Solutions and convergence arguments for systems of hyperbolic time-fractional partial differential equations

Kamel Al-Khaled[1], **Adel Almalki**[2], **Amer H. Darweesh**[1], **Amal Sawalmeh**[1]
[1]Department of Mathematics & Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan
[2]Department of Mathematics, Al-Gunfudah University College, Umm Al-Qura University, Mecca 21955, Saudi Arabia
Emails: kamel@just.edu.jo; aaamalki@uqu.edu.sa; ahdarweesh@just.edu.jo; Amaalsawalmeh@gmail.com

Abstract

In order to solve hyperbolic fractional partial differential equations, this paper develops the Sumudu decomposition method. This method is based on solving time-fractional hyperbolic partial differential equations either individually or in systems using the Sumudu transform. Adomian polynomials whose values are chosen by a specific formula are used to solve the non-linear terms. The developed method's convergence and stability are discussed. Example such as the shallow water equations, which serve as illustrations of the fractional derivatives as defined by Caputo, is used to show the validity and applicability of the proposed method. It is discovered that the procedure is rapid and precise.

Keywords: Approximate Solutions; Fractional Partial Differential Equation; Adomian Decomposition; Sumudu transform; Shallow Water Equations

1 Introduction

Partial differential equations, or PDEs, are especially important in pure and applied mathematics. Since most natural processes are explained by nonlinear partial differential equations (PDEs), there has been a recent surge in interest in PDEs. Fractional calculus-based models are a helpful tool for accurately modeling a variety of phenomena in physics, engineering, and other sciences. Compared to derivatives of integer order, fractional derivatives offer more accurate representations of real-world issues. The scientific community has discovered that fractional partial differential equations are a helpful tool for explaining a variety of physical phenomena, such as diffusion processes,[1] electrical and rheological material properties, and viscoelasticity theories.[2] Time fractional partial differential equations must be solved. It is difficult to determine an exact solution to these equations due to their nonlinear nature.

Fractional partial differential equations have been solved using a variety of methods, such as the variational iteration method (VIM), homotopy perturbation method (HPM), Adomian decomposition method (ADM), homotopy analysis methodology,[3] and others. The Adomian decomposition technique (ADM) was proposed by Adomian[4] in 1980. This approach is effective for both linear and nonlinear equations with deterministic or stochastic operators, such as integral equations, partial differential equations (PDEs), and ordinary differential equations (ODEs). Based on decomposing the nonlinear operator into a series, this method computes the terms of the operator recursively using Adomian polynomials. The variable-depth shallow water equations containing the source term are solved in[5] using the Adomian decomposition method.

DOI: https://doi.org/10.54216/IJNS.250335

Received: March 22, 2024 Revised: June 25, 2024 Accepted: November 04, 2024

The Sumudu transform was shown to be the theoretical equivalent of the Laplace transform. In order to construct effective and understandable solutions for differential equations,[6–8] established several important properties of the Sumudu transform. One of the most crucial transformation methods and a powerful instrument for PDE solving is the Sumudu transform. First, as stated by the authors of,[9] the Sumudu transform is defined on general time scales, after which its salient features are elucidated.

The Sumudu transform is used in this work to solve nonlinear fractional partial differential equations of generic form.

$$D_t^{\alpha}\phi(x,t) = \mathbf{L}\phi(x,t) + \mathbf{N}\phi(x,t) + g(x,t) \tag{1}$$

under the presumption that $n-1 < \alpha \leqslant n$.

$$\frac{\partial^{(r)}\phi(x,0)}{\partial t^r} = \phi^{(r)}(x,0) = f_r(x), r = 0,1,...n-1. \tag{2}$$

where $D_t^{\alpha}\phi(x,t)$ is the Caputo fractional derivatives, $g(x,t)$ is the source term, $\mathbf{L}$ is the linear operator and $\mathbf{N}$ is the general nonlinear operator. The goal of this work is to replicate method for solving nonlinear fractional PDEs of the hyperbolic type. Our primary goal is to solve PDEs with nonlinear systems. In particular, if certain initial conditions are satisfied, we will focus on solving conservation laws also called shallow water equations that have fractional time derivatives. In order to solve hyperbolic fractional partial differential equations, the Sumudu decomposition method was created. The suggested approach is based on the Sumudu transform applied to nonlinear fractional partial differential equations. For managing the nonlinear term, algebraic polynomials are a useful tool,[4].[10] The Caputo sense characterizes the fractional derivatives.

This paper is formatted as follows: A review of partial differential equations and the fundamental fractional calculus utilized in this work is covered in the part one conclusion. A review of the Sumudu transform and a few key concepts and features come next. The Sumudu decomposition method is covered in Section 3, and the solution to the shallow water equations is covered in Section 4. Section 5 provides proof of the obtained solution's stability and convergence. The final section presents the numerical results.

2 Basic of Fractional Calculus

This section goes through some of the fundamental definitions and characteristics of several fractional operators that will be utilized in this thesis. Fractional calculus has been the subject of numerous definitions and studies over the past 200 years. These definitions include Riemman-Liouville, Weyl, Campos, Caputo, and Nishimoto fractional operators, which will be re-introduced in this study. We refer the reader for more details to.[11–14] The fractional derivative operator J_a^{α} has the following Riemann-Liouville definition:

Definition 1. [15] *let $\alpha \in \boldsymbol{R}^+$, The operator J^{α}, defined on the usual Lebesque space $L^1[a,b]$ by*

$$J_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad J_a^0 f(x) = f(x)$$

The Riemann-Liouville derivative presents unique challenges when attempting to simulate real-world events using fractional differential equations, as detailed in[16] and.[17] Following Caputo's lead, we ought to utilize the modified fractional differentiation operator D^{α} proposed in his viscoelasticity theory work.[18]

Definition 2. *In the Caputo interpretation, the fractional derivative of $f(x)$ is defined as*

$$D^{\alpha} f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \tag{3}$$

$m-1 < \alpha \leqslant m,\ m \in \mathbb{N},\ x > 0$

Two of its qualities are required here.

DOI: https://doi.org/10.54216/IJNS.250335
Received: March 22, 2024 Revised: June 25, 2024 Accepted: November 04, 2024

Lemma 1. *Provided that* $m-1<\alpha\leqslant m,\ f\in L^1[a,b]$*, then* $D_a^\alpha J_a^\alpha f(x)=f(x)$*, and*

$$J_a^\alpha D_a^\alpha f(x)=f(x)-\sum_{k=0}^{m-1} f^{(k)}(0^-)\frac{(x-a)^k}{k!},\ \ x>0.$$

The Caputo fractional derivative is studied in the context of the Caputo notion. The reason we went with the Caputo definition is as follows:,[16] in numerical form. In order for differential equations to have a unique solution, we need to provide additional criteria. These extra prerequisites are well known to us in the context of Caputo fractional differential equations since they are essentially the standard conditions that are applied to classical differential equations. In the context of Riemann-Liouville fractional differential equations, these additional conditions are fractional derivatives of the unknown solution at the initial point $x=0$, which are functions of x. The unknown function $\phi(x,t)$ is thought to be a time-dependent function.

Definition 3. *For m to be the smallest integer that exceeds* α*, the Caputo fractional derivatives of order* $\alpha>0$ *is defined as*

$$D^\alpha\phi(x,t)=\frac{\partial^\alpha\phi(x,t)}{\partial t^\alpha}=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_0^t(t-\tau)^{m-\alpha-1}\frac{\partial^m\phi(x,\tau)}{\partial\tau^m}d\tau, & m-1<\alpha<m\\ \frac{\partial^m\phi(x,t)}{\partial t^m}, & \alpha=m\in\boldsymbol{N}.\end{cases}$$

3 Sumudu Transform

Watugala[19] created a novel integral transform, termed the Sumudu transform, in the early 1990s and applied it to the solution of ordinary differential equations in control engineering applications. The Sumudu transform is currently not well-known or regularly utilized. The Sumudu transform, which has scale and unit-preserving features, can be utilized to address issues without turning to a new frequency domain. Belgacem et al[20] shown it to be the theoretical counterpart to the Laplace transform.

Definition 4. *Sumudu transform over the following set of functions*

$$\mathbf{A}=\{f(t):\exists M,\tau_1,\tau_2>0,|f(t)|<Me^{t/\tau_j},if\quad t\in(-1)^j\times[0,\infty)\}. \tag{4}$$

Defined as follows: given $\psi\in(\tau_1,\tau_2)$, $\mathbb{S}[f(t)]G(\psi)=\int_0^\infty f(\psi t)e^{-t}dt=\int_0^\infty\frac{1}{\psi}f(t)e^{\frac{-t}{\psi}}dt$*,*

While[7,21] lists many of the properties of the Sumudu transform, where the traits utilized in this paper can be consulted by the reader.

Theorem 2. *As in Belgacem,[7] let G(u) represent f (t)'s Sumudu transform in such a way that*

- *A meromorphic function,* $\frac{G(\frac{1}{s})}{s}$*, has singularities where* $Re(s)<\gamma$*.*
- *If* $|\frac{G(\frac{1}{s})}{s}|<MR^{-K}$*, a circular region* Γ *with radius R and positive constants,* M *and* K*, exists, then we have*

$$\begin{aligned}\mathbb{S}^{-1}[G(s)]&=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{st}G(\frac{1}{s})\frac{ds}{s}\\&=\sum residues[e^{st}\frac{G(\frac{1}{s})}{s}].\end{aligned}$$

To solve fractional differential equation we need the following lemma.

Lemma 3. [22] *The Sumudu transform* $\boldsymbol{S}[f(t)]$ *of the fractional derivative introduced by Caputo is given by*

$$\mathbb{S}[D_t^\alpha f(t)]=\frac{G(\psi)}{\psi^\alpha}-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{\psi^{\alpha-k}},\ \ where\ \ G(\psi)=\mathbb{S}[f(t)].$$

DOI: https://doi.org/10.54216/IJNS.250335
Received: March 22, 2024 Revised: June 25, 2024 Accepted: November 04, 2024

Because we are working with the numerical solution of fractional partial differential equations, we must deduce the result of the preceding lemma.

