Irreversible k-Threshold Conversion Number of Strong Grids for k>3

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Abstract

An irreversible k-threshold conversion process on a graph $G = (V, E)$ is a dynamic, iterative process which begins by choosing a set $S_0 \subseteq V$. For each step $t(t = 1,2, \dots ,)$, S_t is obtained from S_{t-1} by adjoining all vertices that have at least k neighbors in S_{t-1}. We call S_0 the seed set of the k-threshold conversion process and if $S_t = V(G)$ for some $t \geq 0$, then S_0 is called an irreversible k-threshold conversion set (IkCS) of G. The k-threshold conversion number of G (denoted by $(C_k(G))$ is the minimum cardinality of all the IkCSs of G. In this paper, we study Irreversible k-threshold conversion processes on strong grids $P_m \boxtimes P_n$. We determine $C_k(P_3 \boxtimes P_n)$ for $k = 5,6,7$ and $C_k(P_4 \boxtimes P_n)$ for $k = 6,7$. We also present upper bounds for $C_4(P_3 \boxtimes P_n)$, $C_4(P_4 \boxtimes P_n)$, $C_5(P_3 \boxtimes P_n)$, then we determine $C_8(P_m \boxtimes P_n)$ for arbitrary m, n.

Keywords: Strong grid; graph conversion process; k-threshold conversion set

1. Introduction

Let $G(V, E)$ be a graph with $|V| = n$ vertices and $|E| = m$ edges. The open neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V: uv \in E\}$ and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The degree of a vertex v (denoted by $\deg(v)$) is the number of all vertices that are adjacent to v. Therefore, $\deg(v) = |N(v)|$. Let $Y \subseteq V$ and let F be a subset of E such that F consists of all edges of G which have endpoints in Y, then $H = (Y, F)$ is called an induced subgraph of G by Y and is denoted by G_Y. An independent vertex set of a graph $G(V, E)$ is a subset of V such that no two vertices in the subset represent and edge of G. The independence number, denoted by $\alpha(G)$, is the cardinality of the largest independent vertex set of G. Let P_m, P_n be two paths. We define the strong product of $P_m$and P_n as the graph $P_m \boxtimes P_n$ such that $V(P_m \boxtimes P_n) = V(P_m \times P_n) = \{(i, j): 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and two vertices (i, i'), (j, j') are adjacent if and only if:

- i is adjacent to j and $i' = j'$.
- $i = j$ and i' is adjacent to j'.
- i is adjacent to j and i' is adjacent to j'.

A strong product $P_m \boxtimes P_n$ is called a strong grid. For more information on strong grids see [5]. Irreversible Conversion processes study the spread of a one way change of state (from state 0 to state 1) through a specified society (the spread of disease through populations, the spread of opinion through social networks, etc.) where the conversion rule is determined at the beginning of the study. The study of these processes has important use in real life problems like Anti-Bioterrorism and viral marketing. In the graph theoretical model of these processes, the vertex set $V(G)$ represents the set of individuals on which the conversion is spreading. The irreversible k-threshold conversion process on a graph $G = (V, E)$ is an iterative process which begins by choosing a set $S_0 \subseteq V$. For each step $t(t = 1,2, \dots ,)$, S_t is obtained from S_{t-1} by adjoining all vertices that have at least k neighbors in S_{t-1}. S_0 is called the seed set of the k-threshold conversion process and if $S_t = V(G)$ for some $t \geq 0$, then S_0 is an irreversible k-threshold conversion set (IkCS) of G. The k-threshold conversion number of G (denoted by $(C_k(G))$ is the minimum cardinality of all the IkCSs of G. The first graph model of the Irreversible k-threshold conversion problem was presented by Dreyer and Roberts in [3] where they determined the value of $C_2(G)$ for paths and

DOI: https://doi.org/10.54216/GJMSA.0110208

Received: December 24, 2023 Revised: April 19, 2024 Accepted: August 10, 2024

cycles. They also determined $C_2(G)$ and $C_3(G)$ for grid graphs $P_3 \square P_n$. Adams *et al.*, [1] presented an upper bound for $C_k(G)$ on the tensor product of two arbitrary graphs G and H. For further information on the irreversible k-threshold conversion problem on graphs see Centeno *et al.* [2], Kynčl *et al.* [7], Frances *et al.* [4], Takaoka and Ueno [13]. Mynhardt and Wodlinger presented a lower bound for $C_k(G)$ on graphs of maximum degree $k+1$ in [8]. Mynhardt and Wodlinger [9] gave an upper bound for $C_k(G)$ on k-regular graphs. Shaheen *et al.* studied irreversible k-threshold conversion processes on circulant graphs in [10]. We also studied the problem on Jahangir Graph in [11]. Shaheen *et al.* [12] determined $C_2(G)$ and $C_3(G)$ for the strong grid graphs $P_m \boxtimes P_n$ when $m = 2,3$. In this paper, we extend our study in [12] by determining $C_k(P_3 \boxtimes P_n)$ for $k = 5,6,7$ and $C_k(P_4 \boxtimes P_n)$ for $k = 6,7$. We also present upper bounds for $C_4(P_3 \boxtimes P_n)$, $C_4(P_4 \boxtimes P_n)$, $C_5(P_3 \boxtimes P_n)$, then we determine $C_8(P_m \boxtimes P_n)$ for arbitrary m, n.

2. Preliminary Results

In this section, we present some notations and preliminary facts which are used throughout this paper.

Proposition 2.1 [12]: For $n \geq 1$; $C_2(P_3 \boxtimes P_n) = 2$.

Proposition 2.2 [12]: For $n \geq 1$; $C_3(P_3 \boxtimes P_n) = 3$.

Proposition 2.3 [3]: For $n \geq 3$; $\alpha(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

Proposition 2.4 [6]: For $m, n \geq 2$; $\alpha(P_m \boxtimes P_n) = \left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil$.

Remark 2.1: As immediate consequences of the definition, we conclude the following statements:

- For any graph G; $C_k(G) \geq k$.
- For any graph G; $1 \leq k \leq \Delta(G)$ where $\Delta(G) = max\{deg(v): v \in V(G)\}$.
- When studying an Irreversible k-threshold conversion process on a graph $G = (V, E)$ all vertices $\{v \in V ; \deg(v) < k\}$ must be included in the seed set S_0, otherwise the process will fail because none of these vertices can satisfy the conversion rule.

Note 2.1: In every figure of this article, we represent the vertices as white circles and we assign every converted vertex the number of the conversion step in which it gets converted by placing the number inside the circle.

3. Main Results

In this section, we determine $C_k(P_3 \boxtimes P_n)$ for $k = 5,6,7$ and $C_k(P_4 \boxtimes P_n)$ for $k = 6,7$. We also present upper bounds for $C_4(P_3 \boxtimes P_n)$, $C_4(P_4 \boxtimes P_n)$, $C_5(P_3 \boxtimes P_n)$, then we determine $C_8(P_m \boxtimes P_n)$ when m, n are arbitraries.

3.1. $C_k(P_3 \boxtimes P_n)$.

In this sub-section we present an upper bound for $C_4(P_3 \boxtimes P_n)$. We also determine $C_k(P_3 \boxtimes P_n)$ when $k = 5,6,7$.

