u_0^j - \frac{h^2}{16^2}\right) - \frac{1}{16^2}\right)\right] \quad \ldots(24)$$

As for i=n, we substitute equation (11) into equation (12), we get:

$$u_n^{j+1} = u_n^j \, exp\left[\frac{k}{u_n^j}\left(-u_n^j\left[\frac{u_{n+1}^j - u_{n-1}^j}{2h}\right] - \left[\frac{u_{n+1}^j - 2u_n^j + u_{n-1}^j}{h^2}\right] - \left[\frac{12u_{n+1}^j - 24u_n^j + 12u_{n-1}^j}{h^4}\right] - \frac{12}{16^2h^2}\right)\right]$$
....(25)

And we substitute equation (9) into equation (25) to get:

$$u_n^{j+1} = u_n^j \, exp\left[\frac{k}{u_n^j}\left(\frac{u_n^j}{2h}\left(2u_n^j - 2u_{n-1}^j - \frac{h^2}{16^2}\right) - \frac{1}{16^2}\right)\right] \quad \ldots(26)$$

4- Numerical stability of the Kuramoto-Sivashinsky equation:

The theory of numerical stability of partial differential equations with the most general boundary conditions is often very difficult, as the duality between slicing boundary conditions and slicing partial differential equations. So, they can be unnoticeable [9].

4- 1 numerical stability analysis of the full implicit finite difference method:

Using the Fourier (Von-Neumann) method, we study the numerical stability of the complete finite difference method of the K.S. equation. And that this method requires converting the matter into a linear equation, so we delete the non-linear part of the equation as follows:

At the second node i=1 we use the following differential equation:

$$\frac{u_i^{j+1} - u_i^j}{k} = -\left[\frac{2u_i^{j+1} - 5u_{i+1}^{j+1} + 4u_{i+1}^{j+1} - u_{i+3}^{j+1}}{h^2}\right] - \left[\frac{3u_i^{j+1} - 14u_{i+1}^{j+1} + 26u_{i+2}^{j+1} - 24u_{i+3}^{j+1} + 11u_{i+4}^{j+1} - 2u_{i+5}^{j+1}}{h^4}\right]$$
...(27)

We substitute ${}_i^j$ with $\varphi(t)e^{m\beta x}$in equation (27), since $\varphi(t) \neq 0$ ، $\beta > 0$ *and* $m = \sqrt{-1}$ and multiplying the sides of the equation by $\frac{k}{e^{m\beta x}}$ and assuming that $r = \frac{k}{h^2}$, we get:

$$\frac{\varphi(t+k)}{\varphi(t)} = \frac{1}{a}$$

And that:

$$a = 1 + r\left[2 - 5e^{m\beta h} + 4e^{2m\beta h} - e^{3m\beta h}\right] + \frac{r}{h^2}\left[3 - 14e^{m\beta h} + 26e^{2m\beta h} - 24e^{3m\beta h} + 11e^{4m\beta h} - 2e^{5m\beta h}\right]$$

Using the binomial screwdriver of the denominator and simplifying, we get:

$$\frac{\varphi(t+k)}{\varphi(t)} = \frac{1}{a1}$$

$$a1 = 1 + r\left[(1 - e^{m\beta h})^3 + (1 - e^{m\beta h})^2\right] + \frac{r}{h^2}\left[2(1 - e^{m\beta h})^5 + (1 - e^{m\beta h})^4\right] \Rightarrow \frac{\varphi(t+k)}{\varphi(t)} = \frac{1}{a2}$$
...(28)

$$a2 = 1 + r(1 - e^{m\beta h})^3[1 + \frac{2}{h^2}(1 - e^{m\beta h})^2] + r(1 - e^{m\beta h})^2[1 + \frac{1}{h^2}(1 - e^{m\beta h})^2]$$

By simplifying equation (28) and rewriting it as follows:

$\frac{\varphi(t+k)}{\varphi(t)} = \frac{1}{A+mB} = \psi$ …(29)