Lemma 4. $\mathbb{S}[f(x,t)]$ *is the Sumudu transform, as of Caputo's fractional derivative is given by*

$$\mathbb{S}[D_t^{\alpha} f(x,t)] = \frac{\mathbb{S}[f(x,t)]}{\psi^{\alpha}} - \sum_{k=0}^{n-1} \frac{f^{(k)}(x,0)}{\psi^{\alpha-k}},\ n-1 < \alpha \leqslant n \tag{5}$$

Proof: As in the above definition, we obtain the Sumudu transform of both sides for the partial derivative of $D_t^{\alpha} f(x,t)$.

$$\mathbb{S}[D_t^{\alpha} f(x,t)] = \mathbb{S}[\frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m f(x,\tau)}{\partial \tau^m} \partial\tau]$$

By the convolution theorem of Sumudu transform,[22] we obtain

$$\mathbb{S}[D_t^{\alpha} f(x,t)] = \frac{\psi}{\Gamma(m-\alpha)} \mathbb{S}[f^m(x,t)]\mathbb{S}[t^{-\alpha+m-1}].$$

We get the required outcome by employing the Sumudu transform of multiple differentiation.

4 The Sumudu Decomposition Method (SDM)

This section explains the steps involved in solving temporal fractional hyperbolic partial differential equations using the Sumudu decomposition method. The nonlinear term in the Sumudu decomposition approach can be easily broken down using Adamian polynomials. The Sumudu transform is used in the method to solve nonlinear fractional partial differential equations. The reader can view additional prior research using the Sumudu method by reading.[23,24]

4.1 The Scheme Applied to Nonlinear Systems of Fractional PDEs

The Sumudu decomposition method for solving a time fractional system of hyperbolic partial differential equations is described in this section. Examine the α order partial differential equations' time fractional system.

$$\begin{aligned} D_t^{\alpha}\phi(x,t) &= \mathbf{L}_1(\phi(x,t),\psi(x,t)) + \mathbf{N}_1(\phi(x,t),\psi(x,t)) + g_1(x,t) = 0 \\ D_t^{\alpha}\psi(x,t) &= \mathbf{L}_2(\phi(x,t),\psi(x,t)) + \mathbf{N}_2(\phi(x,t),\psi(x,t)) + g_2(x,t) = 0 \end{aligned} \tag{6}$$

with $n-1 < \alpha \leqslant n,\ \phi(x,t),\ \psi(x,t)$ are unknown functions and subject to the initial conditions

$$\begin{aligned} \frac{\partial^{(r)}\phi(x,0)}{\partial t^r} &= \phi^{(r)}(x,0) = f_r(x), r = 0,1,...n-1. \\ \frac{\partial^{(r)}\psi(x,0)}{\partial t^r} &= \psi^{(r)}(x,0) = g_r(x), r = 0,1,...n-1. \end{aligned} \tag{7}$$

Where $D_t^{\alpha}\phi(x,t)$ is the Caputo fractional derivatives, $g_1(x,t), g_2(x,t)$ are the source terms, $\mathbf{L}_1, \mathbf{L}_2$ are the linear operators and $\mathbf{N}_1, \mathbf{N}_2$ are the general nonlinear operators. We apply the Sumudu transform on both sides of equations (6)

$$\begin{aligned} \mathbb{S}[D_t^{\alpha}\phi(x,t)] &= \mathbb{S}[\mathbf{L}_1(\phi(x,t),\psi(x,t)) + \mathbf{N}_1(\phi(x,t),\psi(x,t)) + g_1(x,t)] = 0. \\ \mathbb{S}[D_t^{\alpha}\psi(x,t)] &= \mathbb{S}[\mathbf{L}_2(\phi(x,t),\psi(x,t)) + \mathbf{N}_2(\phi(x,t),\psi(x,t)) + g_2(x,t)] = 0. \end{aligned} \tag{8}$$

DOI: https://doi.org/10.54216/IJNS.250335

Received: March 22, 2024 Revised: June 25, 2024 Accepted: November 04, 2024

Utilizing the sumudu transform's differential property, we obtain

$$\begin{aligned}\psi^{-\alpha}\mathbb{S}[\phi(x,t)]-\sum_{k=0}^{m-1}\psi^{-(\alpha-k)}\phi^{(k)}(x,0)&=\mathbb{S}[\mathbf{L}_1(\phi(x,t),\psi(x,t))+\mathbf{N}_1(\phi(x,t),\psi(x,t))+g_1(x,t)].\\ \psi^{-\alpha}\mathbb{S}[\psi(x,t)]-\sum_{k=0}^{m-1}\psi^{-(\alpha-k)}\psi^{(k)}(x,0)&=\mathbb{S}[\mathbf{L}_2(\phi(x,t),\psi(x,t))+\mathbf{N}_2(\phi(x,t),\psi(x,t))+g_2(x,t)].\end{aligned} \tag{9}$$

After simplifying and multiplying both sides of the aforementioned equation by ψ^{α}, we get

$$\begin{aligned}\mathbb{S}[\phi(x,t)]&=\sum_{k=0}^{m-1}\psi^{k}\phi^{(k)}(x,0)+\psi^{\alpha}\mathbb{S}[\mathbf{L}_1(\phi(x,t),\psi(x,t))+\mathbf{N}_1(\phi(x,t),\psi(x,t))+g_1(x,t)]\\ \mathbb{S}[\psi(x,t)]&=\sum_{k=0}^{m-1}\psi^{k}\psi^{(k)}(x,0)+\psi^{\alpha}\mathbb{S}[\mathbf{L}_2(\phi(x,t),\psi(x,t))+\mathbf{N}_2(\phi(x,t),\psi(x,t))+g_2(x,t)]\end{aligned} \tag{10}$$

We present the solution as an infinite series given by[4,10,25]

$$\phi(x,t)=\sum_{n=0}^{\infty}\phi_n(x,t),\quad \psi(x,t)=\sum_{n=0}^{\infty}\psi_n(x,t). \tag{11}$$

Decomposing the nonlinear operators yields

$$\mathbf{N}_1(\phi(x,t),\psi(x,t))=\sum_{n=0}^{\infty}A_n(\phi,\psi),\quad \mathbf{N}_2(\phi(x,t),\psi(x,t))=\sum_{n=0}^{\infty}B_n(\phi,\psi). \tag{12}$$

Where $A_n(\phi,\psi), B_n(\phi,\psi)$ are the Adomian polynomials[25,26] of $\phi_0,\phi_1,...,\phi_n,\psi_0,\psi_1,...,\psi_n$ that are given by

$$\begin{aligned}A_n(\phi_0,\phi_1,...,\phi_2,\psi_0,\psi_1,...,\psi_n)&=\frac{1}{n!}\frac{d^n}{d\lambda^n}\mathbf{N}_1(\sum_{K=0}^{\infty}\lambda^k\phi_k,\lambda^k\psi_k)]|_{\lambda=0}, n\geqslant 0\\ B_n(\phi_0,\phi_1,...,\phi_2,\psi_0,\psi_1,...,\psi_n)&=\frac{1}{n!}\frac{d^n}{d\lambda^n}\mathbf{N}_2(\sum_{K=0}^{\infty}\lambda^k\phi_k,\lambda^k\psi_k)]|_{\lambda=0}, n\geqslant 0\end{aligned} \tag{13}$$

Substituting equations (11), (12) into (10), we get

$$\begin{aligned}\mathbb{S}[\sum_{n=0}^{\infty}\phi_n(x,t)]&=\sum_{k=0}^{m-1}\psi^{k}\phi^{(k)}(x,0)+\psi^{\alpha}\mathbb{S}[\mathbf{L}_1(\sum_{n=0}^{\infty}\phi_n(x,t))+(\sum_{n=0}^{\infty}A_n(\phi,\psi))+g_1(x,t)]\\ \mathbb{S}[\sum_{n=0}^{\infty}\psi_n(x,t)]&=\sum_{k=0}^{m-1}\psi^{k}\psi^{(k)}(x,0)+\psi^{\alpha}\mathbb{S}[\mathbf{L}_2(\sum_{n=0}^{\infty}\phi_n(x,t))+(\sum_{n=0}^{\infty}B_n(\phi,\psi))+g_2(x,t)]\end{aligned} \tag{14}$$

Matching both sides of (14) we get the following iterative algorithm

$$\begin{aligned}&\mathbb{S}[\phi_0(x,t)]=\sum_{k=0}^{\infty}\psi^{k}\phi^{(k)}(x,0)\\ &\mathbb{S}[\phi_1(x,t)]=\psi^{\alpha}\mathbb{S}[\mathbf{L}_1(\phi_0(x,t),\psi_0(x,t))+A_0(\phi(x,t),\psi(x,t))+g_1(x,t)]\\ &\mathbb{S}[\phi_{n+1}(x,t)]=\psi^{\alpha}\mathbb{S}[\mathbf{L}_1(\phi_n(x,t),\psi_n(x,t))+A_n(\phi(x,t),\psi(x,t))+g_1(x,t)], n\geqslant 1.\\ &\mathbb{S}[\psi_0(x,t)]=\sum_{k=0}^{\infty}\psi^{k}\psi^{(k)}(x,0)\\ &\mathbb{S}[\psi_1(x,t)]=\psi^{\alpha}\mathbb{S}[\mathbf{L}_2(\phi_0(x,t),\psi_0(x,t))+B_0(\phi(x,t),\psi(x,t))+g_2(x,t)]\\ &\mathbb{S}[\psi_{n+1}(x,t)]=\psi^{\alpha}\mathbb{S}[\mathbf{L}_2(\phi_n(x,t),\psi_n(x,t))+B_n(\phi(x,t),\psi(x,t))+g_2(x,t)], n\geqslant 1.\end{aligned}$$