We notice that $V(P_3 \boxtimes P_n) = W \cup Q_1 \cup Q_2$ which are determined as:

For every $u \in W; deg(u) = 3$. Therefore, $W = \{(1,1), (1,n), (3,1), (3,n)\}$ and $|W| = 4$.

For every $u \in Q_1; deg(u) = 5$. Which means that $Q_1 = \{(1,j), (3,j): 2 \leq j \leq n-1\} \cup \{(2,1), (2,n)\}$ and $|Q_1| = 2n - 2$.

For every $u \in Q_2; deg(u) = 8$. Therefore, $Q_2 = \{(2,j): 2 \leq j \leq n-1\}$. Which means $|Q_2| = n - 2$.

Theorem 3.1 *For* $n \geq 2$:

i. $C_4(P_3 \boxtimes P_n) = n + 2\ if\ 2 \leq n \leq 4$.

ii. $C_4(P_3 \boxtimes P_n) \leq n + 1\ if\ n > 4$.

Proof: As a consequence of the definition of Irreversible k-threshold conversion processes, and since $deg(u) = 3 < 4$ for all $u \in W$, then any 4-threshold conversion seed set of $P_3 \boxtimes P_n$ must contain W or else the process fails automatically. We consider the following cases for n:

Case 1. $n = 2$. It is obvious that $S_0 = W = \{(1,1), (1,2), (3,1), (3,2)\}$. Then $S_1 = S_0 \cup \{(2,1), (2,2)\} = V(P_3 \boxtimes P_2)$ and the process succeeds. Therefore, $C_4(P_3 \boxtimes P_2) = 2$.

DOI: https://doi.org/10.54216/GJMSA.0110208
Received: December 24, 2023 Revised: April 19, 2024 Accepted: August 10, 2024

Case 2. $n = 3$. Let $S_0 = W = \{(1,1), (1,3), (3,1), (3,3)\}$. Then $S_1 = S_0 \cup \{(2,2)\}$. However, $S_2 = S_1 \neq V(P_3 \boxtimes P_3)$ and the process fails. Now let $S_0 = W \cup \{(1,2)\}$. The process goes as follows:

$$S_1 = S_0 \cup \{(2,2)\},$$

$$S_2 = S_1 \cup \{(2,1), (2,3)\},$$

$$S_3 = S_2 \cup \{(3,2)\} = V(P_3 \boxtimes P_3).$$

Therefore, $S_0 = W \cup \{(1,2)\}$ is a I4CS of $P_3 \boxtimes P_3$ which means $4 < C_4(P_3 \boxtimes P_3) \leq 5$. We conclude that $C_4(P_3 \boxtimes P_3) = 5$.

Case 3. $n = 4$. Let $S_0 = W = \{(1,1), (1,3), (3,1), (3,3)\}$, then $S_1 = S_0$ and the process fails. Now let $S_0 = W \cup \{(1,2)\}$, then $S_1 = S_0$ and the process also fails. Without loss of generality, the process fails for any $S_0 = W \cup \{v : v \in V(P_3 \boxtimes P_4) - W\}$, this means $C_4(P_3 \boxtimes P_4) > 5$.

Now let $S_0 = W \cup \{(1,2), (1,3)\}$. The process goes as follows:

$$S_1 = S_0 \cup \{(2,2), (2,3)\},$$

$$S_2 = S_1 \cup \{(2,1), (2,4)\},$$

$$S_3 = S_2 \cup \{(3,2), (3,3)\} = V(P_3 \boxtimes P_4).$$

Therefore $S_0 = W \cup \{(2,2), (2,3)\}$ is a I4CS of $P_3 \boxtimes P_4$ which means $5 < C_4(P_3 \boxtimes P_4) \leq 6$. We conclude that $C_4(P_3 \boxtimes P_4) = 6$.

Case 4. $n \geq 5$. We consider the following subcases for n:

Case 4.1. n is odd.

Let $S_0 = \{(1,2l+1), (3,2l+1) : 0 \leq l \leq \frac{n-1}{2}\}$ which is of cardinality $n+1$ be the seed set. The process goes as follows:

$$S_1 = S_0 \cup \{(2,2l) : 1 \leq l \leq \frac{n-1}{2}\},$$

$$S_2 = S_1 \cup \{(2,2l+1) : 1 \leq l \leq \frac{n-3}{2}\},$$

$$S_3 = S_2 \cup \{(1,2l), (3,2l) : 1 \leq l \leq \frac{n-1}{2}\},$$

$$S_4 = S_3 \cup \{(2,1), (2,n)\} = V(P_3 \boxtimes P_n).$$

This means S_0 is a I4CS of cardinality $n+1$ on $P_3 \boxtimes P_n$. We conclude that for $n \geq 5$ and n is odd; $C_4(P_3 \boxtimes P_n) \leq n+1$.

Case 4.2. n is even.

Let $S_0 = \{(1,2l+1), (3,2l+1) : 0 \leq l \leq \frac{n}{2} - 2\} \cup \{(2,n-1), (1,n), (3,n)\}$ which is of cardinality $n+1$ be the seed set. The process goes as follows:

$$S_1 = S_0 \cup \{(2,2l) : 1 \leq l \leq \frac{n}{2} - 2\},$$

$$S_2 = S_1 \cup \{(2,2l+1) : 1 \leq l \leq \frac{n}{2} - 3\},$$

$$S_3 = S_2 \cup \{(1,2l), (3,2l) : 1 \leq l \leq \frac{n}{2} - 2\},$$

$$S_4 = S_3 \cup \{(2,1), (2,n-3)\},$$

$$S_5 = S_4 \cup \{(2,n-2)\},$$

$$S_6 = S_5 \cup \{(1,n-2), (3,n-2)\},$$

$$S_7 = S_6 \cup \{(1,n-1), (3,n-1)\},$$

DOI: https://doi.org/10.54216/GJMSA.0110208
Received: December 24, 2023 Revised: April 19, 2024 Accepted: August 10, 2024

$$S_8 = S_7 \cup \{(2,n)\} = V(P_3 \boxtimes P_n).$$

which means S_0 is a I4CS of cardinality $n+1$ on $P_3 \boxtimes P_n$. Therefore for $n > 5$ and n is even; $C_4(P_3 \boxtimes P_n) \leq n+1$. Fig.1 shows that $C_4(P_3 \boxtimes P_{16}) \leq 17$.

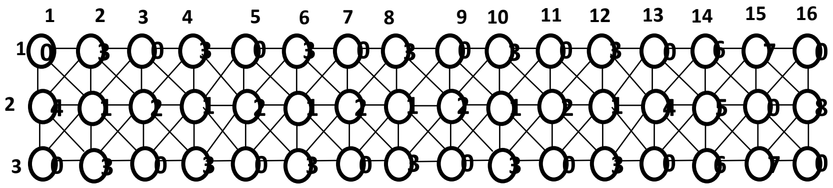

Figure 1. $C_4(P_3 \boxtimes P_{16}) \leq 17$.