Where E is the inflation factor and:

$$A = 1 + r\left(8rsin^6\left(\frac{\beta h}{2}\right) - 6sin^2(\beta h)sin^2\left(\frac{\beta h}{2}\right) + \frac{64}{h^2}sin^{10}\left(\frac{\beta h}{2}\right) - \frac{160}{h^2}sin^2(\beta h)sin^6\left(\frac{\beta h}{2}\right) + \frac{20}{h^2}sin^4(\beta h)sin^2\left(\frac{\beta h}{2}\right) + 4sin^4\left(\frac{\beta h}{2}\right) - sin^2(\beta h) + \frac{16}{h^2}sin^8\left(\frac{\beta h}{2}\right) - \frac{24}{h^2}sin^2(\beta h)sin^4\left(\frac{\beta h}{2}\right) + \frac{1}{h^2}sin^4(\beta h)\right)$$

$B = rsin(\beta h)\left(-12sin^4\left(\frac{\beta h}{2}\right) + sin^2(\beta h) - \frac{160}{h^2}sin^8\left(\frac{\beta h}{2}\right) + \frac{80}{h^2}sin^2(\beta h)sin^4\left(\frac{\beta h}{2}\right) - \frac{2}{h^2}sin^4(\beta h) - 4sin^2\left(\frac{\beta h}{2}\right) - \frac{32}{h^2}sin^6\left(\frac{\beta h}{2}\right) + \frac{8}{h^2}sin^2(\beta h)sin^2\left(\frac{\beta h}{2}\right)\right)$A necessary and sufficient condition for numerical stability is:

$$\left|\frac{\varphi(t+k)}{\varphi(t)}\right| = |\psi| \leq 1$$

For some values of (βh) being $sin^2\left(\frac{\beta h}{2}\right) = 1$ [8], we get:

$$A = 1 + r\left(5 - \frac{83}{h^2}\right)$$

$$B = rsin(\beta h)(-15 - \frac{106}{h^2})$$

Therefore, equation (29) becomes as follows:

$$|\psi| = \frac{1}{\sqrt{A^2 + B^2}} = \frac{1}{\sqrt{1 + 2r(5 - \frac{83}{h^2}) + r^2(5 - \frac{83}{h^2})^2 + r^2(-15 - \frac{106}{h^2})^2}}$$

It is obvious that $\sqrt{A^2 + B^2} \geq 1$ on it we get the fulfillment of the condition $|\psi| \leq 1$

As for the study of the stability of solutions at nodes i =2,3,..., n-2, we use the following differential equation:

$$\frac{u_i^{j+1} - u_i^j}{k} = -[\frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2}] - [\frac{u_{i+2}^{j+1} - 4u_{i+1}^{j+1} + 6u_i^{j+1} - 4u_{i-1}^{j+1} + u_{i+2}^{j+1}}{h^4}]$$

We substitute ${}_i^j$ with $\varphi(t)e^{m\beta x}$ into the equation above and assuming that $r = \frac{k}{h^2}$, we get:

$$\frac{\varphi(t+k)}{\varphi(t)} = \frac{1}{a}$$

That:

$$a = 1 + r\left[e^{m\beta h} - 2 + e^{-m\beta h}\right] + \frac{r}{h^2}\left[e^{2m\beta h} - 4e^{m\beta h} + 6 - 4e^{-m\beta h} + e^{-2m\beta h}\right]$$

$\frac{\varphi(t+k)}{\varphi(t)} = \frac{1}{a1} = \frac{1}{A} = \psi$ …(30)

$$a1 = 1 + 4rsin^2(\frac{\beta h}{2}) - \frac{8r}{h^2}sin^4(\frac{\beta h}{2}) + \frac{2r}{h^2}sin^2(\beta h) - \frac{8r}{h^2}sin^2(\frac{\beta h}{2})$$

For some values of (βh) being $sin^2\left(\frac{\beta h}{2}\right) = 1$, we get:

$$|\psi| = \frac{1}{\sqrt{A^2}} = \frac{1}{\sqrt{1 + r(-8 + \frac{28}{h^2}) + r^2(4 - \frac{14}{h^2})^2}}$$

It is obvious that $\sqrt{1 + r(-8 + \frac{28}{h^2}) + r^2(4 - \frac{14}{h^2})^2} \geq 1$ on it the condition $|\psi| \leq 1$ is fulfilled.

