Characteristics Neutrosophic Homomorphism for Neutrosophic Rings: On Review

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Abstract

The objective of this paper is to present and study elementary properties for concept a neutrosophic ring homomorphism and isomorphism which introduced by Florentine Smarandache in 2006. We will use a concept ring homomorphism and isomorphism in classical ring.

Keywords: Neutrosophic ring; Neutrosophic ideal ring; Neutrosophic quotient ring; Neutrosophic ring homomorphism; Neutrosophic ring isomorphism

1. Introduction

In 2006 [1], the concept of neutrosophic rings was defined by Kandasamy and Smarandache, as an extension of a classical ring R. Also, they introduced neutrosophic algebra structure, N-algebra structure, and neutrosophic ring in their works. They studied the structure of Neutrosophic Groups, Neutrosophic N-Groups structure, Neutrosophic rings, and Neutrosophic group rings in [1]and [2]. Moreover, Kandasamy has written a book about Smarandache neutrosophic algebraic structure, he showed it through neutrosophic groups (semigroups), Bi-groups, and N-groups in [2]. Previous work has motivated other researchers such Agboola, Adeleke, and Oyebola, they introduced some elementary properties of neutrosophic rings with neutrosophic polynomial rings in [3]. In 2012, they studied ideals of neutrosophic rings and Neutrosophic quotient rings in [4]. Later on, in 2023, Al-Odhari presented a review of the neutrosophic ring and neutrosophic subring with properties of neutrosophic elements of the neutrosophic ring and some algebraic structure of neutrosophic matrices in [5]and [6]. Through this paper, we present the properties of Neutrosophic Ring Homomorphism, Neutrosophic Ring Isomorphism and their properties.

2. Preliminaries

In this section, we introduce a review of the neutrosophic ring, neutrosophic subring, neutrosophic ideal, and neutrosophic quotient ring which will be used in the next sections.

Definition 2.1 [1]Let R be any ring. The neutrosophic ring is also a ring generated by R and

I under the operations of R and denoted by $N(R)$, such that:

$$N(R) = \langle R \cup I \rangle = \{a + bI : a, b \in R\}.$$

Definition 2.2 [1,3,4] Let R be any commutative ring, such that for all $a, b \in R \ is \ ab = ba$, then

$\langle R \cup I \rangle$ is a commutative neutrosophic ring such for all $x, y \in \langle R \cup I \rangle \ is \ xy = yx.$

Theorem 2.1 [6]Let $\langle R \cup I \rangle$ be a neutrosophic ring and $x, y \in \langle R \cup I \rangle$. then:

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1. $x.0 = 0.x = 0$,
2. $x.(-y) = (-x).y = -(xy)$, and
3. $(-x).(-y) = xy$.

Definition 2.3[1,3,4] Let $\langle R \cup I \rangle$ be a neutrosophic ring. A proper subset P of $\langle R \cup I \rangle$ is said to be a neutrosophic subring if P itself is a neutrosophic ring under the operations of $\langle R \cup I \rangle$. It is essential that $P = \langle S \cup nI \rangle$, n a positive integer where S is a subring of R. i.e. $\{ P$ is generated by the subring S with $nI, (n \in Z^{+})\}$. Note: Even if P is a ring and cannot be represented as

$\langle S \cup nI \rangle$ where S is a subring of R. then we do not call P a neutrosophic subring of ⟨R ∪ I⟩.

Theorem 2.2 [6]Let $\langle R \cup I \rangle$ be a neutrosophic ring and $N(S) \subseteq N(R)$, then N(S) is called a neutrosophic subring of $N(R)$ **iff** :

1. $\forall\, a, b \in$ N(S) ⇒ a − $b \in$ N(S).
2. $\forall\, a, b \in$ N(S) ⇒ ab ∈ N(S).

Definition 2.4 [1] Let ⟨R ∪ I⟩ be any neutrosophic ring. A non-empty neutrosophic subset P of $\langle R \cup I \rangle$ is called a neutrosophic left ideal of ⟨R ∪ I⟩, if the following conditions are satisfied

1. P is a neutrosophic subring ⟨R ∪ I⟩ , and
2. $rp \in P$ For every $p \in P$, and $r \in \langle R \cup I \rangle$. Also, P is called a neutrosophic right ideal of ⟨R ∪ I⟩ , if the following conditions are satisfied
3. P is a neutrosophic subring ⟨R ∪ I⟩ , and
4. $pr \in P$ For every $p \in P$, and $r \in \langle R \cup I \rangle$.