From Case 4.1 and Case 4.2 we conclude that $C_4(P_3 \boxtimes P_n) = n+1$ if $n \geq 4$. From all cases and subcases we conclude the requested.□

Theorem 3.2 *For* $n \geq 2$; $C_5(P_3 \boxtimes P_n) = \begin{cases} \frac{3n+3}{2} \; if\ n\ is\ odd; \\ \frac{3}{2}n + 2 \; if\ n\ is\ even. \end{cases}$

Proof: As we mentioned in Theorem 3.1, $W = \{(1,1),(1,n),(3,1),(3,n)\}$ must be contained in the seed set S_0 or else the process fails automatically. We divide $V - W$ into two sets, Q_1 which consists of all the vertices of degree 5 and Q_2 which consists of all the vertices of degree 8 which means:

$$Q_1 = \{(1,j),(3,j): 1 \leq j \leq n\} \cup \{(2,1),(2,n)\},$$

$$Q_2 = \{(2,j): 2 \leq j \leq n-1\}.$$

We notice that the process automatically fails in the following conditions:

Condition 1. There are two adjacent unconverted vertices from Q_1 at $t = 0$, then neither of these two vertices will satisfy the conversion rule at any step of the process. Therefore, it fails.

Condition 2. For some $2 \leq i \leq n-1$**;** there is a set $U = \{(1,i-1),(3,i-1),(2,i),(1,i+1),(3,i+1)\}$ and $U \cap S_0 = \emptyset$. We notice that $(1,i-1),(3,i-1),(1,i+1),(3,i+1)$ are of degree 5 and each one of them is adjacent to $(2,i)$ which is unconverted. On the other hand, $(2,i)$ is of degree 8 and it is adjacent to the four unconverted vertices mentioned above. We conclude that no vertex of U can satisfy the conversion rule at any step of the process, therefore U is 5-unconvertible. Fig.2 illustrates that for $k = 5$; $Z = \{(1,2),(1,4),(2,3),(3,2),(3,4)\}$ (which is a version of U) is unconvertible on $P_3 \boxtimes P_8$ if $Z \cap S_0 = \emptyset$ even if $S_0 = V - Z$.

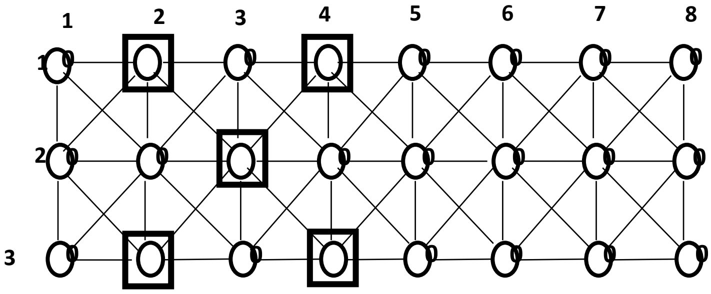

Figure 2. $Z = \{(1,2),(1,4),(2,3),(3,2),(3,4)\}$ is unconvertible on $P_3 \boxtimes P_8$ for $k = 5$ and $S_0 = V - Z$.

When assigning the seed set S_0 we try to avoid having any version of U. To avoid Condition 1, $Q_1 - S_0$ must be independent. We realize that G_{Q_1} is isomorphic to a cycle C_{2n+2}. Due to Proposition 2.3, $\alpha(C_{2n+2}) = n+1$. This means we need to convert $2n+2-(n+1) = n+1$ vertices of Q_1 or else the process automatically fails. We consider the following cases for n:

DOI: https://doi.org/10.54216/GJMSA.0110208
Received: December 24, 2023 Revised: April 19, 2024 Accepted: August 10, 2024

Case 1. n is odd.

Let $S_0 = \{(1,2l+1),(2,2l+1),(3,2l+1): 0 \le l \le \frac{n-1}{2}\}$ be the seed set. It is obvious that S_0 is of cardinality $\frac{3n+3}{2}$. The process goes as follows:

$$S_1 = S_0 \cup \{(2,2l): 1 \le l \le \frac{n-1}{2}\},$$

$$S_2 = S_1 \cup \{(1,2l),(3,2l): 1 \le l \le \frac{n-1}{2}\} = V(P_3 \boxtimes P_n).$$

This means S_0 is a I5CS of cardinality $\frac{3n+3}{2}$ on $P_3 \boxtimes P_n$. We conclude that if n is odd, then:

$$C_5(P_3 \boxtimes P_n) \le \frac{3n+3}{2} \tag{3.1}$$

Fig.3 illustrates that $C_5(P_3 \boxtimes P_{17}) \le 27$.

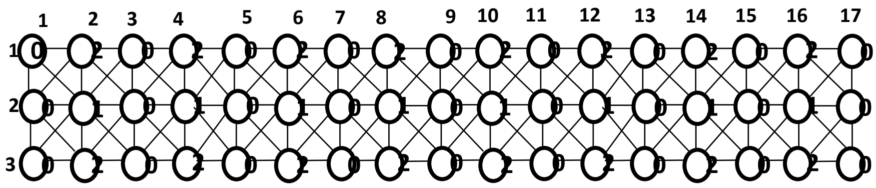

Figure 3. $C_5(P_3 \boxtimes P_{17}) \le 27$.

Now according to the same 5-threshold conversion process, let D_0 be a seed set of cardinality $\frac{3n+1}{2}$ on $P_3 \boxtimes P_n$. We consider the following two subcases:

Case 1.1. $D_0 \subset S_0$. This means $D_0 = S_0 - \{x\}$. We consider the following situations:

- $x \in W$ and the process fails automatically.
- $x \in Q_1$ which means we end up with two adjacent unconverted vertices from Q, therefore Condition 1 is realized and the process fails.
- $x \in Q_2$ which means $x = (2,i)$. We notice that $(1,i-1),(3,i-1),(2,i),(1,i+1),(3,i+1)$ form a version of U on $P_3 \boxtimes P_n$ and Condition 2 is realized. Therefore, the process fails.

From all the previous situations we conclude that D_0 cannot be a I5CS if $D_0 \subset S_0$.

Case 1.2. $D_0 \not\subset S_0$. However, since W must be contained in D_0, that leaves $\frac{3n-7}{2}$ vertices of D_0 to be distributed on Q_1 and Q_2. We found that $Q_1 \cap D_0$ must contain at least $n-1$ vertices. This means we need to distribute the remaining $\frac{n-5}{2}$ vertices of D_0 on Q_2 without leaving any version of U, which is impossible because we need at least $\frac{n-3}{2}$ vertices to achieve that. We conclude that D_0 cannot be a I5CS if $D_0 \not\subset S_0$.

From Case 1.1 and Case 1.2 we conclude that if n is odd, then:

$$C_5(P_3 \boxtimes P_n) > \frac{3n+1}{2} \tag{3.2}$$

From (3.1) and (3.2) we conclude that if n is odd; then $C_5(P_3 \boxtimes P_n) = \frac{3n+3}{2}$.

Case 2. n is even.