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As for the point i=n-1, we use the following differential equation:

$$\frac{u_i^{j+1}-u_i^j}{k}=-\left[\frac{2u_i^{j+1}-5u_{i-1}^{j+1}+4u_{i-2}^{j+1}-u_{i-3}^{j+1}}{h^2}\right]$$

$$=-\left[\frac{2u_i^{j+1}-5u_{i-1}^{j+1}+4u_{i-2}^{j+1}-u_{i-3}^{j+1}}{h^2}\right]$$

$$-\left[\frac{3u_i^{j+1}-14u_{i-1}^{j+1}+26u_{i-2}^{j+1}-24u_{i-3}^{j+1}+11u_{i-4}^{j+1}-2u_{i-5}^{j+1}}{h^4}\right] \quad \ldots(31)$$

We substitute $_i^j$ with $\varphi(t)e^{m\beta x}$ into equation (31) and assuming that $r=\frac{k}{h^2}$, we get:

$$\frac{\varphi(t+k)}{\varphi(t)}=\frac{1}{a}$$

Thus:

$$a=1+r[2-5e^{-m\beta h}+4e^{-2m\beta h}-e^{-3m\beta h}]$$
$$+\frac{r}{h^2}[3-14e^{-m\beta h}+26e^{-2m\beta h}-24e^{-3m\beta h}+11e^{-4m\beta h}-2e^{-5m\beta h}]$$

Using the binomial screwdriver of the denominator and simplifying, we get:

$$\frac{\varphi(t+k)}{\varphi(t)}=\frac{1}{a1}$$

$$a1=1+r[(1-e^{-m\beta h})^3+(1-e^{-m\beta h})^2]+\frac{r}{h^2}[2(1-e^{-m\beta h})^5+(1-e^{-m\beta h})^4]$$

$$\Rightarrow \frac{\varphi(t+k)}{\varphi(t)}=\frac{1}{a2} \quad \ldots(32)$$

$$a2=1+r(1-e^{-m\beta h})^3[1+\frac{2}{h^2}(1-e^{-m\beta h})^2]$$
$$+r(1-e^{-m\beta h})^2[1+\frac{1}{h^2}(1-e^{-m\beta h})^2]$$

By simplifying equation (32) and rewriting it as follows:

$$\frac{\varphi(t+k)}{\varphi(t)}=\frac{1}{A+mB}=\psi \quad \ldots(33)$$

Where is the inflation factor and if:

$$A=1+r(8rsin^6(\frac{\beta h}{2})-6sin^2(\beta h)sin^2(\frac{\beta h}{2})+\frac{64}{h^2}sin^{10}(\frac{\beta h}{2})$$
$$-\frac{160}{h^2}sin^2(\beta h)sin^6(\frac{\beta h}{2})+\frac{20}{h^2}sin^4(\beta h)sin^2(\frac{\beta h}{2})+4sin^4(\frac{\beta h}{2})$$
$$-sin^2(\beta h)+\frac{16}{h^2}sin^8(\frac{\beta h}{2})-\frac{24}{h^2}sin^2(\beta h)sin^4(\frac{\beta h}{2})+\frac{1}{h^2}sin^4(\beta h))$$

$$B=rsin(\beta h)\left(12sin^4\left(\frac{\beta h}{2}\right)-sin^2(\beta h)+\frac{160}{h^2}sin^8\left(\frac{\beta h}{2}\right)-\frac{80}{h^2}sin^2(\beta h)sin^4\left(\frac{\beta h}{2}\right)+\frac{2}{h^2}sin^4(\beta h)\right.$$
$$\left.+4sin^2\left(\frac{\beta h}{2}\right)+\frac{32}{h^2}sin^6\left(\frac{\beta h}{2}\right)-\frac{8}{h^2}sin^2(\beta h)sin^2\left(\frac{\beta h}{2}\right)\right)$$