Definition 2.5 [1]Let $\langle R \cup I \rangle$ be any neutrosophic ring, A none-empty neutrosophic subset P of

$\langle R \cup I \rangle$ is defined to be a neutrosophic ideal (two sided) of ⟨R ∪ I⟩ if the following conditions are satisfied;

1. P is a neutrosophic subring of $\langle R \cup I \rangle$.
2. Every all p ∈ P and r ∈ $\langle R \cup I \rangle$, then rp and $pr \in P$.

Theorem 2.3 Let R be any ring, S is ideal of R. if $\langle R \cup I \rangle$ neutrosophic ring. Then $\langle S \cup I \rangle$

is a neutrosophic ideal of $\langle R \cup I \rangle$.

Proposition 2.1 If H a neutrosophic ideal (neutrosophic left ideal/ neutrosophic right ideal/ neutrosophic-two–sided) of $\langle R \cup I \rangle$ and J is a neutrosophic ideal (neutrosophic left ideal/ neutrosophic right ideal/neutrosophic-two–sided) of H . then not necessarily J is ideal (neutrosophic left ideal/ neutrosophic right ideal/ neutrosophic-two –sided) of ⟨R ∪ I⟩ .

Proposition 2.2 Let P and J are neutrosophic ideals of a neutrosophic ring ⟨R ∪ I⟩.

then:

1. $P + J$ is a neutrosophic ideals of a neutrosophic ring ⟨R ∪ I⟩.

2. PJ is a neutrosophic ideals of a neutrosophic ring ⟨R ∪ I⟩.

3. $P \cap J$ is neutrosophic ideals of a neutrosophic ring ⟨R ∪ I⟩.

Definition 2.6 Let $\langle R \cup I \rangle$ be a neutrosophic ring, P a pseudo neutrosophic subring of $\langle R \cup I \rangle$.

If for all $p \in P$ and $r \in$⟨R ∪ I⟩ is rp and $pr \in P$. then P is called be a pseudo neutrosophic ideal of ⟨R ∪ I⟩.

Definition 2.7 Let ⟨R ∪ I⟩ be a neutrosophic ring, let P be a neutrosophic ideal of ⟨R ∪ I⟩. The neutrosophic quotient ring is defined to be the following: ⟨R ∪ I⟩/P = {r + P: r ∈ ⟨R ∪ I⟩, such that:

For all $r_1 + P$, $r_2 + P \in \langle R \cup I \rangle/P$ is $(r_1 + P) + (r_2 + P) = (r_1 + r_2) + P$, and

$(r_1 + P)(r_2 + P) = (r_1 r_2) + P$.

Definition 2.8 Let $\langle R \cup I \rangle$ be a neutrosophic ring, P be a pseudo neutrosophic ideal of $\langle R \cup I \rangle$.

The pseudo neutrosophic quotient ring is defined to be the following:

$\langle R \cup I \rangle/P = \{r + P : r \in \langle R \cup I \rangle\}$.

3. Neutrosophic Ring Homomorphism

In this section, we introduce neutrosophic ring homomorphism with properties, and some important examples.

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Definition 3.1 [1]Let $R[I]$ and $S[I]$ be two neutrosophic rings. A map $\phi: \langle R \cup I \rangle \longrightarrow \langle S \cup I \rangle$ is said to be a neutrosophic ring homomorphism if the following conditions satisfying:

1. $\phi(x+y) = \phi(x) + \phi(y)\ for\ x, y \in \langle R \cup I \rangle$.
2. $\phi(xy) = \phi(x)\phi(y)\ for\ x, y \in \langle R \cup I \rangle$, and
3. $\phi(I) = I\ for\ I\ the\ neutrosophic\ element\ of\ \langle R \cup I \rangle$