Let $S_0 = \{(1,2l+1),(2,2l+1),(3,2l+1): 0 \le l \le \frac{n}{2}-1\} \cup \{(1,n),(3,n)\}$ be the seed set. It is obvious that S_0 is of cardinality $\frac{3}{2}n+2$. The process goes as follows:

$$S_1 = S_0 \cup \{(2,2l): 1 \le l \le \frac{n}{2}\},$$

$$S_2 = S_1 \cup \{(1,2l),(3,2l): 1 \le l \le \frac{n}{2}-1\} = V(P_3 \boxtimes P_n).$$

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Received: December 24, 2023 Revised: April 19, 2024 Accepted: August 10, 2024

This means S_0 is a I5CS of cardinality $\frac{3}{2}n + 2$ on $P_3 \boxtimes P_n$. We conclude that if n is even, then $C_5(P_3 \boxtimes P_n) \leq \frac{3}{2}n + 2$. In a similar way to Case 1, we can prove that S_0 is the smallest 6-threshold seed set that contains W and does not allow Condition 1 and Condition 2 to occur. Therefore, $C_5(P_3 \boxtimes P_n) = \frac{3}{2}n + 2$ if n is even. From Case 1 and Case 2 we conclude the requested.□

Theorem 3.3. *For* $n \geq 3$; $C_6(P_3 \boxtimes P_n) = 2n + 2$.

Proof: As we mentioned in the previous Theorems, W must be contained in the seed set S_0. However, since for every $v \in Q_1; deg(v) = 5 < 6 = k$ then Q_1 must be contained in S_0 as well. Let $S_0 = W \cup Q_1$ be the seed set. $|S_0| = |W| + |Q_1| = 2n - 2 + 4 = 2n + 2$. The process goes as follows:

$S_0 = W \cup Q_1 = \{(1,j),(3,j): 1 \leq j \leq n\} \cup \{(2,1),(2,n)\}$,

$$S_1 = S_0 \cup \{(2,j): 2 \leq j \leq n-1\} = V(P_3 \boxtimes P_n).$$

Therefore S_0 is a I6CS of $P_3 \boxtimes P_n$. However, since S_0 is the smallest seed set that contains both W and Q_1, then $|S_0|$ is the minimum cardinality of all I6CSs on $P_3 \boxtimes P_n$ which means $C_6(P_3 \boxtimes P_n) = |S_0| = 2n + 2$.□

Theorem 3.4. *For* $n \geq 4$; $C_7(P_3 \boxtimes P_n) = 2n + 2$.

Proof: In a similar way to Theorem 3.3, both W and Q_1 must be contained in the seed set S_0. Let $S_0 = W \cup Q_1 = \{(1,j),(3,j): 1 \leq j \leq n\} \cup \{(2,1),(2,n)\}$ be the seed set. The process goes as follows:

$S_1 = S_0 \cup \{(2,2),(2,n-1)\}$,

$S_2 = S_1 \cup \{(2,3),(2,n-2)\}$,

For $3 \leq t < \left\lfloor \frac{n-1}{2} \right\rfloor$: $S_t = S_{t-1} \cup \{(2,t+1),(2,n-t)\}$.

$$t = \left\lfloor \frac{n-1}{2} \right\rfloor: S_{\left\lfloor \frac{n-1}{2} \right\rfloor} = \begin{cases} S_{\left\lfloor \frac{n-1}{2} \right\rfloor - 1} \cup \{(2,\frac{n+1}{2})\} if\ n\ is\ odd; \\ S_{\left\lfloor \frac{n-1}{2} \right\rfloor - 1} \cup \{(2,\frac{n}{2}),(2,\frac{n}{2}+1)\} if\ n\ is\ even. \end{cases}$$

In both cases, $S_{\left\lfloor \frac{n-1}{2} \right\rfloor} = V(P_3 \boxtimes P_n)$. Therefore, S_0 is a I7CS on $P_3 \boxtimes P_n$ which means $C_7(P_3 \boxtimes P_n) \leq 2n + 2$. Since S_0 is the smallest 7-threshold conversion seed set that contains $W \cup Q_1$, we conclude that $C_7(P_3 \boxtimes P_n) = 2n + 2$.□

3.2. $C_k(P_4 \boxtimes P_n)$.

In this sub-section we present upper bounds for $C_k(P_4 \boxtimes P_n)$ when $k = 4,5$. We also determine $C_k(P_4 \boxtimes P_n)$ when $k = 6,7$.

We notice that $V(P_4 \boxtimes P_n) = W_1 \cup Q_3 \cup Q_4$. We determine these sets as:

For every $u \in W_1; deg(u) = 3$. Therefore, $W_1 = \{(1,1),(1,n),(4,1),(4,n)\}$ and $|W| = 4$.

For every $u \in Q_3; deg(u) = 5$. Which means that $Q_3 = \{(1,j),(4,j): 2 \leq j \leq n-1\} \cup \{(2,1),(2,n),(3,1),(3,n)\}$ and $|Q_3| = 2n$.

For every $u \in Q_4; deg(u) = 8$. Therefore, $Q_4 = \{(2,j),(3,j): 2 \leq j \leq n-1\}$. Which means $|Q_4| = 2n - 4$.

Theorem 3.5. *For* $n \geq 3$; $C_4(P_4 \boxtimes P_n) \leq n + 3$.

Proof: We consider the following cases for n:

Case 1. n is odd.

Let the seed set be $S_0 = \{(1,2l+1),(3,2l+1): 0 \leq l \leq \frac{n-1}{2}\} \cup \{(4,1),(4,n)\}$ which is of cardinality $n + 3$. The process goes as follows:

$S_1 = S_0 \cup \{(2,2l): 1 \leq l \leq \frac{n-1}{2}\}$,

$S_2 = S_1 \cup \{(2,2l+1): 1 \leq l \leq \frac{n-3}{2}\}$,

$S_3 = S_2 \cup \{(1,2l),(3,2l): 1 \leq l \leq \frac{n-1}{2}\}$,

DOI: https://doi.org/10.54216/GJMSA.0110208
Received: December 24, 2023 Revised: April 19, 2024 Accepted: August 10, 2024

$S_4 = S_3 \cup \{(2,1), (2,n), (4,2), (4, n-1)\}$,

For $5 \le t \le \frac{n+3}{2}$: $S_t = S_{t-1} \cup \{(4, t-2), (4, n-t+3)\}$.

The process ends at step $t = \frac{n+5}{2}$ and $S_{\frac{n+5}{2}} = S_{\frac{n+3}{2}} \cup \{(4, \frac{n+1}{2})\} = V(P_4 \boxtimes P_n)$, which means S_0 is a I4CS on $P_4 \boxtimes P_n$. Therefore, $C_4(P_4 \boxtimes P_n) \le n+3$ if n is odd. Fig.4 shows that $C_4(P_4 \boxtimes P_{15}) \le 18$.

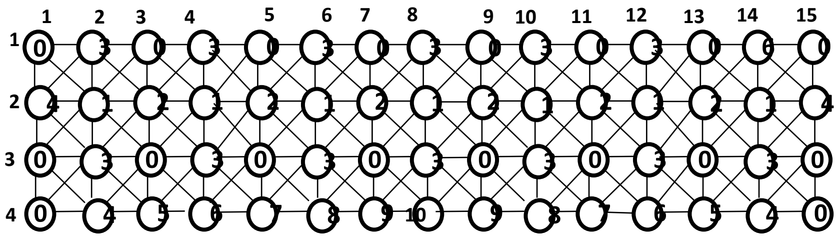

Figure 4. $C_4(P_4 \boxtimes P_{15}) \le 18$.