DOI: https://doi.org/10.54216/IJNS.250335

Received: March 22, 2024 Revised: June 25, 2024 Accepted: November 04, 2024

Take the Sumudu inverse on both sides of the above equation then we have

$$\phi_0(x,t) = \mathbb{S}^{-1}[\sum_{k=0}^{\infty} \psi^k \phi^{(k)}(x,0)]$$
$$\phi_1(x,t) = \mathbb{S}^{-1}[\psi^\alpha \mathbb{S}[\mathbf{L}_1(\phi_0(x,t), \psi_0(x,t)) + A_0(\phi(x,t), \psi(x,t)) + g_1(x,t)]]$$
$$\phi_{n+1}(x,t) = \mathbb{S}^{-1}[\psi^\alpha \mathbb{S}[\mathbf{L}_1(\phi_n(x,t), \psi_n(x,t)) + A_n(\phi(x,t), \psi(x,t)) + g_1(x,t)]], n \geqslant 1.$$
$$\psi_0(x,t) = \mathbb{S}^{-1}[\sum_{k=0}^{\infty} \psi^k \psi^{(k)}(x,0)]$$
$$\psi_1(x,t) = \mathbb{S}^{-1}[\psi^\alpha \mathbb{S}[\mathbf{L}_2(\phi_0(x,t), \psi_0(x,t)) + B_0(\phi(x,t), \psi(x,t)) + g_2(x,t)]]$$
$$\psi_{n+1}(x,t) = \mathbb{S}^{-1}[\psi^\alpha \mathbb{S}[\mathbf{L}_2(\phi_n(x,t), \psi_n(x,t)) + B_n(\phi(x,t), \psi(x,t)) + g_2(x,t)]], n \geqslant 1.$$

The truncated series can be used to approximate the solutions $\phi_n(x,t;\alpha), \psi_n(x,t;\alpha)$.

$$\phi_n(x,t) = \sum_{j=0}^{n-1} \phi_j(x.t), \quad \psi_n(x,t) = \sum_{j=0}^{n-1} \psi_j(x.t) \tag{15}$$

such that

$$\lim_{n\to\infty} \phi_n(x,t) = \phi(x,t), \quad \lim_{n\to\infty} \psi_n(x,t) = \psi(x,t) \tag{16}$$

In real physical problems, the decomposition series solutions typically converge very quickly. Numerous writers have looked into the convergence of the decomposition series. Section 6 presents the numerical example results, where accurate solutions can be obtained by requiring a minimal number of terms. α will be involved in $\phi_n(x,t)$ as a variable so we may varies between 0 and 1. By selecting large values for n in the computation of $\phi(x,t)$, the expressions of A_n, B_n contains more terms. The Adomian decomposition approach simplifies the massive calculation. Section 6 presents the results of numerical applications, and just a few terms are necessary to produce precise solutions. While in the next section, we present the convergence proof together with the stability analysis of the procedure.

5 Stability analysis of STADM

In this section, we analyse the stability of STADM when it is applied to solve fractional shallow water equations. Let $(\mathcal{B}, ||.||)$ be a Banach space and Υ be a self-map of $\mathcal{B}$ ($\Upsilon : \mathcal{B} \to \mathcal{B}$), satisfying $||\Upsilon_x - \Upsilon_y|| \leqslant k||x - \Upsilon_x|| + \beta||x - y||$, for all $x, y \in \mathcal{B}, k \geqslant 0$, and $0 < \beta < 1$, then Υ is Picard,$\Upsilon-$stable, for more details we refer readers to.[27] Since our equations in this work are in terms of the unknowns $\phi(x,t), \psi(x,t)$, we will use $\omega(x,t)$ for both of the unknowns without sacrificing generality.

Theorem 5. *Let $(\mathcal{B}, \|\ .\ \|)$ be a Banach space and $\Upsilon : \mathcal{B} \to \mathcal{B}$ be a self-map of $\mathcal{B}$. Then, the iteration procedure of STADM defined by*

$$\Upsilon(\omega_m(x,t)) = \omega_{m+1}(x,t) = \omega(x,0) + S^{-1}[\psi^\alpha S[f(x,t) - R\omega_m(x,t) - A_m(\omega)]],$$

is Υ-stable if $\epsilon = (\epsilon_0 + \epsilon_1)||\frac{t^\alpha}{\Gamma(\alpha+1)}|| < 1$ for some constants $\epsilon_0, \epsilon_1 \in \mathbb{R}^+$ and for any t in the domain.

Proof. We first demonstrate the fixed point of Υ. To accomplish this, we have for $n, m \in \mathbb{N}$

$$\Upsilon(\omega_n(x,t)) = \omega(x,0) + S^{-1}[\psi^\alpha S[f(x,t) - R\omega_n(x,t) - A_n(\omega)]],$$
$$\Upsilon(\omega_m(x,t)) = \omega(x,0) + S^{-1}[\psi^\alpha S[f(x,t) - R\omega_m(x,t) - A_m(\omega)]].$$

The result of subtracting the two equations above is

$$\Upsilon(\omega_n(x,t)) - \Upsilon(\omega_m(x,t)) = S^{-1}[\psi^\alpha S[R\omega_m(x,t) + A_m(\omega)] - S^{-1}[\psi^\alpha S[R\omega_n(x,t) + A_n(\omega)]$$

Next, taking the norm on both sides of the previously mentioned result, we have, without sacrificing generality,

$$||\Upsilon(\omega_n(x,t)) - \Upsilon(\omega_m(x,t))|| = ||S^{-1}[\psi^\alpha S[R\omega_m(x,t) + A_m(\omega)] - S^{-1}[\psi^\alpha S[R\omega_n(x,t) + A_n(\omega)]||$$

DOI: https://doi.org/10.54216/IJNS.250335

Received: March 22, 2024 Revised: June 25, 2024 Accepted: November 04, 2024

By applying the Sumudu transform's linearity property and its inverse, we are able to obtain

$$\| \Upsilon(\omega_n(x,t)) - \Upsilon(\omega_m(x,t)) \| = \| S^{-1}[\psi^\alpha S[R\omega_m(x,t)]] + S^{-1}[\psi^\alpha S[A_m(\omega)]] \quad - S^{-1}[\psi^\alpha S[R\omega_n(x,t)]] - S^{-1}[\psi^\alpha S[A_n(\omega)]] \| .$$

We move forward by using the norm's properties

$$\| \Upsilon(\omega_n(x,t)) - \Upsilon(\omega_m(x,t)) \| \leqslant \| S^{-1}[\psi^\alpha S[R\omega_m(x,t) - R\omega_n((x,t))] \| + \| S^{-1}[\psi^\alpha S[A_m(\omega) - A_n(\omega)] \| . \tag{17}$$

Assuming for the moment for $\epsilon_0 \in \mathbb{R}^+$ that

$$\| R\omega_m(x,t) - R\omega_n((x,t)) \| \leqslant \epsilon_0 \| \omega_m(x,t) - \omega_n((x,t) \|,$$

and that for some $\epsilon_1 \in \mathbb{R}^+$, we have

$$\| A_m(\omega) - A_n(\omega) \| \leqslant \epsilon_1 \| \omega_m(x,t) - \omega_n((x,t) \| .$$

Equation (17) then turns into

$$\| \Upsilon(\omega_n(x,t)) - \Upsilon(\omega_m(x,t)) \| \leqslant \Big(\epsilon_0 ||\omega_m(x,t) - \omega_n((x,t) \| + \epsilon_1 \| \omega_m(x,t) - \omega_n((x,t) \| \Big) \| S^{-1}[\psi^\alpha S[1]].$$

But based on the Sumudu transform's characteristics, we have

$$\| S^{-1}[\psi^\alpha S[1]] \| = \| S^{-1}[\psi^\alpha(1)] \| = S^{-1}[\psi^\alpha] = \| \frac{t^\alpha}{\Gamma(\alpha+1)} \| .$$

Therefore, we obtain

$$\begin{aligned} \| \Upsilon(\omega_n(x,t)) - \Upsilon(\omega_m(x,t)) \| &\leqslant \Big(\epsilon_0 \| \omega_m(x,t) - \omega_n((x,t) \| + \epsilon_1 \| \omega_m(x,t) - \omega_n((x,t) \| \Big) \| \frac{t^\alpha}{\Gamma(\alpha+1)} \| \\ &\leqslant (\epsilon_0 + \epsilon_1) \| \frac{t^\alpha}{\Gamma(\alpha+1)} \| \| \| \omega_m(x,t) - \omega_n((x,t) \| \\ &\leqslant \epsilon \| \omega_m(x,t) - \omega_n((x,t) \|, \end{aligned}$$

where $\epsilon = (\epsilon_0 + \epsilon_1) \| \frac{t^\alpha}{\Gamma(\alpha+1)} \|$. Consequently, the self-mapping Υ has a fixed point. Now, we prove that Υ satisfies the condition given in Theorem 5. Considering this, we have

$$\| \Upsilon(\omega_n(x,t)) - \Upsilon(\omega_m(x,t)) \| \leqslant k \| \omega_n(x,t) - \Upsilon(\omega_m(x,t)) + \epsilon \| v_n(x,t) - v_m(x,t) \|$$

for $k = 0, \epsilon = (\epsilon_0 + \epsilon_1)||\frac{t^\alpha}{\Gamma(\alpha+1)}|| < 1$. This shows that the conditions of Theorem 5 hold for self-mapping Υ. Hence, Υ is Picard and Υ-stable. This indicates that the recurrence iterations $\phi_{n+1}(x,t),\ \psi_{n+1}(x,t)$ are Υ-stable.