For some values of (βh) being $sin^2\left(\frac{\beta h}{2}\right)=1$, we get:

$$A=1+r(5-\frac{83}{h^2})$$

$$B=rsin(\beta h)(15+\frac{106}{h^2})$$

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Using the necessary and sufficient condition for numerical stability in equation (33):

$$|\psi| = \frac{1}{\sqrt{A^2+B^2}} = \frac{1}{\sqrt{1+2r(5-\frac{83}{h^2})+r^2(5-\frac{83}{h^2})^2+r^2(15+\frac{106}{h^2})^2}}$$

It is obvious that $\sqrt{A^2+B^2} \geq 1$ on it we get the fulfillment of the condition $|\psi| \leq 1$

Therefore, the full implicit method is unconditionally stable at all points of the clamp.

4-2 Numerical stability analysis of the exponential finite difference method using the Fourier (Von-Neumann) method:

To study the numerical stability of the exponential difference method, we delete the nonlinear part of equation (19).

The equation with the nonlinear part is:

$$u_i^{j+1} = u_i^j exp\left[\frac{k}{u_i^j}\left(-u_i^j\left(\frac{\partial u}{\partial x}\right)_i^j - \left(\frac{\partial^2 u}{\partial x^2}\right)_i^j - \left(\frac{\partial^4 u}{\partial x^4}\right)_i^j\right)\right]$$

…(34)

Removing the nonlinear part from equation (34), we have:

$$u_i^{j+1} = u_i^j exp\left[\frac{k}{u_i^j}\left(-\left(\frac{\partial^2 u}{\partial x^2}\right)_i^j - \left(\frac{\partial^4 u}{\partial x^4}\right)_i^j\right)\right]$$

…(35)

To study numerical stability, starting with the second node when i=1, we use the following differential equation:

$$u_i^{j+1} = u_i^j exp[\frac{k}{u_i^j}(-(\frac{2u_i^j - 5u_{i+1}^j + 4u_{i+2}^j - u_{i+3}^j}{h^2})$$

$$-(\frac{3u_i^j - 14u_{i+1}^j + 26u_{i+2}^j - 24u_{i+3}^j + 11u_{i+4}^j - 2u_{i+5}^j}{h^4}))] \qquad \text{…(36)}$$

We substitute u_i^j with $\varphi(t)e^{m\beta x}$into equation (36) and after simplification, we get:

$$\frac{\varphi(t+k)}{\varphi(t)} = exp\{r(1-e^{m\beta h})^3[-1-\frac{2}{h^2}(1-e^{m\beta h})^2]$$

$$+r(1-e^{m\beta h})^2[-1-\frac{1}{h^2}(1-e^{m\beta h})^2]$$

Simplifying the above equation and using the same method as in the other methods, we obtain:

$$\frac{\varphi(t+k)}{\varphi(t)} = exp\{A+mB\} = \psi \qquad \text{…(37)}$$

Since:

$$A = r(-5+\frac{83}{h^2})$$

$$B = rsin(\beta h)(15+\frac{106}{h^2})$$

$$|\psi| = |exp\{A+iB\}| = e^A \leq 1$$

It should be $|\psi| = 1$ only if $A \leq 0$,so:

$$r(-5+\frac{83}{h^2}) \leq 0$$

Since $r > 0$ is always true on it.