Example 3.1 We have a neutrosophic ring of integers $\langle Z \cup I \rangle$. The mapping $\phi\ \langle Z \cup I \rangle \longrightarrow \langle Z \cup I \rangle\ via\ \phi(x) = x, for\ all\ x \in Z[I]$ is neutrosophic ring homomorphism where:

1. $$\begin{aligned} \phi(x+y) &= \phi\big((a+bI)+(c+dI)\big) \\ &= \phi\big((a+c)+(b+d)I\big) \\ &= (a+c)+(b+d)I \\ &= a+c+bI+dI \\ &= (a+bI)+(c+dI) \\ &= x+y = \phi(x)+\phi(y). \end{aligned}$$
2. $$\begin{aligned} \phi(xy) &= \phi\big((a+bI)(c+dI)\big) \\ &= \phi(ac+(ad+bc+bd)I) \\ &= ac+(ad+bc+bd)I \\ &= ac+adI+bcI+bdI \\ &= (a+bI)c+(a+bI)dI \\ &= (a+bI)(c+dI) \\ &= \phi(x)\phi(y). \end{aligned}$$
3. $\phi(I) = I\ \ for\ \ I \in Z[I]$.

Example 3.2 Let R be real numbers set. Define $\phi : \langle R \cup I \rangle \longrightarrow \langle M_{2\times 2}(R) \cup I \rangle$ such as for all

$$x \in \langle R\ \cup\ I \rangle \text{ is: } \phi(x) = \begin{cases} \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}, if\ \ x \neq I \\ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, f\ \ x = I \end{cases}$$

Let $x_1, x_2 \in \langle R \cup I \rangle$ such that $x_1 = a + bI\ , x_2 = c + dI\ ; a, b, c, d$ are real numbers with $a \neq 0\ , c \neq 0$, then

1. $\phi(x_1 + x_2) = \phi\left(\begin{bmatrix} a+bI & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} c+dI & 0 \\ 0 & 0 \end{bmatrix}\right) = \phi\left(\begin{bmatrix} e+fI & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} e+fI & 0 \\ 0 & 0 \end{bmatrix}$
   where $e, f\ are\ real\ numbers,$ and
   $\phi(x_1) + \phi(x_2) = \begin{bmatrix} a+bI & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} c+dI & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e+fI & 0 \\ 0 & 0 \end{bmatrix}$, then
   $\phi(x_1 + x_2) = \phi(x_1) + \phi(x_2)$,
2. $\phi(x_1\, x_2) = \left(\begin{bmatrix} a+bI & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c+dI & 0 \\ 0 & 0 \end{bmatrix}\right) = \phi\left(\begin{bmatrix} g+kI & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} g+kI & 0 \\ 0 & 0 \end{bmatrix}$;
   $g, k\ \ are\ real\ numbers$, and
   $\phi(x_1)\, \phi(x_2) = \begin{bmatrix} a+bI & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c+dI & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} g+kI & 0 \\ 0 & 0 \end{bmatrix}$, hence
   $\phi(x_1\, x_2) = \phi(x_1)\, \phi(x_2)$, and
3. When $x = I$, we have $\phi(I) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$, emplies that ϕ is a neutrosophic ring homomorphism.

Proposition 3.1 A map $\phi : \langle R \cup I \rangle \longrightarrow \langle R \cup I \rangle\ via\ \phi(x) = x\ and\ x \in \langle R \cup I \rangle$ is neutrosophic ring homomorphism, for all $R \in \{\mathbb{Q}, \mathfrak{R}, \mathbb{C}\}$.

Proof clear that $\phi(I) = I, and\ for\ all\ x, y \in \langle R\ \cup\ I \rangle\ is: \phi(x+y) = \phi\big((a+bI)+(c+dI)\big) = \phi(e+fI) = e+fI\ ; a, b, c, d, e, f \in R$,and
$\phi(x) + \phi(y) = (a+bI)+(c+dI) = e+fI :\ a, b, c, d, e, f \in R;\ implies\ that :$
$\phi(x+y) = \phi(x)+\phi(y)$. $Also\ \phi(xy) = \phi((a+bI)(c+dI)) = \phi(g+kI\ g+kI,$
$\phi(x)\phi(y) = \phi(a+bI)\phi(c+dI) = (a+bI)(c+dI) = (g+kI), g, k \in R$, implies that:
$\phi(xy) = \phi(x)\phi(y)$. this means that ϕ is a neutrosophic ring homomorphism.