5.1 Convergence analysis of STADM

The convergence of STADM applied to a time fractional system of hyperbolic partial differential equations is demonstrated in this section. Given a mapping associated with STADM (defined in section 4), also we may refer to,[27] let $\Upsilon : \mathcal{B} \to \mathcal{B}$ be the mapping, and let $(\mathcal{B}, ||.||)$ be a Banach space. Afterwards, if $0 < \epsilon < 1$ such that $||\omega_{n+1}(x,t)|| \leqslant \epsilon||\omega_n(x,t)||$, then Υ has a unique fixed point, and the sequence solution $\{\omega_n(x,t)\}_{n=0}^{\infty}$ converges to the fixed point of $\mathcal{B}$, starting at $\omega_0 \in \mathcal{B}$.

Theorem 6. *Given a mapping associated with STADM (defined in section 4), let $\Upsilon : \mathcal{B} \to \mathcal{B}$ be the mapping, and let $(\mathcal{B}, ||.||)$ be a Banach space. Afterwards, if $0 < \epsilon < 1$ such that $||\omega_{n+1}(x,t)|| \leqslant \epsilon||\omega_n(x,t)||$, then Υ has a unique fixed point, and the sequence solution $\{\omega_n(x,t)\}_{n=0}^{\infty}$ converges to the fixed point of $\mathcal{B}$, starting at $\omega_0 \in \mathcal{B}$.*

DOI: https://doi.org/10.54216/IJNS.250335

Received: March 22, 2024 Revised: June 25, 2024 Accepted: November 04, 2024

Proof. First, we show the existence of a fixed point of Υ. For this, define $\{\eta_n\}$ as the sequence of partial sums of the series equation:

$$\begin{aligned}
\eta_0 &= \omega_0(x,t),\\
\eta_1 &= \omega_0(x,t) + \omega_1(x,t),\\
\eta_2 &= \omega_0(x,t) + \omega_1(x,t) + \omega_2(x,t),\\
\eta_n &= \omega_0(x,t) + \omega_1(x,t) + \omega_2(x,t) + \ldots + \omega_n(x,t).
\end{aligned}$$

We now demonstrate that in Banach space $(\mathcal{B}, ||.||)$, $\{\eta_n\}_{n=0}^{\infty}$ is a Cauchy sequence. For this reason, we consider

$$||\eta_{n+1} - \eta_n|| = ||\omega_{n+1}(x,t)|| \leqslant \epsilon||\omega_n(x,t)|| \leqslant \epsilon^2||\omega_{n-1}(x,t)|| \leqslant \ldots \leqslant \epsilon^{n+1}||\omega_0(x,t)||.$$

Now, using the prior equation and the triangle inequality, we obtain for each $m, n \in \mathbb{N}$ where $n > m$

$$||\eta_n - \eta_m|| = ||(\eta_n - \eta_{n-1}) + (\eta_{n-1} - \eta_{n-2}) + \ldots + (\eta_{m+1} - \eta_m)||$$

$$\leqslant ||\eta_n - \eta_{n-1}|| + ||\eta_{n-1} - \eta_{n-2}|| + \ldots + ||\eta_{m+1} - \eta_m||$$

$$\leqslant \epsilon^n||\omega_0(x,t)|| + \epsilon^{n-1}||\omega_0(x,t)|| + \ldots + \epsilon^{m+1}||\omega_0(x,t)||$$

$$\leqslant \epsilon^{m+1}||\omega_0(x,t)||(1 + \epsilon + \epsilon^2 + \ldots + \epsilon^{n-m}). \tag{18}$$

A finite geometric sequence with a total sum of $\frac{1-\epsilon^{n-m}}{1-\epsilon}$ is represented by the sum $1 + \epsilon + \epsilon^2 + \ldots + \epsilon^{n-m}$ since $0 < \epsilon < 1$. With $1 - \epsilon^{n-m} < 1$, then the above equation becomes

$$||\eta_n - \eta_m|| \leqslant \frac{\epsilon^{m+1}}{1-\epsilon}||\omega_0(x,t)||. \tag{19}$$

Since $\omega_0(x,t)$ is bounded, $\lim_{n,m\to\infty} ||\eta_n - \eta_m|| = 0$. Hence, $\eta_n = \eta_m$ is implied. As a result, $\{\eta_n\}_{n=0}^{\infty}$ is a convergent Cauchy sequence. Thus, there is a fixed point for Υ.

To complete this proof, we must demonstrate that ω is the only fixed point of Υ after letting $\{\omega_n(x,t)\}$ converge to $\omega \in \mathcal{B}$. Let ℓ be an additional fixed point of Υ. Next, by (18), we possess $||\omega - \ell|| = ||\Upsilon\omega - \Upsilon\ell|| \leqslant \epsilon||\omega - \ell||$, which implies that $(1-\epsilon)||u-l|| \leqslant 0$. The inequality mentioned above can only be true if $||u-l|| = 0$ implies $u = l$, since $(1-\epsilon) < 0$ for $0 < \epsilon < 1$. This completes the proof that ω is the only fixed point of Υ.

5.2 Error bounds

The system in (6) has an exact solution, $\lim_{n\to\infty} \phi_n(x,t) = \phi(x,t), \quad \lim_{n\to\infty} \psi_n(x,t) = \psi(x,t)$. There is a finite number of terms in the series (15) that can be truncated to obtain the numerical solution. If $\sum_{j=0}^{n-1} \phi_j(x.t)$ (similarly $\sum_{j=0}^{n-1} \psi_j(x.t)$) gives the n terms approximate solution, then the absolute point-wise error bound depends on the partial sum $\sum_{j=0}^{n-1} \phi_j(x.t)$ and which is bounded by $\frac{\epsilon^{m+1}}{1-\epsilon}||\phi_0(x,t)||$, where ϵ as defined in the above theorem.

DOI: https://doi.org/10.54216/IJNS.250335

Received: March 22, 2024 Revised: June 25, 2024 Accepted: November 04, 2024

6 The Equations for Shallow Wave Water

The one-dimensional shallow water equations regulate the flow of water in a sizable, frictionless channel with a rectangular cross-section and a constantly shifting bottom surface. According to,[28] these equations are derived under the assumptions of a hydrostatic pressure distribution and a short bottom slope.[29] illustrates the fluid flow in an infinitely broad channel in the one-dimensional case.

$$\frac{\partial^\alpha}{\partial t^\alpha}\begin{pmatrix}\phi\\ \psi\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\phi\psi\\ \frac{1}{2}\psi^2+\phi\end{pmatrix}=\begin{pmatrix}0\\ H^{'}\end{pmatrix}, x\in\mathbb{R}, t>0 \tag{20}$$

based on the initial circumstances

$$\begin{pmatrix}\phi(x,0)\\ \psi(x,0)\end{pmatrix}=\begin{pmatrix}g_1(x)\\ g_2(x)\end{pmatrix}, x\in\mathbb{R} \tag{21}$$

The function $\psi(x,t)$ represents the total height above the bottom of the channel, and $\phi(x,t)$ represents the fluid velocity. $H(x)$ represents the depth of a point in relation to a fixed reference level of the water. The two independent variables are x and t, which stand for the time and the distance along the flow direction, respectively. In a more condensed form, (20) can be written as

$$\frac{\partial^\alpha U}{\partial t^\alpha}+\mathcal{F}^{'}(U)\frac{\partial U}{\partial x}=\mathcal{G}(x,t) \tag{22}$$

where

$$U(x,t)=\begin{pmatrix}\phi(x,t)\\ \psi(x,t)\end{pmatrix}, \mathcal{F}^{'}(U)=\begin{pmatrix}\phi & \psi\\ 1 & \psi\end{pmatrix}, \mathcal{G}(x,t)=\begin{pmatrix}0\\ H^{'}(x).\end{pmatrix}$$

By rewriting the system as, the shallow water equations can be solved.

$$\frac{\partial^\alpha \phi}{\partial t^\alpha}+\psi\frac{\partial\phi}{\partial x}+\phi\frac{\partial\psi}{\partial x}=0,\quad \phi(x,0)=g_1(x)$$
$$\frac{\partial^\alpha \psi}{\partial t^\alpha}+\frac{\partial\phi}{\partial x}+\psi\frac{\partial\phi}{\partial x}+\psi\frac{\partial\psi}{\partial x}=H^{'}(x),\quad \psi(x,0)=g_2(x)$$

Next, both sides of the previously given equation are subjected to the Sumudu transform, producing the following outcome:

$$\begin{aligned}\mathbb{S}\Big[\frac{\partial^\alpha \phi}{\partial t^\alpha}+\psi\frac{\partial\phi}{\partial x}+\phi\frac{\partial\psi}{\partial x}\Big]=0\\ \mathbb{S}\Big[\frac{\partial^\alpha \psi}{\partial t^\alpha}+\frac{\partial\phi}{\partial x}+\psi\frac{\partial\phi}{\partial x}+\psi\frac{\partial\psi}{\partial x}\Big]=\mathbb{S}[H^{'}(x)]\end{aligned} \tag{23}$$

Let us assume that $\Phi_1(\phi,\psi)=\psi\phi_x$, $\Phi_2(\phi,\psi)=\phi\psi_x$, and $\Phi_3(\psi)=\psi\psi_x$. Make use of the initial condition and the sumudu transform property. As we have

$$\begin{aligned}\mathbb{S}[\phi(x,t)]=\frac{1}{\psi^\alpha}\sum_{k=0}^{m-1}\psi^{-\alpha+k}\phi^{(k)}(x,0)-\frac{1}{\psi^\alpha}\mathbb{S}[\phi_1+\phi_2]\\ \mathbb{S}[\psi(x,t)]=\frac{1}{\psi^\alpha}\sum_{k=0}^{m-1}\psi^{-\alpha+k}\psi^{(k)}(x,0)=\frac{1}{\psi^\alpha}\mathbb{S}[H^{'}x]-\frac{1}{\psi^\alpha}\mathbb{S}[\phi_x+\phi_3]\end{aligned} \tag{24}$$