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$$-5+\frac{83}{h^2}\leq 0\ \Rightarrow \frac{83}{h^2}\leq 5\Rightarrow h>\sqrt{\frac{83}{5}}$$

As for the study of stability at points i=2,..., n-2, we use the following differential equation:

$$u_i^{j+1}=u_i^j exp[\frac{k}{u_i^j}(-(\frac{u_{i+1}^j-2u_i^j+u_{i-1}^j}{h^2})$$

$$-(\frac{u_{i+2}^j-4u_{i+1}^j+6u_i^j-4u_{i-1}^j+u_{i-2}^j}{h^4}))] \qquad ..(38)$$

Substituting u_i^j with $\varphi(t)e^{m\beta x}$into equation (38) after simplification, we obtain:

$$\frac{\varphi(t+k)}{\varphi(t)}=exp\{-r[e^{m\beta h}-2+e^{-m\beta h}]$$

$$-\frac{r}{h^2}[e^{2m\beta h}-4e^{m\beta h}+6-4e^{-m\beta h}+e^{-2m\beta h}]\}$$

Simplifying, we get:

$$\frac{\varphi(t+k)}{\varphi(t)}=exp\{4rsin^2(\frac{\beta h}{2})-\frac{8r}{h^2}sin^4(\frac{\beta h}{2})+\frac{2r}{h^2}sin^2(\beta h)-\frac{8r}{h^2}sin^2(\frac{\beta h}{2})\}=e^A=\psi$$
…(39)

So that:

$$A=4rsin^2(\frac{\beta h}{2})-\frac{8r}{h^2}sin^4(\frac{\beta h}{2})+\frac{2r}{h^2}sin^2(\beta h)-\frac{8r}{h^2}sin^2(\frac{\beta h}{2})$$

Then:

$$|\psi|=|e^A|$$

In order for the condition to be fulfilled, it must be $A\leq 0$,so:

$$A=r(4-\frac{14}{h^2})$$

$$r(4-\frac{14}{h^2})\leq 0$$

Since $r>0$ is always true on it.

$$4-\frac{14}{h^2}\leq 0\ \Rightarrow \frac{14}{h^2}\leq 4$$

$$\Rightarrow \frac{1}{h^2}\leq\frac{4}{14}\Rightarrow h\geq\sqrt{\frac{14}{4}}$$

For node i=n-1, we use the following differential equation:

$$u_i^{j+1}=u_i^j exp[\frac{k}{u_i^j}(-(\frac{2u_i^j-5u_{i-1}^j+4u_{i-2}^j-u_{i-3}^j}{h^2})$$

$$-(\frac{3u_i^j-14u_{i-1}^j+26u_{i-2}^j-24u_{i-3}^j+11u_{i-4}^j-2u_{i-5}^j}{h^4}))] \qquad ..(40)$$

Substituting u_i^j with $\varphi(t)e^{m\beta x}$into equation (40) and simplifying the equation, we obtain:

$$\frac{\varphi(t+k)}{\varphi(t)}=exp\{-r[2-5e^{-m\beta h}+4e^{-2m\beta h}-e^{-3m\beta h}]$$

$$-\frac{r}{h^2}[3-14e^{-m\beta h}+26e^{-2m\beta h}-24e^{-3m\beta h}+11e^{-4m\beta h}-2e^{-5m\beta h}]\}$$

Using a binomial screw driver, we get:

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$$\frac{\varphi(t+k)}{\varphi(t)} = exp\{r\left(1-e^{-m\beta h}\right)^{3}[-1-\frac{2}{h^{2}}\left(1-e^{-m\beta h}\right)^{2}]$$

$$+r(1-e^{-m\beta h})^{2}[-1-\frac{1}{h^{2}}(1-e^{-m\beta h})^{2}]\}$$

Simplifying the above equation by the same method as in the other methods, we obtain:

$\frac{\varphi(t+k)}{\varphi(t)} = exp\{A+mB\} = \psi$ …(41)

Then:

$$A = r(-5+\frac{83}{h^{2}})$$

$$B = rsin(\beta h)(-15-\frac{106}{h^{2}})$$

From equation (41), we find that:

$$|\psi| = |exp\{A+mB\}| = e^{A} \leq 1$$

Since $r > 0$ is always true on it.

$$-5+\frac{83}{h^{2}} \leq 0 \quad \Rightarrow \frac{1}{h^{2}} \leq \frac{5}{83} \Rightarrow h \geq \sqrt{\frac{83}{5}}$$

We note that the exponential difference method is for solving the equation K.S. For a time step k is unconditionally stable for all points.