Definition 3.2 Let $\phi : \langle R\ \cup\ I \rangle \longrightarrow \langle S\ \cup\ I \rangle$ be a neutrosophic ring homomorphism. the kernel of ϕ is $ker(\phi) = \{x \in \langle R\ \cup\ I \rangle : \phi(x) = 0\}$.

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Example 3.3 Let $\langle R \cup I\rangle, \langle S \cup I\rangle$ be neutrosophic rings, let the mapping $\phi : \langle R \cup I\rangle \longrightarrow \langle S \cup I\rangle$ be define by:
$\phi(x) = \begin{cases} 0 & ,if \;\; x \neq 0 \\ I & ,if \;\; = I \end{cases}$, Its clear , $\phi(I) = I$,and for all $x , y \in \langle R \cup I\rangle$ is
$\phi(x + y) = \phi((a + bI) + (c + dI)) = 0 \;, \phi(x) + \phi(y) = \phi(a + bI) + \phi(c + dI) = 0 + 0 = 0$,
$And \;, \phi(x\, y) = \phi((a + bI)(c + dI)) = 0 \;, \phi(x)\phi(y) = \phi(a + bI)\phi(a + bI) = (0)(0) = 0 = \phi(x)\phi(y)$,
then ϕ is a neutrosophic ring homomorphism and $ker(\phi) = \langle R \cup I\rangle / \{I\}$.

Proposition 3.2 For any neutrosophic ring homomorphism ϕ. If the kernel of ϕ is ker(ϕ). then $I \notin ker(\phi)$.

Proposition 3.3 [3] Let $\phi : \langle R \cup I\rangle \longrightarrow \langle S \cup I\rangle$, and $ker(\phi)$ is the kernel of ϕ. then

1. $ker(\phi)\; is\; subring\; of\; \langle R \cup I\rangle$.
2. $ker(\phi)$ cannot be a neutrosophic subring of $\langle R \cup I\rangle$.
3. $ker(\phi)$ is an ideal of $\langle R \cup I\rangle$.
4. $ker(\phi)$ cannot be a neutrosophic ideal of $\langle R \cup I\rangle$.

Proof firstly: $ker(\phi)$ is not empty set, (because $0 \in ker(\phi)$).
1. $let\; (a + bI), (c + dI) \in ker(\phi) \implies \phi(a + bI) = 0 \;, \phi(c + dI) = 0 \;, then$
$\phi\big((a + bI) - (c + dI)\big) = \phi(a + bI) - \phi(c + dI) = 0 - 0 = 0,\; and$
$\phi\big((a + bI)(c + dI)\big) = \phi(a + bI)\phi(c + dI) = (0)(0) = 0, implies\; that$
$(a + bI) - (c + dI) \in ker(\phi)\; and\; (a + bI)(c + dI) \in ker(\phi), So$
$ker(\phi)\; is\; subring\; of\; \langle R \cup I\rangle$.
2. It's clear, because that $ker(\phi)$ can not write its by $ker(\phi) = \langle S \cup I\rangle$ for some S is subring of R.
3. We want prove that $xr, rx\; ker(\phi)$ for all $x \in ker(\phi), r \in \langle R \cup I\rangle$.
$Let\; x = (a + bI) \in ker(\phi), r = (e + fI) \in \langle R \cup I\rangle, then\; \phi(a + bI) = 0 \;, and$
$\phi((a + bI)(e + fI)) = \phi(a + bI)\phi(e + fI) = 0\, \phi(e + fI)$. $Since\; 0x = x0 = 0\; for\; all\; x \in \langle R \cup I\rangle, then\; 0\phi(e + fI) = 0$, this means that $xr \in ker(\phi)$, Also
By way Similar $rx \in ker(\phi)$, then $ker(\phi)\; is\; an\; ideal\; of\; \langle R \cup I\rangle$.
4. It's clear, by way similar for number (2).

Definition 3.3 Let $\phi : \langle R \cup I\rangle \longrightarrow \langle S \cup I\rangle$ be a neutrosophic ring homomorphism.the image of ϕ is $im(\phi) = \{y \in \langle S \cup I\rangle \colon\; y = \phi(x)\; for\; some\; x \in \langle R \cup I\rangle\}$.