Using the inverse Sumudu transform, convert both sides of equation (24).

$$\begin{aligned}\phi(x,t)=\mathbb{S}^{-1}[\frac{1}{\psi}\phi(x,0)]-\mathbb{S}^{-1}[\frac{1}{\psi^\alpha}\mathbb{S}[\Phi_1+\Phi_2]]\\ \psi(x,t)=\mathbb{S}^{-1}[\frac{1}{\psi}\psi(x,0)]+\mathbb{S}^{-1}[\mathbb{S}[H^{'}-\phi_x-\Phi_3]].\end{aligned} \tag{25}$$

DOI: https://doi.org/10.54216/IJNS.250335

Received: March 22, 2024 Revised: June 25, 2024 Accepted: November 04, 2024

Let $\phi(x,t)$ and $\psi(x,t)$ be unknown functions with an infinite series solution provided by

$$\phi(x,t) = \sum_{n=0}^{\infty} \phi_n(x,t); \quad \psi(x,t) = \sum_{n=0}^{\infty} \psi_n(x,t) \tag{26}$$

and the infinite series of Adomian polynomials Φ_1, Φ_2, and Φ_3 by the nonlinear operators

$$\Phi_1(\phi,\psi) = \sum_{n=0}^{\infty} A_n, \;\; \Phi_2(\phi,\psi) = \sum_{n=0}^{\infty} B_n, \;\; \Phi_3(\psi) = \sum_{n=0}^{\infty} C_n. \tag{27}$$

Where the Adomian's polynomials, $A_n,\ B_n$, and C_n, are constructed based on an algorithm. In this instance, we utilize the general form of the Adomian polynomial formulas for A_n, B_n, and C_n as

$$\begin{aligned} A_n(\phi_0,\phi_1,...,\phi_2,\psi_0,\psi_1,...,\psi_n) &= \frac{1}{n!}\frac{d^n}{d\lambda^n}\Phi_2(\sum_{i=0}^{\infty}\lambda^k\phi_k,\lambda^k\psi_k)]|_{\lambda=0},\ n \geqslant 0 \\ B_n(\phi_0,\phi_1,...,\phi_2,\psi_0,\psi_1,...,\psi_n) &= \frac{1}{n!}\frac{d^n}{d\lambda^n}\Phi_1(\sum_{i=0}^{\infty}\lambda^k\phi_k,\lambda^k\psi_k)]|_{\lambda=0},\ n \geqslant 0 \\ c_n(\psi_0,\psi_1,...,\psi_2) &= \frac{1}{n!}\frac{d^n}{d\lambda^n}[\Phi_3(\sum_{i=0}^{\infty}\lambda^k\psi_k)]|_{\lambda=0},\ n \geqslant 0 \end{aligned} \tag{28}$$

To obtain as many polynomials as required for the computation of the numerical solution, these are simple formulas to set up in computer code. We can offer the first few Adomian polynomials for the nonlinearity for $\Phi_1 = \psi\phi_x, \Phi_2 = \phi\psi_x, \Phi_3 = \psi\psi_x$ as an aid to the reader.

$A_0 = \psi_0\phi_{0x},\ A_1 = \psi_1\phi_{0x}+\psi_0\phi_{1x},\ \ A_2 = \psi_2\phi_{0x}+\psi_1\phi_{1x}+\psi_0\phi_{2x},\ \ A_3 = \psi_3\phi_{0x}+\psi_2\phi_{1x}+\psi_1\phi_{2x}+\psi_0\phi_{3x},$

$B_0 = \phi_0\psi_{0x},\ \ B_1 = \phi_1\psi_{0x}+\phi_0\psi_{1x},\ \ B_2 = \phi_2\psi_{0x}+\phi_1\psi_{1x}+\phi_0\psi_{2x},\ \ B_3 = \phi_3\psi_{0x}+\phi_2\psi_{1x}+\phi_1\psi_{2x}+\phi_0\psi_{3x},$

$C_0 = \psi_0\psi_{0x},\ \ C_1 = \psi_1\psi_{0x}+\psi_0\psi_{1x},\ \ C_2 = \psi_2\psi_{0x}+\psi_1\psi_{1x}+\psi_0\psi_{2x},\ \ C_3 = \psi_3\psi_{0x}+\psi_2\psi_{1x}+\psi_1\psi_{2x}+\psi_0\psi_{3x}.$

The recursive relations provided by are used to construct the nonlinear system.

$$\begin{aligned} &\phi_0(x,t) = g_1(x) \\ &\phi_{n+1} = -\mathbb{S}^{-1}[\frac{1}{\psi^{\alpha}}\mathbb{S}[A_n + B_n]] \\ &\psi_0(x,t) = g_2(x) \\ &\psi_{n+1} = -\mathbb{S}^{-1}[\frac{1}{\psi^{\alpha}}\mathbb{S}[H^{'} - \phi_{nx} - C_n]], n \geqslant 1. \end{aligned} \tag{29}$$

where the initial conditions are where the functions $g_1(x)$ and $g_2(x)$ originate. We build $\phi(x,t)$ and $\psi(x,t)$ as the solutions.

$$\lim_{n\to\infty}\phi_n(x,t) = \phi(x,t), \quad \lim_{n\to\infty}\psi_n(x,t) = \psi(x,t) \tag{30}$$

$$\phi_n(x,t) = \sum_{k=0}^{n-1}\phi_k(x,t), \quad \psi_n(x,t) = \sum_{k=0}^{n-1}\psi_k(x,t), n \geqslant 0 \tag{31}$$

Now in Equation (20), we take

$$H(x) = \frac{e^{-x^2}}{1+e^{-x^2}}$$

and assume that the water's initial height $g_2(x)$ and velocity $g_1(x)$ are known.

$$\begin{aligned} &\phi(x,0) = g_1(x) = H(x) + \frac{sech(x)}{4}, x\ \in \mathbb{R}. \\ &\psi(x,0) = g_2(x) = 0,\ x \in \mathbb{R} \end{aligned} \tag{32}$$

DOI: https://doi.org/10.54216/IJNS.250335

Received: March 22, 2024 Revised: June 25, 2024 Accepted: November 04, 2024

When we apply the Sumudu transform to the above equation on both sides, we obtain

$$\begin{aligned} \mathbb{S}[\phi_\alpha + \psi\phi_x + \phi\psi_x] &= 0 \\ \mathbb{S}[\psi_\alpha + \phi_x + \psi\psi_x] &= \mathbb{S}[H^{'}(x)] \end{aligned} \tag{33}$$

Assuming $\phi_1(\phi,\psi) = \psi\phi_x$, $\phi_2(\phi,\psi) = \phi\psi_x$, and $\phi_3(\psi) = \psi\psi_x$, simplifying and applying the Sumudu transform property and the initial condition. As we have

$$\begin{aligned} \mathbb{S}[\phi(x,t)] &= \frac{1}{\psi^\alpha}\sum_{k=0}^{m-1}\psi^{-\alpha+k}\phi^{(k)}(x,0) - \frac{1}{\psi^\alpha}\mathbb{S}[\phi_1+\phi_2] \\ \mathbb{S}[\psi(x,t)] &= \frac{1}{\psi^\alpha}\sum_{k=0}^{m-1}\psi^{-\alpha+k}\psi^{(k)}(x,0) = \frac{1}{\psi^\alpha}\mathbb{S}[H^{'}(x)] - \frac{1}{\psi^\alpha}\mathbb{S}[\phi_x+\phi_3] \end{aligned} \tag{34}$$

Taking the inverse Sumudu transform for the both sides of equation (34)

$$\begin{aligned} \phi(x,t) &= \mathbb{S}^{-1}[\frac{1}{\psi}\phi(x,0)] - \mathbb{S}^{-1}[\frac{1}{\psi^\alpha}\mathbb{S}[\phi_1+\phi_2]] \\ \psi(x,t) &= \mathbb{S}^{-1}[\frac{1}{\psi}\psi(x,0)] + \mathbb{S}^{-1}[\mathbb{S}[H^{'} - \phi_x - \phi_3]] \end{aligned} \tag{35}$$

Assume that the infinite series solution for the unknown functions $\phi(x,t)$ and $\psi(x,t)$ is of the following form.

$$\phi(x,t) = \sum_{n=0}^{\infty}\phi_n(x,t), \quad \psi(x,t) = \sum_{n=0}^{\infty}\psi_n(x,t). \tag{36}$$

And the nonlinear operators ϕ_1, ϕ_2 and ϕ_3 by the infinite series of Adomian polynomials given by

$$\Phi_1(\phi,\psi) = \sum_{n=0}^{\infty}A_n, \;\; \Phi_2(\phi,\psi) = \sum_{n=0}^{\infty}B_n, \;\; \Phi_3(\phi,\psi) = \sum_{n=0}^{\infty}C_n \tag{37}$$

where the Adomian's polynomials, which are built using an algorithm, are denoted by the variables A_n, B_n, and C_n. The general form of the formulas for A_n, B_n and C_n Adomian polynomials is applied in this instance. As we mentioned in a previous section, we can provide the first few Adomian polynomials for the nonlinearity for $\phi_1 = \psi\phi_x, \phi_2 = \phi\psi_x, \phi_3 = \psi\psi_x$ respectively.