5- Numerical Results

In this paragraph, we will discuss the numerical results of the finite difference methods represented by the following methods: the fully implicit Scheme method and the Exponential Scheme method, the solution of equation (1) and boundary conditions (2) have been discussed taking the period [0,32 π] with the initial condition in equation (3), and the period [-30,30] with the initial condition in Equation (4):

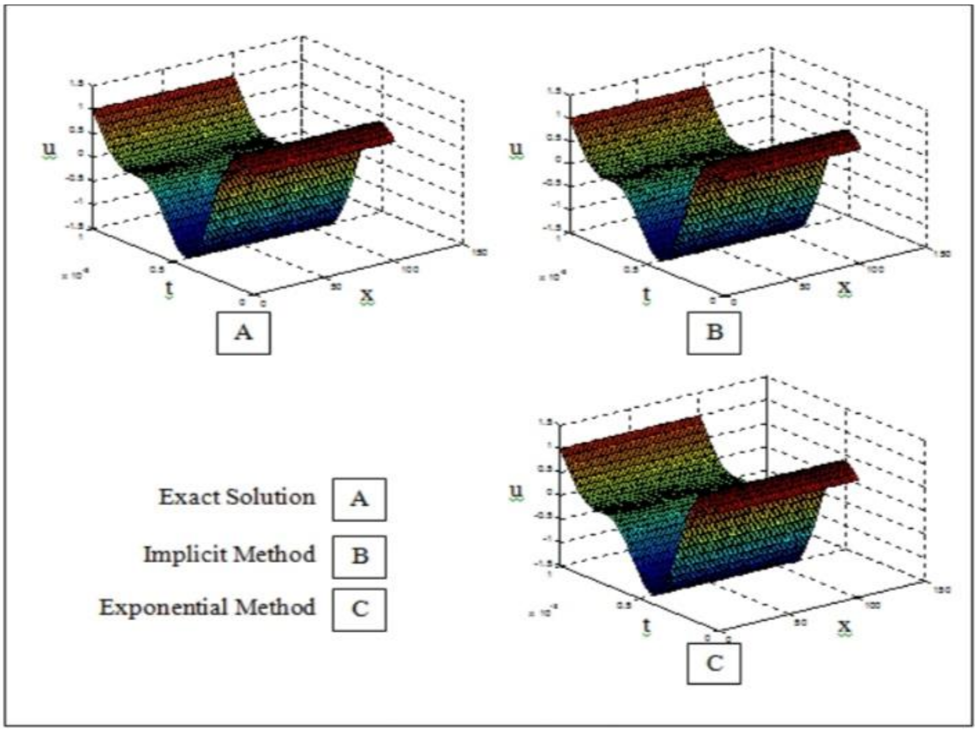

Figure 2. shows the numerical solutions of the two methods used and the exact solution of the period [0,32 π] and the initial condition in equation (3)