Proposition 3.4 Let $\phi : \langle R \cup I\rangle \longrightarrow \langle S \cup I\rangle$ be a neutrosophic ring homomorphism, and $im(\phi)$ is the image of ϕ then

1. $the\; im(\phi)\; is\; subring\; of\; \langle S \cup I\rangle$.
2. $im(\phi)\; can\; not\; be\; a\; neutrosophic\; subring\; of\; \langle S \cup I\rangle$.
3. $im(\phi)\; can\; not\; be\; a\; neutrosophic\; ideal\; of\; \langle S \cup I\rangle$.

Proof firstly, since $\phi(0) = 0 \implies im(\phi)$ is nonempty set, Now
1.We want prove that $\big(\phi(x = a + bI) - \phi(y = c + dI)\big), \phi(x = a + bI)\phi(y = c + dI) \in im(\phi) for\; all\; \phi(x), \phi(y) \in im(\phi).\; Let\; \phi(x = a + bI), \phi(y = c + dI) \in im(\phi), this\; mean\; that\; (a + bI), (c + dI), ((a + bI) - (c + dI))\;, (a + bI)(c + dI) \in \langle R \cup I\rangle,$ $then\; \phi((a + bI) - (c + dI))\;, \phi((a + bI)(c + dI)) \in im(\phi), but$
$\phi((a + bI) - (c + dI)) = \phi(a + bI) - \phi(c + dI), and$
$\phi((a + bI)(c + dI)) = \phi(a + bI)\phi(c + dI)$, this means that
$\phi(x) - \phi(y) \in im(\phi)$, and $\phi(x)\phi(y) \in im(\phi)$, then $im(\phi)$ is subring $of\; \langle S \cup I\rangle$.
2. It is clear, because that $im(\phi)$ can not write its by $im(\phi) = \langle H \cup I\rangle, for\; some$
subring H of S.
3. by way Similar for (2).

Proposition 3.5 If $\phi : \langle R \cup I\rangle \longrightarrow \langle S \cup I\rangle$ is a neutrosophic ring homomorphism. then for all $x \in \langle R \cup I\rangle$:
1. $\phi(0) = 0$.
2. $\phi(-x) = -(\phi(x))$.
3. $\phi(nx) = n\phi(x)\; for\; all\; integer\; n \geq 1$.
4. $\phi(xn) = (\phi(x))n\; for\; all\; integer\; n \geq 1$.

proof
1. $\phi(0) = \phi(0 + 0) = \phi(0) + \phi(0)\; (because\; \phi\; hom)$
$\Rightarrow \phi(0) = \phi(0) + \phi(0) \Rightarrow \phi(0) = 0$,
2. $Let\; x \in \langle R \cup I\rangle,\; since\; x + (-x) = x - x = -x + x = 0 = \phi(0)\;, Also$
$\phi(-x + x) = \phi(-x) + \phi(x) = 0 \Rightarrow \phi\big(-(a + bI) + (a + bI)\big) = \phi\big(-(a + bI)\big) + \phi(a + bI) = 0 \Rightarrow$
$\phi\big(-(a + bI)\big) + \phi(a + bI) = 0 \Rightarrow \phi\big(-(a + bI)\big) = -\phi(a + bI)$.
$this\; means\; that\; \phi(-x) = -\phi(x)$
3. $for\; any\; x = a + bI \in \langle R \cup I\rangle;\; n \geq 1\; is\; \phi(nx) = \phi(n(a + bI)) =$

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$\phi\left(\underbrace{(a+bI)+(a+bI)+\cdots+(a+bI)}_{n..times}\right)=\underbrace{\phi(a+bI)+\phi(a+bI)+\cdots+\phi(a+bI)}_{n..times}=n\phi(a+bI)=n\phi(x)$

4. $\phi(x^n)=\phi((a+bI)^n)=$

$\phi\left(\underbrace{(a+bI)(a+bI)\dots(a+bI)}_{n..times}\right)=\underbrace{\phi(a+bI)\,\phi(a+bI)\dots\phi(a+bI)}_{n..times}=$
$(\phi(a+bI))^n=(\phi(x))^n$

Proposition 3.6 Let $\langle R\cup I\rangle$ be a neutrosophic ring has an identity element 1, $\langle S\cup I\rangle$ is a neutrosophic ring has an identity element 1,
If $\phi:\langle R\cup I\rangle\longrightarrow\langle S\cup I\rangle$ is a neutrosophic ring homomorphism. then
$\phi(x^{-1})=(\phi(x))^{-1}\ for\ all\ x\in\langle R\cup I\rangle$ such that x is unit.