The construction of the nonlinear system takes the form of the recursive relations provided by

$$\phi_0(x,t) = g_1(x), \; \phi_{n+1} = -\mathbb{S}^{-1}[\frac{1}{\psi^\alpha}\mathbb{S}[A_n + B_n]]$$

$$\psi_0(x,t) = g_2(x), \psi_{n+1} = -\mathbb{S}^{-1}[\frac{1}{\psi^\alpha}\mathbb{S}[H^{'} - \phi_{nx} - C_n]], n \geqslant 1$$

where the initial conditions give rise to the functions $g_1(x)$, $g_2(x)$. So, we get

$$\phi_0(x,t) = \frac{e^{-x^2}}{1+e^{-x^2}} + \frac{sech(x)}{4}$$

$$\phi_1(x,t) = 0$$

$$\phi_2(x,t) = \frac{t^{2\alpha}sech^4(x)}{16(1+e^{x^2})^2\Gamma(1+2\alpha)}(-2 - 4e^{x^2} - 2e^{2x^2} - 5(1+e^{x^2})\,cosh(x) + (1+e^{x^2})^2 cosh(2x)$$

$$+ cosh(3x) + e^{x^2}cosh(3x) + 2e^{x^2}x\; sinh(x) + 2e^{x^2}xsinh(3x).$$

$$\phi_3(x,t) = \frac{e^{x^2}t^{2\alpha}}{2(1+e^{x^2})^4\Gamma(1+2\alpha)}(-4 - 8x^2 + 4e^{x^2}(4x^2-1) + (1+e^{x^2})sech(x)(-1-2x^2$$

$$+ e^{x^2}(-1+2x^2) + (1+e^{x^2})\;x\;tanh(x)))$$

.

.

DOI: https://doi.org/10.54216/IJNS.250335

Received: March 22, 2024 Revised: June 25, 2024 Accepted: November 04, 2024

and,

$$\psi_0(x,t) = 0$$

$$\psi_1(x,t) = \frac{t^\alpha sech(x)\ tanh(x)}{4\Gamma(1+\alpha)}$$

$$\psi_2(x,t) = \frac{\frac{2e^{-2x^2}x}{(1+e^{-2x^2})^2} - \frac{2e^{-x^2}x}{1+e^{-x^2}}}{\Gamma(1+\alpha)}$$

$$\psi_3(x,t) = \frac{2e^{x^2}t^\alpha x}{(1+e^{x^2})^2\Gamma(1+\alpha)} + \frac{t^{3\alpha}\Gamma(1+2\alpha)\ sech(x)^5(sinh(3x)-7sinh(x))}{64\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}$$

$$-\frac{t^{3\alpha}sech(x)^5}{32(1+e^{x^2})^3\Gamma(1+3\alpha)}(20xe^{x^2} + 20e^{2x^2} + 16e^{x^2}(1+e^{x^2})x\ cosh(2x) - 4e^{x^2}(1+e^{x^2})x\ cosh(4x)$$

$$+22sinh(x) + 66e^{x^2}sinh(x) + 66e^{2x^2}sinh(x) + 22e^{3x^2}sinh(x) + 22sinh(2x) + 48e^{x^2}sinh(2x)$$

$$+26e^{2x^2}sinh(2x) + 8e^{x^2}x^2sinh(2x) - 2sinh(3x) - 6e^{x^2}sinh(3x)$$

$$-6e^{2x^2}sinh(3x) - 2e^{3x^2}sinh(3x) - sinh(4x) + e^{2x^2}sinh(4x) + 4e^{x^2}x^2sinh(4x) - 4e^{2x^2}sinh(4x)$$

.

and so on; in this manner, the remaining elements of the decomposition series can be obtained. The approximations for responses are given by

$$\begin{aligned}
&\phi(x,t) = \phi_0(x,t) + \phi_1(x,t) + \phi_2(x,t) + ...\\
=&\frac{e^{-x^2}}{1+e^{-x^2}} + \frac{sech(x)}{4} + \frac{t^{2\alpha}sech^4(x)}{16(1+e^{x^2})^2\Gamma(1+2\alpha)}(-2 - 4e^{x^2} - 2e^{2x^2}\\
&-5(1+e^{x^2})\ cosh(x) + (1+e^{x^2})^2cosh(2x) + cosh(3x) + e^{x^2}cosh(3x)\\
&+2e^{x^2}x\ sinh(x) + 2e^{x^2}x\ sinh(3x) + ...\\
&\psi(x,t) = \psi_0(x,t) + \psi_1(x,t) + \psi_2(x,t) + ...\\
=&\frac{t^\alpha sech(x)\ tanh(x)}{4\Gamma(1+\alpha)} + \frac{\frac{2e^{-2x^2}x}{(1+e^{-2x^2})^2} - \frac{2e^{-x^2}x}{1+e^{-x^2}}}{\Gamma(1+\alpha)} + \frac{2e^{x^2}t^\alpha x}{(1+e^{x^2})^2\Gamma(1+\alpha)} + ...
\end{aligned} \tag{38}$$

The $3D$ surface for the shallow water equations' solution $\phi(x,t), \psi(x,t)$ are depicted in Figures 5-8 when $\alpha = 1, 0.9, 0.75, 0.5$.

7 Numerical Tables for Approximated Solutions

To provide a clear understanding of the behavior of the approximated solutions, we present numerical tables for different values of α, x, and t. These tables contain values for both $\phi(x,t)$ and $\psi(x,t)$, providing insight into how each function evolves over time and space.

DOI: https://doi.org/10.54216/IJNS.250335
Received: March 22, 2024 Revised: June 25, 2024 Accepted: November 04, 2024

7.1 Table: Numerical Values of the Approximated Solution $\phi(x,t)$

Table 1: Numerical values of the approximated solution $\phi(x,t)$ for different α

x	t	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
0.25	0.25	0.722156	0.719349	0.714821
0.25	0.50	0.707046	0.698092	0.684532
0.25	0.75	0.677591	0.658374	0.630266
0.50	0.25	0.658079	0.657375	0.656481
0.50	0.50	0.654650	0.653855	0.654049
0.50	0.75	0.651725	0.653460	0.658843
0.75	0.25	0.557334	0.558232	0.559908
0.75	0.50	0.562527	0.566659	0.574103
0.75	0.75	0.575823	0.587593	0.607233
1.00	0.25	0.433350	0.434823	0.437201
1.00	0.50	0.441263	0.445999	0.453233
1.00	0.75	0.456801	0.467097	0.482293

7.2 Table: Numerical Values of the Approximated Solution $\psi(x,t)$

Table 2: Numerical values of the approximated solution $\psi(x,t)$ for different α

x	t	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
0.25	0.25	-0.0786442	-0.0938054	-0.111046
0.25	0.50	-0.156250	-0.173514	-0.191164
0.25	0.75	-0.231778	-0.246598	-0.260304
0.50	0.25	-0.157955	-0.187708	-0.220747
0.50	0.50	-0.309720	-0.340178	-0.368450
0.50	0.75	-0.449102	-0.468060	-0.479673
0.75	0.25	-0.227511	-0.269856	-0.316258
0.75	0.50	-0.443306	-0.484080	-0.519415
0.75	0.75	-0.635667	-0.654944	-0.659589
1.00	0.25	-0.262605	-0.312197	-0.367323
1.00	0.50	-0.516416	-0.567760	-0.615527
1.00	0.75	-0.752639	-0.785616	-0.806103

8 Visualizations

The visualization of the solutions for $\phi(x,t)$ and $\psi(x,t)$ for different values of α is crucial for understanding their behavior in 3D. The solutions are plotted over a range of x values from -3 to 3 and a range of t values to illustrate the wave patterns that emerge due to the fractional derivative's influence.

DOI: https://doi.org/10.54216/IJNS.250335

Received: March 22, 2024 Revised: June 25, 2024 Accepted: November 04, 2024

8.1 2D Plots for $\phi(x,t)$ and $\psi(x,t)$

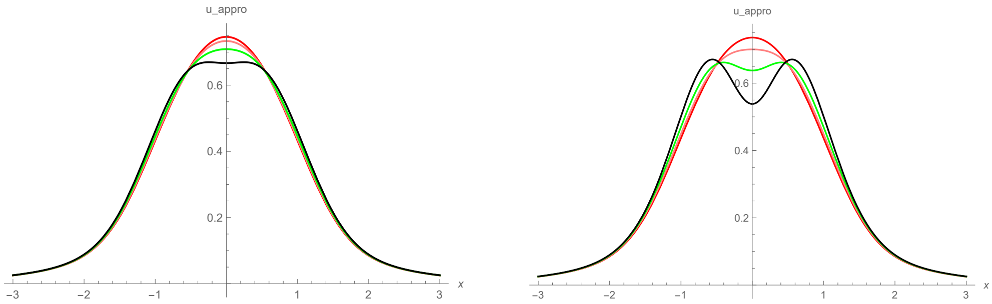

Figure 1: Plots of $\phi(x,t)$ for different values of t and α, $-3 \leqslant x \leqslant 3$, (left) $\alpha = 1$, (right) $\alpha = 0.75$.

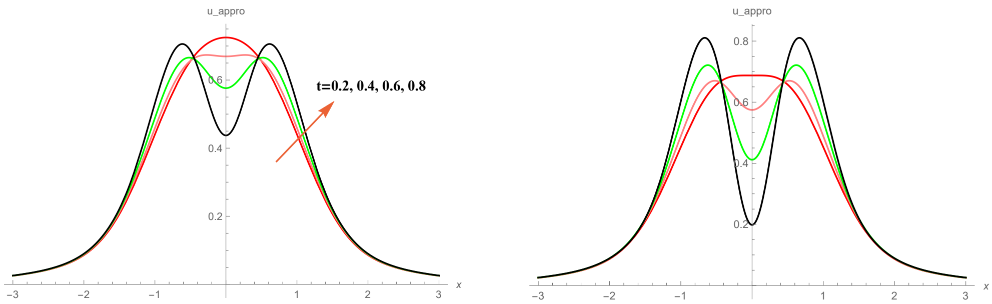

Figure 2: Plots of $\phi(x,t)$ for different values of t and α, $-3 \leqslant x \leqslant 3$, (left) $\alpha = 0.65$, (right) $\alpha = 0.5$.