Case 2. n is even.

Let the seed set be $S_0 = \{(1,2l+1), (3,2l+1): 0 \le l \le \frac{n-1}{2}\} \cup \{(1,n), (4,1), (4,n)\}$ which is of cardinality $n+3$. The process goes as follows:

$S_1 = S_0 \cup \{(2,2l): 1 \le l \le \frac{n}{2} - 1\}$.

$S_2 = S_1 \cup \{(2,2l+1): 1 \le l \le \frac{n}{2} - 1\}$.

$S_3 = S_2 \cup \{(1,2l), (3,2l): 1 \le l \le \frac{n}{2} - 1\} \cup \{(2,n)\}$.

$S_4 = S_3 \cup \{(2,1), (3,n), (4,2)\}$.

For $5 \le t \le \frac{n}{2} + 2$: $S_t = S_{t-1} \cup \{(4, t-2), (4, n-t+4)\}$.

The process ends at step $t = \frac{n}{2} + 2$ and $S_{\frac{n}{2}+3} = S_{\frac{n}{2}+2} \cup \{(4, \frac{n}{2}+1)\} = V(P_4 \boxtimes P_n)$, which means S_0 is a I4CS on $P_4 \boxtimes P_n$. Therefore, $C_4(P_4 \boxtimes P_n) \le n+3$ if n is even. From Case 1 and Case 2 we conclude the requested.□

Theorem 3.6. *For* $n \ge 2$; $C_5(P_4 \boxtimes P_n) \le \begin{cases} 2n+2 & \text{if } n \text{ is odd;} \\ 2n+1 & \text{if } n \text{ is even.} \end{cases}$

Proof: As we implied in Theorem 3.2, there cannot be two adjacent unconverted vertices at $t = 0$ if we are studying a 5-threshold conversion process on $P_4 \boxtimes P_n$ or else the process will fail. We consider the following cases for n:

Case 1. n is odd.

Let the seed set be $S_0 = W_1 \cup \{(1,2l), (2,2l), (3,2r+1), (4,2d+1): 0 \le l \le \frac{n-1}{2}; 0 \le r \le \frac{n-1}{2}; 1 \le d \le \frac{n-3}{2}\}$, which is of cardinality $2n+2$. The process goes as follows:

$S_1 = S_0 \cup \{(2,2d+1), (3,2l): 1 \le d \le \frac{n-3}{2}; 1 \le l \le \frac{n-1}{2}\}$.

$S_2 = S_1 \cup \{(1,2d+1), (4,2l): 1 \le d \le \frac{n-3}{2}; 1 \le l \le \frac{n-1}{2}\} \cup \{(2,2), (2,n)\} = V(P_4 \boxtimes P_n)$.

Therefore, $C_5(P_4 \boxtimes P_n) \le 2n+2$ if n is odd.

Case 2. n is even.

Let the seed set be $S_0 = W_1 \cup \{(1,2d), (2,2p), (3,2l+1), (4,2d+1): 1 \le d \le \frac{n}{2} - 1; 2 \le p \le \frac{n}{2}; 0 \le l \le \frac{n}{2} - 1\}$, which is of cardinality $2n+1$. The process goes as follows:

DOI: https://doi.org/10.54216/GJMSA.0110208
Received: December 24, 2023 Revised: April 19, 2024 Accepted: August 10, 2024

$S_1 = S_0 \cup \{(2,2l+1),(3,2d): 2 \le l \le \frac{n}{2}-1; 2 \le d \le \frac{n}{2}-1\}$.

$S_2 = S_1 \cup \{(1,2l+1),(4,2d): 2 \le l \le \frac{n}{2}-1; 2 \le d \le \frac{n}{2}-1\} \cup \{(2,3),(3,n)\}$.

$S_3 = S_2 \cup \{(2,2),(3,2)\}$.

$S_4 = S_3 \cup \{(1,3),(2,1),(4,2)\} = V(P_4 \boxtimes P_n)$.

Therefore $C_5(P_4 \boxtimes P_n) \le 2n+1$ if n is even. Fig.5 shows that $C_5(P_4 \boxtimes P_{14}) \le 29$.

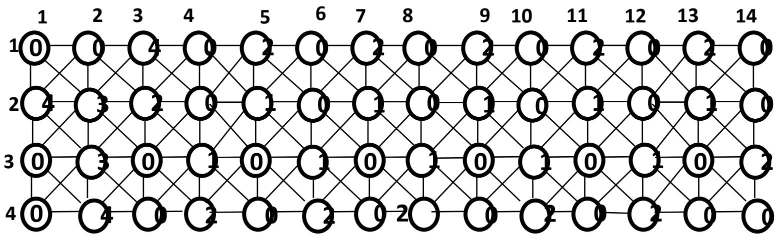

Figure 5. $C_5(P_4 \boxtimes P_{14}) \le 29$.

From Case 1 and Case 2 we conclude the requested. □

Theorem 3.7. *For* $n \ge 2$; $C_6(P_4 \boxtimes P_n) = \begin{cases} \frac{5n+5}{2} & if\ n\ is\ odd; \\ \frac{5n}{2}+3 & if\ n\ is\ even. \end{cases}$

Proof: In a similar way to Theorem 3.3, both W_1, Q_3 must be contained in S_0 or else the process automatically fails. Now we try to determine which vertices of Q_4 we should include in S_0. We define some subsets of Q_4 as the following: For $2 \le i \le n-2$: $B_i = \{(2,i),(2,i+1),(3,i),(3,i+1)\}$. We notice that for any $B_i: 2 \le i \le n-2$, if $B_i \cap S_0 = \emptyset$ then no vertex of B_i can be converted at any step of the process because every $u \in B_i$ is of degree 8 and is adjacent to 3 unconverted vertices of B_i itself. Fig. 6 shows that for $k=6$; if $B_3 \cap S_0 = \emptyset$ then B_3 cannot be converted on $P_4 \boxtimes P_9$ even if $S_0 = V - B_3$.

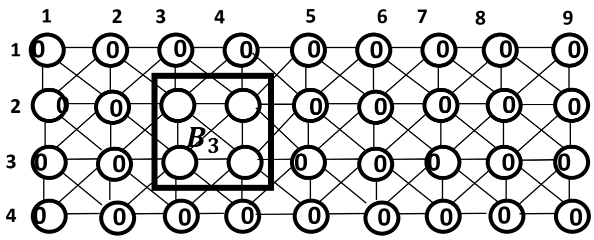

Figure 6. B_3 is unconvertible on $P_4 \boxtimes P_9$ for $k=6$ and $S_0 = V - B_3$.