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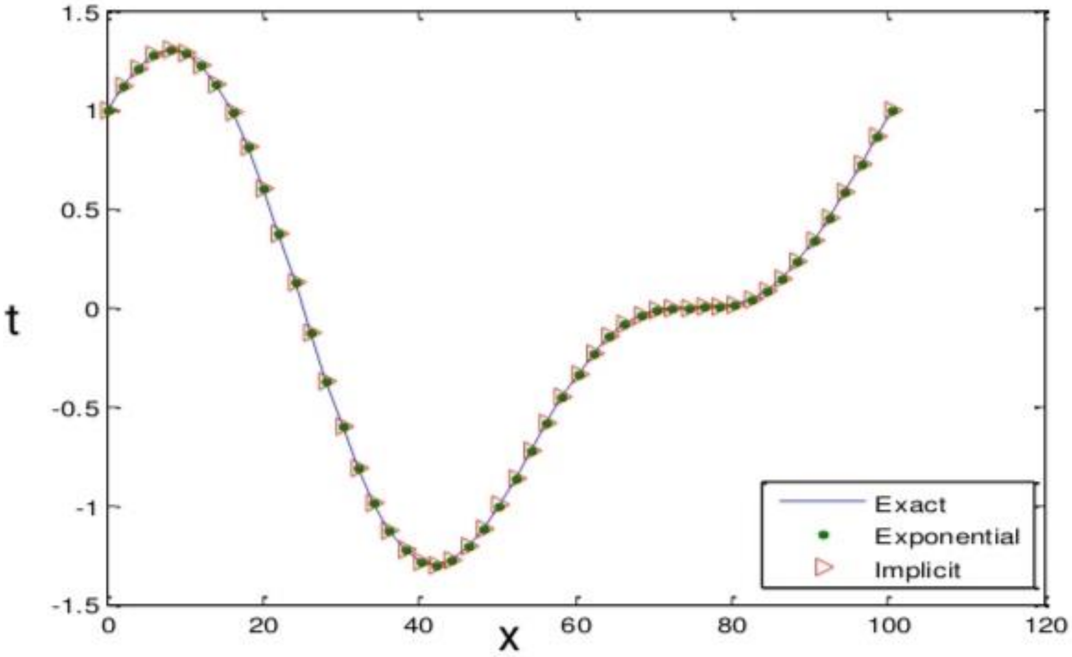

Figure 3. shows the comparison of the numerical solution of the two methods used with the exact solution of the period [0,32 π] and the initial condition in Equation (3)