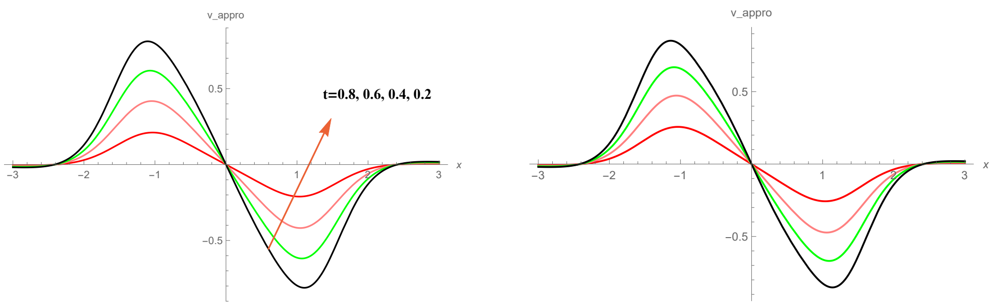

Figure 3: Plots of $\psi(x,t)$ for different values of t and α, $-3 \leqslant x \leqslant 3$, (left) $\alpha = 1$, (right) $\alpha = 0.9$.

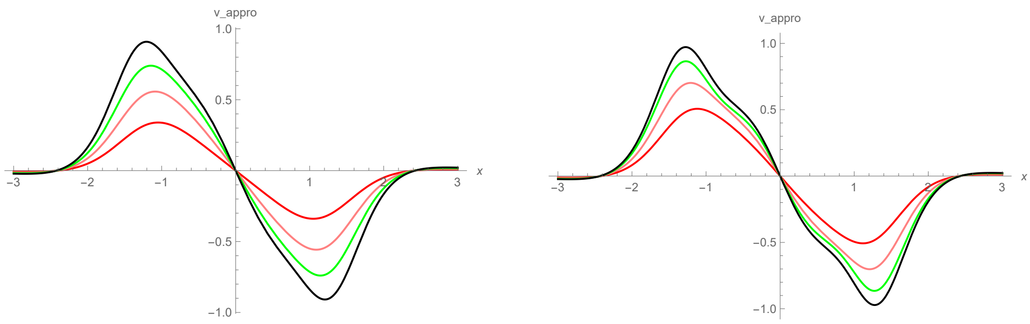

Figure 4: Plots of $\psi(x,t)$ for different values of t and α, $-3 \leqslant x \leqslant 3$, (left) $\alpha = 0.75$, (right) $\alpha = 0.5$.

DOI: https://doi.org/10.54216/IJNS.250335

Received: March 22, 2024 Revised: June 25, 2024 Accepted: November 04, 2024

8.2 3D Surface Plots for $\phi(x,t)$ and $\psi(x,t)$

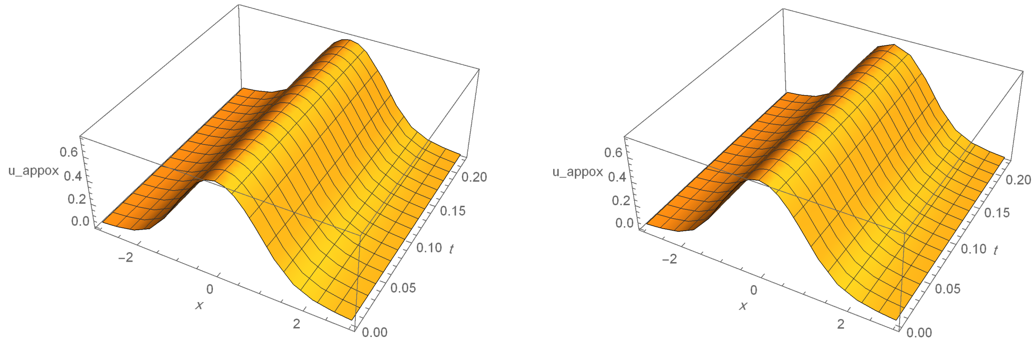

Figure 5: Surface plot of $\phi(x,t)$, for $-3 \leqslant x \leqslant 3$, (left) $\alpha = 1$, (right) $\alpha = 0.9$.

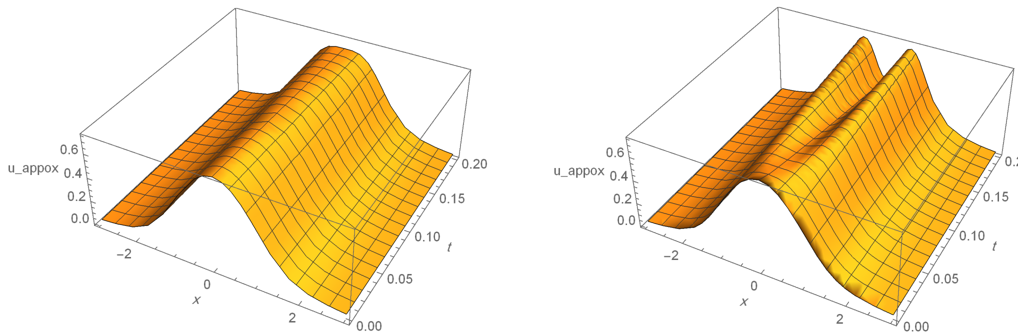

Figure 6: Surface plot of $\phi(x,t)$, for $-3 \leqslant x \leqslant 3$, (left) $\alpha = 0.9$, (right) $\alpha = 0.5$.

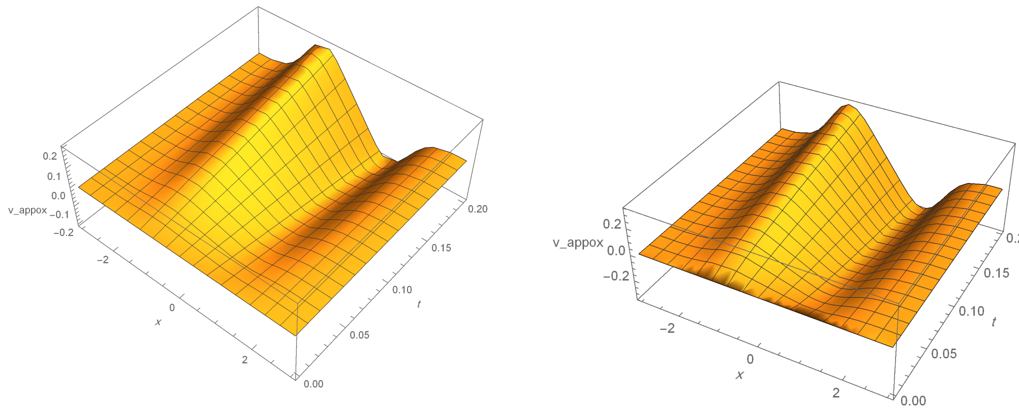

Figure 7: Surface plot of $\psi(x,t)$, for $-3 \leqslant x \leqslant 3$, (left) $\alpha = 1$, (right) $\alpha = 0.9$.

DOI: https://doi.org/10.54216/IJNS.250335

Received: March 22, 2024 Revised: June 25, 2024 Accepted: November 04, 2024

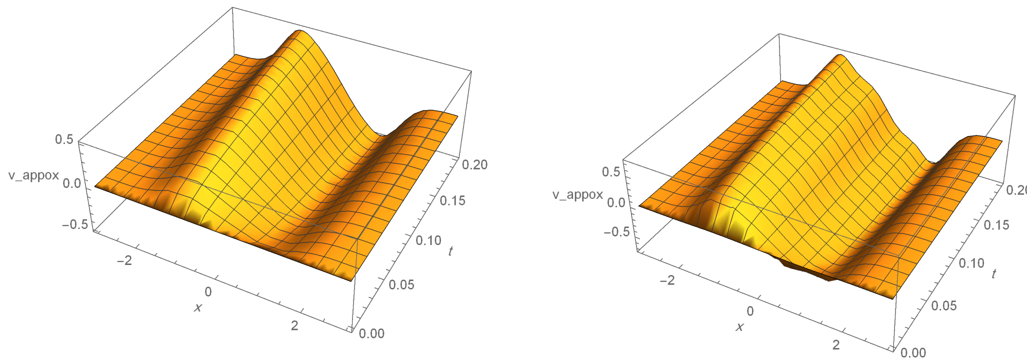

Figure 8: Surface plot of $\psi(x,t)$, for $-3 \leqslant x \leqslant 3$, (left) $\alpha = 0.75$, (right) $\alpha = 0.5$.

8.3 Contour Plots for $\phi(x,t)$ and $\psi(x,t)$

The contour plots in Figures 9-10 are darkly colored, emphasizing a strong central peak that progressively fades to a lighter blue at the edges, indicating the decreasing wave intensity.