This means when choosing S_0 we need to avoid having two adjacent columns of Q_4 that do not include any vertex of S_0. Since Q_4 consists of $n-2$ columns, we need to include at least $\frac{n-3}{2}$ vertices of Q_4 in S_0 if n is odd, while we need to include at least $\left\lceil\frac{n-3}{2}\right\rceil = \frac{n}{2}-1$ vertices of Q_4 in S_0 if n is even or else we end up with at least one unconverted $B_i: 2 \le i \le n-2$ and the process fails. We conclude that $|S_0| \ge |W| + |Q_1| + \left\lceil\frac{n-3}{2}\right\rceil$. Therefore:

$$C_6(P_4 \boxtimes P_n) \ge \begin{cases} \frac{5n+5}{2} & if\ n\ is\ odd; \\ \frac{5n}{2}+3 & if\ n\ is\ even. \end{cases} \quad (3.3)$$

We consider the following cases for n:

Case 1. $n \equiv 0 (mod\ 4)$.

DOI: https://doi.org/10.54216/GJMSA.0110208
Received: December 24, 2023 Revised: April 19, 2024 Accepted: August 10, 2024

Let the seed set be $S_0 = W_1 \cup Q_3 \cup \{(2,3+4l), (3,5+4d): 0 \le l \le \left\lfloor \frac{n}{4} \right\rfloor - 1; 0 \le d \le \left\lfloor \frac{n}{4} \right\rfloor - 2\}$ which is of cardinality $\frac{5n}{2} + 3$. The process goes as follows:

For $0 < t < \frac{n}{2} - 1$:

If $t \equiv 0(mod\ 4)$; $S_t = S_{t-1} \cup \{(2,t+1), (2,n-t), (3,n-t)\}$.

If $t \equiv 1(mod\ 4)$; $S_t = S_{t-1} \cup \{(2,t+1), (3,t+1), (3,n-t)\}$.

If $t \equiv 2(mod\ 4)$; $S_t = S_{t-1} \cup \{(2,n-t), (3,t+1), (3,n-t)\}$.

If $t \equiv 3(mod\ 4)$; $S_t = S_{t-1} \cup \{(2,t+1), (2,n-t), (3,t+1)\}$.

The process ends at $t = \frac{n}{2} - 1$ and:

$$S_{\frac{n}{2}-1} = \begin{cases} S_{\frac{n}{2}-2} \cup \{(2,\frac{n}{2}), (3,\frac{n}{2}), (3,\frac{n}{2}+1)\}\ if\ (\frac{n}{2}-1) \equiv 1(mod\ 4); \\ S_{\frac{n}{2}-2} \cup \{(2,\frac{n}{2}), (2,\frac{n}{2}+1), (3,\frac{n}{2})\}\ if\ (\frac{n}{2}-1) \equiv 3(mod\ 4). \end{cases}$$

In both situations, $S_{\frac{n}{2}-1} = V(P_4 \boxtimes P_n)$ which means $C_6(P_4 \boxtimes P_n) \le \frac{5n}{2} + 3$. Fig.7 illustrates that $C_6(P_4 \boxtimes P_{16}) \le 43$. Due to (3.3) we conclude that $C_6(P_4 \boxtimes P_n) = \frac{5n}{2} + 3$ if $n \equiv 0(mod\ 4)$.

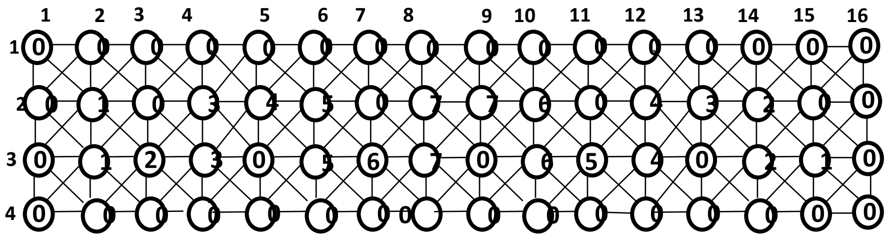

Figure 7. $C_6(P_4 \boxtimes P_{16}) \le 43$.

Case 2. $n \equiv 1(mod\ 4)$.

We choose $S_0 = W_1 \cup Q_3 \cup \{(2,3+4l), (3,5+4d): 0 \le l \le \left\lfloor \frac{n}{4} \right\rfloor - 1; 0 \le d \le \left\lfloor \frac{n}{4} \right\rfloor - 2\}$ which is of cardinality $\frac{5n+5}{2}$. The process goes as follows:

For $0 < t < \frac{n-1}{2}$:

If t is odd; $S_t = S_{t-1} \cup \{(2,t+1), (2,n-t), (3,t+1), (3,n-t)\}$.

If $t \equiv 0(mod\ 4)$; $S_t = S_{t-1} \cup \{(2,t+1), (2,n-t)\}$.

If $t \equiv 2(mod\ 4)$; $S_t = S_{t-1} \cup \{(3,t+1), (3,n-t)\}$.

The process ends at $t = \frac{n-1}{2}$ and $S_{\frac{n-1}{2}} = \begin{cases} S_{\frac{n-3}{2}} \cup \{(2,\frac{n+1}{2})\}\ if\ \frac{n-1}{2} \equiv 0(mod\ 4); \\ S_{\frac{n-3}{2}} \cup \{(3,\frac{n+1}{2})\}\ if\ \frac{n-1}{2} \equiv 2(mod\ 4). \end{cases}$

In both cases, $S_{\frac{n-1}{2}} = V(P_4 \boxtimes P_n)$. Therefore, $C_6(P_4 \boxtimes P_n) \le \frac{5n+5}{2}$. From (3.3) we conclude that $C_6(P_4 \boxtimes P_n) = \frac{5n+5}{2}$ if $n \equiv 1(mod\ 4)$.

Case 3. $n \equiv 2(mod\ 4)$.

We choose $S_0 = W_1 \cup Q_3 \cup \{(2,3+4l), (3,5+4l): 0 \le l \le \left\lfloor \frac{n}{4} \right\rfloor - 1\}$ which is of cardinality $\frac{5n}{2} + 3$. The process goes as follows:

For $0 < t < \frac{n}{2} - 1$:

DOI: https://doi.org/10.54216/GJMSA.0110208
Received: December 24, 2023 Revised: April 19, 2024 Accepted: August 10, 2024

If $t \equiv 0(mod\ 4)$; $S_t = S_{t-1} \cup \{(2, t+1), (2, n-t), (3, n-t)\}$.

If $t \equiv 1(mod\ 4)$; $S_t = S_{t-1} \cup \{(2, t+1), (2, n-t), (3, t+1)\}$.

If $t \equiv 2(mod\ 4)$; $S_t = S_{t-1} \cup \{(2, n-t), (3, t+1), (3, n-t)\}$.

If $t \equiv 3(mod\ 4)$; $S_t = S_{t-1} \cup \{(2, t+1), (3, t+1), (3, n-t)\}$.

The process ends at $t = \frac{n}{2} - 1$ and:

$$S_{\frac{n}{2}-1} = \begin{cases} S_{\frac{n}{2}-2} \cup \{(2, \frac{n}{2}), (2, \frac{n}{2}+1), (3, \frac{n}{2}+1)\}\ if\ (\frac{n}{2} - 1) \equiv 0(mod\ 4); \\ S_{\frac{n}{2}-2} \cup \{(2, \frac{n}{2}+1), (3, \frac{n}{2}), (3, \frac{n}{2}+1)\}\ if\ (\frac{n}{2} - 1) \equiv 2(mod\ 4). \end{cases}$$

In both situations, $S_{\frac{n}{2}-1} = V(P_4 \boxtimes P_n)$. Therefore, $C_6(P_4 \boxtimes P_n) \leq \frac{5n}{2} + 3$. From (3.3) we conclude that $C_6(P_4 \boxtimes P_n) = \frac{5n}{2} + 3$ if $n \equiv 2(mod\ 4)$.