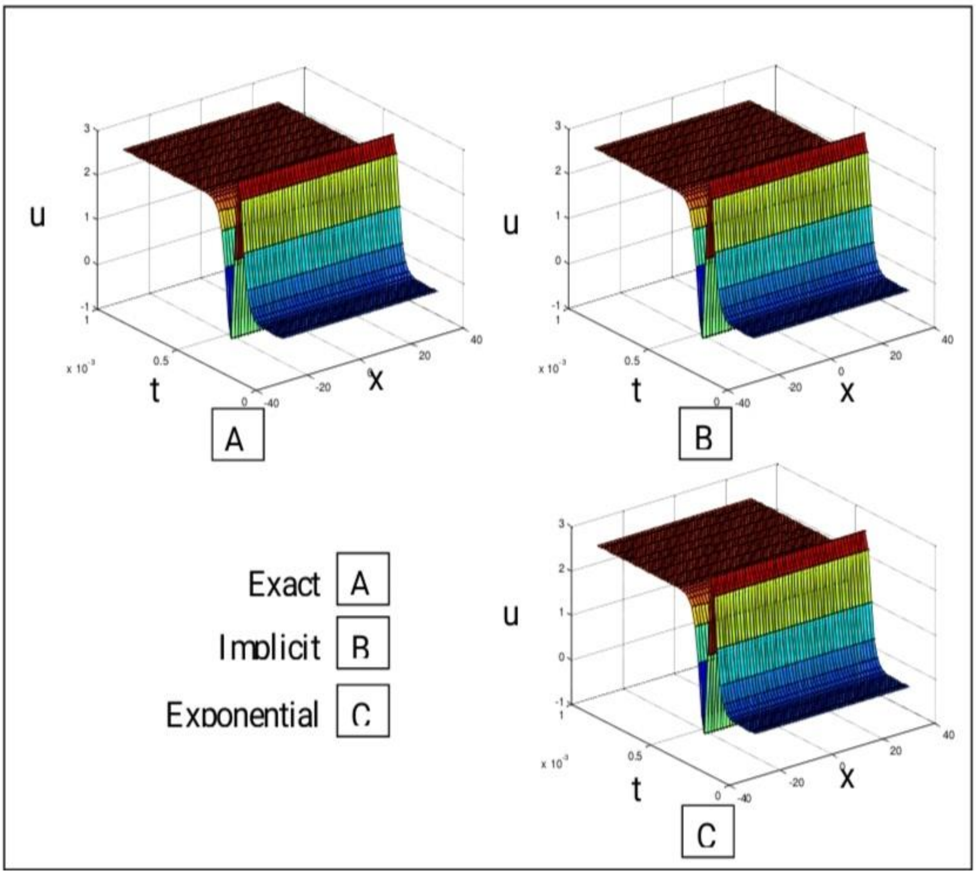

Figure 4. shows the numerical solutions of the two methods used and the exact solution of the period [-30,30] and the initial condition in the Equation (4)

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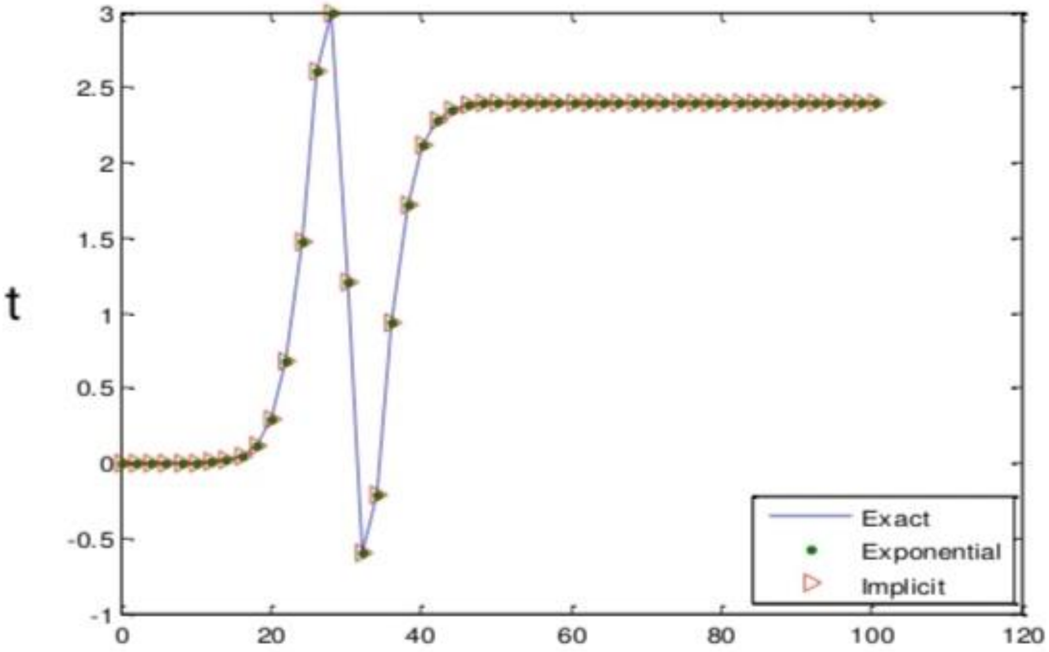

Figure 5. shows the comparison of the numerical solution of the two methods used with the exact solution of the period [-30,30] and the initial condition in the Equation (4)

6- conclusion

Through our study of the K.S equation. We have noticed that it is one of the important thermal equations in engineering and physical applications, and scientists have used it a lot recently, and it is one of the nonlinear equations that need great effort for the purpose of finding a numerical solution to it and need great accuracy in processing the fourth derivative, and we have reached the most important conclusions which are: The fully-Implicit method is unconditionally stable at all points of the clamp. The exponential difference method is for solving the K.S. Equation For a time step k is unconditionally stable for all points.

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