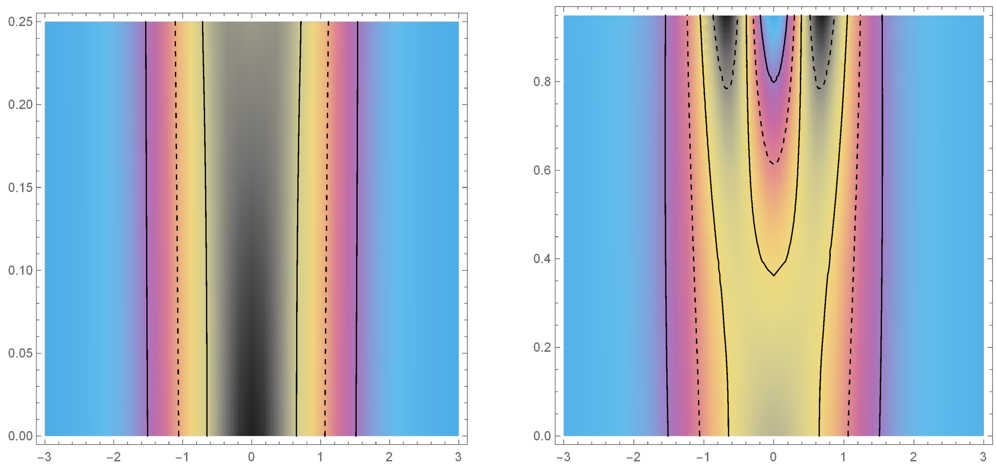

Figure 9: A contour plot for $\phi(x,t)$ (left) when $\alpha = 0.5$ and $0 < t < 0.25$ while the (right) is $\phi(x,t)$ when $\alpha = 0.95$ and $0 < t < 0.5$

DOI: https://doi.org/10.54216/IJNS.250335
Received: March 22, 2024 Revised: June 25, 2024 Accepted: November 04, 2024

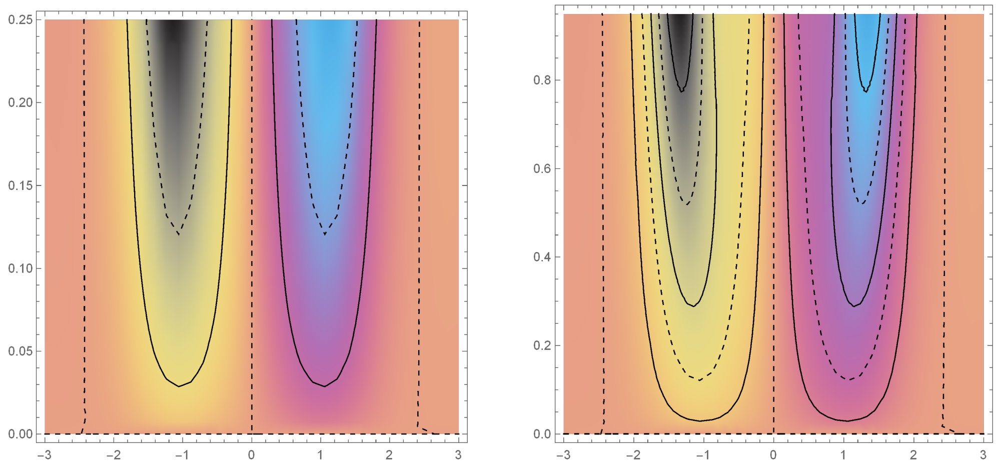

Figure 10: A contour plot for $\psi(x,t)$ (left) when $\alpha = 0.5$ and $0 < t < 0.25$ while the (right) is $\psi(x,t)$ when $\alpha = 0.95$ and $0 < t < 0.5$

9 Discussion

Figures $1-8$, either in 2D or 3D, demonstrate how waveforms in shallow water equations change with different fractional derivative values. In the left Figures, where the fractional derivative is close to $\alpha = 1$, we see a single, well-defined wave. On the other hand, the right Figures, with a fractional derivative of $\alpha = 0.5$, presents a more complex scenario. Here, the wave splits into two distinct parts for the ϕ variable, forming two separate peaks with dark centers and lighter surrounding areas. This dual-wave pattern arises because the lower fractional derivative alters the wave's behavior, making it more complex and less like the single wave seen when the derivative is closer to $\alpha = 1$. The contour plots in Figures $9-10$ highlights a strong central peak, shown by a dark color, which gradually fades to a lighter blue at the edges, indicating the wave's diminishing intensity. This represents a typical wave pattern similar to classical solutions. The Figures clearly show how fractional calculus influences wave structures, transitioning from a single wave at higher derivatives to multiple waves as the derivative decreases.

10 Conclusion

The numerical results and visualizations demonstrate the effectiveness of the Sumudu decomposition method in solving time-fractional hyperbolic partial differential equations, such as the shallow water equations. The solutions reveal different wave behaviors depending on the fractional order α, with more classical behavior for $\alpha = 1$ and increasingly complex wave patterns as α decreases. This method is both stable and convergent, making it a powerful tool for analyzing and solving fractional differential equations in various scientific and engineering applications.

Authors contributions

This article was written in collaboration with all of the contributors. All writers read and approved the final manuscript.

Conflicts of interest

There are no competing interests declared by the authors.

DOI: https://doi.org/10.54216/IJNS.250335

Received: March 22, 2024 Revised: June 25, 2024 Accepted: November 04, 2024

References

[1] Mohamed AE Herzallah, Ahmed MA El-Sayed, and Dumitru Baleanu. On the fractional-order diffusion-wave process. *Rom. J. Phys*, 55(3-4):274–284, 2010.

[2] Mohammad Javidi and Bashir Ahmad. Numerical solution of fractional partial differential equations by numerical laplace inversion technique. *Advances in Difference Equations*, 2013(1):1–18, 2013.

[3] Shijun Liao. *Homotopy analysis method in nonlinear differential equations*. Springer, 2012.

[4] George Adomian. A review of the decomposition method in applied mathematics. *Journal of mathematical analysis and applications*, 135(2):501–544, 1988.

[5] Kamel Al-Khaled and Fathi Allan. Construction of solutions for the shallow water equations by the decomposition method. *Mathematics and computers in simulation*, 66(6):479–486, 2004.

[6] Kamel Al-Khaled. Numerical solution of time-fractional partial differential equations using sumudu decomposition method. *Rom. J. Phys*, 60(1-2):99–110, 2015.

[7] Fethi Bin Muhammed Belgacem and Ahmed Abdullatif Karaballi. Sumudu transform fundamental properties investigations and applications. *Journal of Applied Mathematics and Stochastic Analysis*, 2006(Article ID 91083):1–23.

[8] Kamel Al-Khaled. Solving a generalized fractional nonlinear integro-differential equations via modified sumudu decomposition transform. *Axioms*, 11(398), 2022- https://doi.org/10.3390/axioms 11080398.

[9] Fahd Jarad, Thabet Abdeljawad, and Dumitru Baleanu. Caputo-type modification of the hadamard fractional derivatives. *Advances in Difference Equations*, 2012:142, 2012.

[10] George Adomian. *Solving frontier problems of physics: the decomposition method*, volume 60. Springer Science & Business Media, 2013.

[11] Imad Jaradat, Marwan Alquran, and Kamel Al-Khaled. An analytical study of physical models with inherited temporal and spatial memory. *The European Physical Journal Plus*, 133(4):1–11, 2018.

[12] Yuriy A Rossikhin and Marina V Shitikova. Application of fractional calculus for dynamic problems of solid mechanics: novel trends and recent results. *Applied Mechanics Reviews*, 63(1), 2010.

[13] Zhiwu Liao. A new definition of fractional derivatives based on truncated left-handed grünwald-letnikov formula with and median correction. In *Abstract and Applied Analysis*, volume 2014. Hindawi, 2014.

[14] Kai Diethelm. *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type*. Springer Science & Business Media, 2010.

[15] Igor Podlubny. *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Elsevier, 1998.

[16] Zaid Odibat and Shaher Momani. Numerical methods for nonlinear partial differential equations of fractional order. *Applied Mathematical Modelling*, 32(1):28–39, 2008.

[17] Ahmad El-Ajou, Omar Abu Arqub, and Shaher Momani. Approximate analytical solution of the nonlinear fractional kdv–burgers equation: a new iterative algorithm. *Journal of Computational Physics*, 293:81–95, 2015.

[18] Michele Caputo. Linear models of dissipation whose q is almost frequency independent—ii. *Geophysical Journal International*, 13(5):529–539, 1967.

[19] GK1206847 Watugala. Sumudu transform: a new integral transform to solve differential equations and control engineering problems. *Integrated Education*, 24(1):35–43, 1993.

[20] Fethi Bin Muhammed Belgacem, Ahmed Abdullatif Karaballi, and Shyam L Kalla. Analytical investigations of the sumudu transform and applications to integral production equations. *Mathematical problems in Engineering*, 2003(3):103–118, 2003.

DOI: https://doi.org/10.54216/IJNS.250335

Received: March 22, 2024 Revised: June 25, 2024 Accepted: November 04, 2024

[21] Qutaibeh Deeb Katatbeh and Fethi Bin Muhammad Belgacem. Applications of the sumudu transform to fractional differential equations. *Nonlinear Studies*, 18(1):99–112, 2011.

[22] VG Gupta and Bhavna Sharma. Application of sumudu transform in reaction-diffusion systems and nonlinear waves. *Applied Mathematical Sciences*, 4(9-12):435–446, 2010.

[23] Mathew O. Aibinu, Fazal M. Mahomed, and Palle E. Jorgensen. Solutions of fractional differential models by using sumudu transform method and its hybrid. *Partial Differential Equations in Applied Mathematics*, 11:100872, 2024.

[24] Inderdeep Singh, Nizamul Haque Ansari, and Gurpreet Singh. Solving pdes arising in the formation of liquid drop pattern using sumudu transform based technique. *Partial Differential Equations in Applied Mathematics*, 8:100578, 2023.

[25] Jun-Sheng Duan, Randolph Rach, Dumitru Baleanu, and Abdul-Majid Wazwaz. A review of the adomian decomposition method and its applications to fractional differential equations. *Communications in Fractional Calculus*, 3(2):73–99, 2012.

[26] Abdul-Majid Wazwaz. A new algorithm for calculating adomian polynomials for nonlinear operators. *Applied Mathematics and computation*, 111(1):33–51, 2000.

[27] Alemu Senbeta Bekela1 and Alemayehu Tamirie Deresse. A hybrid yang transform adomian decomposition method for solving time-fractional nonlinear partial diferential equation. *Bekela and Deresse BMC Research Notes*, 17(226):1–20, 2024.

[28] Susan Cole. Partial differential equations: Analytical solution techniques, 1991.

[29] Alfredo Bermudez and Ma Elena Vazquez. Upwind methods for hyperbolic conservation laws with source terms. *Computers & Fluids*, 23(8):1049–1071, 1994.

DOI: https://doi.org/10.54216/IJNS.250335
Received: March 22, 2024 Revised: June 25, 2024 Accepted: November 04, 2024