Case 4. $n \equiv 3(mod\ 4)$.

We choose $S_0 = W_1 \cup Q_3 \cup \{(2, 3+4l), (3, 5+4d): 0 \leq l \leq \left\lfloor\frac{n}{4}\right\rfloor - 1\}$ which is of cardinality $\frac{5n+5}{2}$. The process goes as follows:

For $0 < t < \frac{n-1}{2}$:

If t is odd; $S_t = S_{t-1} \cup \{(2, t+1), (2, n-t), (3, t+1), (3, n-t)\}$.

If $t \equiv 0(mod\ 4)$; $S_t = S_{t-1} \cup \{(2, n-t), (3, t+1)\}$.

If $t \equiv 2(mod\ 4)$; $S_t = S_{t-1} \cup \{(2, t+1), (3, n-t)\}$.

The process ends at $t = \frac{n-1}{2}$ which is odd and $S_{\frac{n-1}{2}} = S_{\frac{n-3}{2}} \cup \{(2, \frac{n+1}{2}), (3, \frac{n+1}{2})\} = V(P_4 \boxtimes P_n)$. Therefore, $C_6(P_4 \boxtimes P_n) \leq \frac{5n+5}{2}$. Due to (3.3), we can conclude that $C_6(P_4 \boxtimes P_n) = \frac{5n+5}{2}$ if $n \equiv 3(mod\ 4)$. From all the cases we conclude the requested.□

Theorem 3.8. *For* $n \geq 4$; $C_7(P_4 \boxtimes P_n) = 3n + 2$.

Proof: In addition to W_1 and Q_3, we try to determine which vertices of Q_4 we should include in S_0. We define some subsets of Q_4 as the following: For $2 \leq i \leq n-2$:

$$E_i = \{(2, i), (3, i), (2, i+1)\},$$

$$F_i = \{(2, i), (3, i), (3, i+1)\},$$

$$H_i = \{(2, i), (2, i+1), (3, i+1)\},$$

$$T_i = \{(2, i+1), (3, i), (3, i+1)\}.$$

We notice that for any $E_i: 2 \leq i \leq n-2$, if $E_i \cap S_0 = \emptyset$ then no vertex of E_i can be converted at any step of the conversion process because every $u \in E_i$ is of degree 8 and is adjacent to 2 unconverted vertices of B_i itself. The same argument applies for $F_i, H_i, T_i: 2 \leq i \leq n-2$. Fig. 8(a) shows that E_2 cannot be converted on $P_4 \boxtimes P_6$ even if $S_0 = V - E_2$, while Fig. 8 (b) shows the same argument for $S_0 = V - F_2$, Fig. 8(c) shows the same situation for $S_0 = V - H_2$ and Fig. 8(d) illustrates how $S_0 = V - T_2$ cannot convert T_2. We conclude that every two adjacent columns of Q_4 must contain at least two vertices of S_0 or else an unconverted version of $E_i, F_i, H_i, T_i: 2 \leq i \leq n-2$ will be created on $P_4 \boxtimes P_n$ and the process will fail. Since Q_4 has $n-2$ columns, then $|S_0| \geq |W_1| + |Q_3| + n - 1$. This means:

$$C_7(P_4 \boxtimes P_n) \geq 3n + 2 \qquad (3.4)$$

DOI: https://doi.org/10.54216/GJMSA.0110208
Received: December 24, 2023 Revised: April 19, 2024 Accepted: August 10, 2024

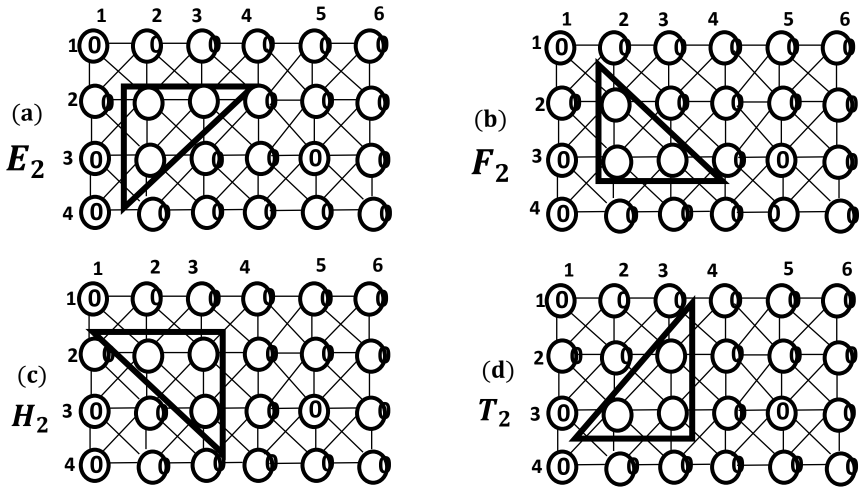

Figure 8. E_2, F_2, H_2, T_2 are unconvertible on $P_4 \boxtimes P_6$ for $k = 7$.

We consider the following cases for n:

Case 1. n is odd.

Let $S_0 = W_1 \cup Q_3 \cup \{(2,1+2l),(3,2d): 1 \leq l \leq \frac{n-3}{2}; 1 \leq d \leq \frac{n-1}{2}\}$ which is of cardinality $3n+2$ be the seed set. The process goes as follows:

For $0 < t < \frac{n-1}{2}$:

If t is odd; $S_t = S_{t-1} \cup \{(2,t+1),(2,n-t)\}$.

If t is even; $S_t = S_{t-1} \cup \{(3,t+1),(3,n-t)\}$.

The process ends at $t = \frac{n-1}{2}$, for which $S_{\frac{n-1}{2}} = \begin{cases} S_{\frac{n-3}{2}} \cup \{(2,\frac{n+1}{2})\} \; if \; \frac{n-1}{2} \; is \; odd; \\ S_{\frac{n-3}{2}} \cup \{(3,\frac{n+1}{2})\} \; if \; \frac{n-1}{2} \; is \; even. \end{cases}$

In both cases, $S_{\frac{n-1}{2}} = V(P_4 \boxtimes P_n)$. Therefore, $C_7(P_4 \boxtimes P_n) \leq 3n+2$. Fig. 9 shows that $C_7(P_4 \boxtimes P_{15}) \leq 47$.

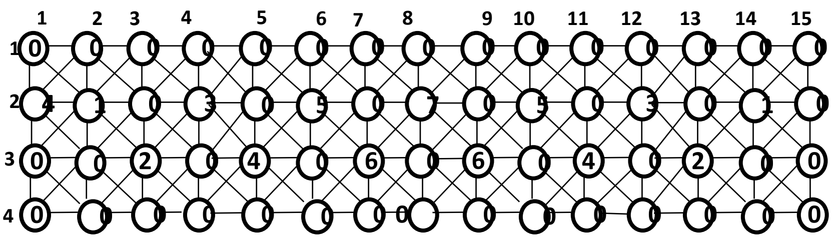

Figure 9. $C_7(P_4 \boxtimes P_{15}) \leq 47$.

From (3.4) we conclude that $C_7(P_4 \boxtimes P_n) = 3n+2$ if n is odd.

Case 2. n is even.

Let $S_0 = W_1 \cup Q_3 \cup \{(2,1+2l),(3,2l): 1 \leq l \leq \frac{n}{2} - 1\}$ which is of cardinality $3n+2$ be the seed set. The process goes as follows:

For $0 < t < \frac{n-1}{2}$:

If t is odd; $S_t = S_{t-1} \cup \{(2,t+1),(3,n-t)\}$.

DOI: https://doi.org/10.54216/GJMSA.0110208
Received: December 24, 2023 Revised: April 19, 2024 Accepted: August 10, 2024

If t is even; $S_t = S_{t-1} \cup \{(2, n-t), (3, t+1)\}$.

The process ends at $t = \frac{n}{2} - 1$, and $S_{\frac{n}{2}-1} = \begin{cases} S_{\frac{n}{2}-2} \cup \{(2, \frac{n}{2}), (3, \frac{n}{2}+1)\}\ if\ \frac{n}{2}-1\ is\ odd; \\ S_{\frac{n}{2}-2} \cup \{(2, \frac{n}{2}+1), (3, \frac{n}{2})\}\ if\ \frac{n}{2}-1\ is\ even. \end{cases}$

In both cases, $S_{\frac{n}{2}-1} = V(P_4 \boxtimes P_n)$. Therefore, $C_7(P_4 \boxtimes P_n) \leq 3n + 2$. From (3.4) we conclude that $C_7(P_4 \boxtimes P_n) = 3n + 2$ if n is even.

From Case 1 and Case 2 we conclude the requested.□

3.3. $C_k(P_m \boxtimes P_n)$.

In this sub-section we determine $C_8(P_m \boxtimes P_n)$ for m, n are arbitraries. We notice that $V(P_4 \boxtimes P_n) = W_2 \cup Q_5 \cup Q_6$. We determine these sets as:

For every $u \in W_2$; $deg(u) = 3$. Therefore, $W_2 = \{(1,1), (1,n), (m,1), (m,n)\}$ and $|W| = 4$.

For every $u \in Q_5$; $deg(u) = 5$, then $Q_5 = \{(1,j), (m,j), (i,1), (i,n): 2 \leq i \leq m-1;\ 2 \leq j \leq n-1\}$. This means $|Q_5| = 2m + 2n - 8$.

Finally, For every $u \in Q_6$; $deg(u) = 8$. Therefore, $Q_6 = \{(i,j): 2 \leq i \leq m-1;\ 2 \leq j \leq n-1\}$. We notice that $|Q_6| = (m-2)(n-2) = mn - 2m - 2n + 4$.

Theorem 3.9. *For $m, n \geq 3$;* $C_8(P_m \boxtimes P_n) = \begin{cases} \frac{3mn+m+n-1}{4}\ if\ m\ is\ odd\ and\ n\ is\ odd; \\ \frac{3mn+2m+n-2}{4}\ if\ m\ is\ odd\ and\ n\ is\ even; \\ \frac{3mn+m+2n-2}{4}\ if\ m\ is\ even\ and\ n\ is\ odd; \\ \frac{3mn+2m+2n-4}{4}\ if\ m\ is\ even\ and\ n\ is\ even. \end{cases}$

Proof: Since $k = 8$, all vertices of $W_2 \cup Q_5$ must be included in S_0. Otherwise, the process automatically fails. Since every $u \in Q_6$ is of degree 8, there cannot be two adjacent unconverted vertices of Q_6 at $t = 0$ or else neither one of these two vertices will satisfy the conversion rule at any step of the process, therefore the process fails. To avoid that, $Q_6 - S_0$ must be independent. In order to make S_0 as small as possible, we try to make $Q_6 - S_0$ as large as possible, thus $Q_6 - S_0$ must be the largest independent set of the graph G_{Q_6} which is induced by Q_6 on $P_m \boxtimes P_n$, which means $|Q_6 - S_0| = \alpha(G_{Q_6})$. We notice that G_{Q_6} is isomorphic to a strong grid of $m-2$ rows and $n-2$ columns $(P_{m-2} \boxtimes P_{n-2})$. Therefore, due to Proposition 2.4, we have $\alpha(G_{Q_6}) = \alpha(P_{m-2} \boxtimes P_{n-2}) = \left\lceil \frac{m-2}{2} \right\rceil \left\lceil \frac{n-2}{2} \right\rceil$ and the smallest seed set S_0 on $P_m \boxtimes P_n$ that contains $W_2 \cup Q_5$ and guarantees not leaving two adjacent unconverted vertices from Q_6 is of cardinality $|S_0| = |W_2| + |Q_5| + |Q_6| - \alpha(P_{m-2} \boxtimes P_{n-2})$. This means:

$$C_8(P_m \boxtimes P_n) = mn - \left\lceil \frac{m-2}{2} \right\rceil \left\lceil \frac{n-2}{2} \right\rceil \qquad (3.5)$$

We consider the following cases for m, n:

Case 1. m is odd and n is odd.

This means $\left\lceil \frac{m-2}{2} \right\rceil = \frac{m-1}{2}$; $\left\lceil \frac{n-2}{2} \right\rceil = \frac{n-1}{2}$. From (3.5) we conclude that $C_8(P_m \boxtimes P_n) = \frac{3mn+m+n-1}{4}$.

Case 2. m is odd and n is even.

We have $\left\lceil \frac{m-2}{2} \right\rceil = \frac{m-1}{2}$; $\left\lceil \frac{n-2}{2} \right\rceil = \frac{n-2}{2}$. From (3.5) we have $C_8(P_m \boxtimes P_n) = \frac{3mn+2m+n-2}{4}$.

Case 3. m is even and n is odd.

This means $\left\lceil \frac{m-2}{2} \right\rceil = \frac{m-2}{2}$; $\left\lceil \frac{n-2}{2} \right\rceil = \frac{n-1}{2}$. From (3.5) we conclude that $C_8(P_m \boxtimes P_n) = \frac{3mn+m+2n-2}{4}$.

Case 4. m is even and n is even.

We have $\left\lceil \frac{m-2}{2} \right\rceil = \frac{m-2}{2}$; $\left\lceil \frac{n-2}{2} \right\rceil = \frac{n-2}{2}$. From (3.5) we have $C_8(P_m \boxtimes P_n) = \frac{3mn+2m+2n-4}{4}$.

From all the four cases we conclude the requested.□

DOI: https://doi.org/10.54216/GJMSA.0110208
Received: December 24, 2023 Revised: April 19, 2024 Accepted: August 10, 2024

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DOI: https://doi.org/10.54216/GJMSA.0110208
Received: December 24, 2023 Revised: April 19, 2024 Accepted: August 10, 2024