4. Neutrosophic Ring Isomorphism

In this section, we introduce neutrosophic ring isomorphism with theories it's the three.

Definition 4.1 Let $\phi:\langle R\cup I\rangle\longrightarrow\langle S\cup I\rangle$ is a neutrosophic ring homomorphism. is said that ϕ a neutrosophic ring isomorphism if: ϕ is injective and surjective, implies that (ϕ is bijective), and is denote by $\langle R\cup I\rangle\approx\langle S\cup I\rangle$.
Proposition 4.1 A neutrosophic ring homomorphism $\phi:\langle R\cup I\rangle\longrightarrow\langle S\cup I\rangle$ is an injective (one - to - one) if and only if $ker(\phi)=\{0\}$.
Proposition 4.2 A neutrosophic ring homomorphism $\phi:\langle R\cup I\rangle\longrightarrow\langle S\cup I\rangle$ is surjective (onto) if and only if $im(\phi)=\langle S\cup I\rangle$.
Example 4.1 In neutrosophic rings $\langle Z\cup I\rangle,\langle Q\cup I\rangle,\langle R\cup I\rangle\ and\ \langle C\cup I\rangle$ of integer, rational, real and complex numbers respectively. the maps:
$\phi:\langle Z\cup I\rangle\longrightarrow\langle Z\cup I\rangle,\ \phi:\langle Q\cup I\rangle\longrightarrow\langle Q\cup I\rangle,$
$\phi:\langle R\cup I\rangle\longrightarrow\langle R\cup I\rangle\ ,\phi:\langle C\cup I\rangle\longrightarrow\langle C\cup I\rangle.with\ \ \phi(x)=x$ are neutrosophic ring homomorphism injective and surjective, implies that (ϕ is bijective).
Example 4.2 The mapping $\phi:\langle Z_4\cup I\rangle\longrightarrow\langle Z_{10}\cup I\rangle$ via: $\phi(x)=\begin{cases}5x & ,x\neq I\\ I & ,x=I\end{cases}$ is neutrosophic ring homomorphism.
$We\ write\ \langle Z_4\cup I\rangle\ as\ following:$
$\langle Z_4\cup I\rangle=\{0,1,2,3,I,2I,3I,1+I,1+2I,1+3I,2+I,2+2I,2+3I,3+I,3+2I,3+3I\},then\ for\ all\ x,y\in\langle Z_4\cup I\rangle,x\neq I\ is$
$\phi(x+y)=\phi(x\oplus_4 y)=5(x+y)=5x\oplus_{10}5y=\phi(x)+\phi(y),and\ \phi(xy)=\phi(x\otimes_4 y)=5(xy)$, also $\phi(x)\phi(y)=\phi(x)\otimes_{10}\phi(y)=(5x\otimes_{10}5y)=5(xy)=\phi(xy)$.
$We\ take\ (3+2I),(3+3I)\in\langle Z_4\cup I\rangle:$
$\phi((3+2I)+(3+3I))=\phi((3+2I)\oplus_4(3+3I))=\phi(2+I)=5(2+I)\ 5I\ ,and$
$\phi(3+2I)+\phi(3+3I)=\phi(3+2I)\oplus_{10}\phi(3+3I)=5(3+2I)\oplus_{10}5(3+3I)=5(5+5I)=5I=\phi\big((3+2I)+(3+3I)\big),\ Also\ \ \phi\big((3+2I)(3+3I)\big)=$
$\phi((3+2I)\otimes_4(3+3I))=\phi(1+I)=5(1+I)=5+5I\ ,and$
$\phi(3+2I)\otimes_{10}\phi(3+3I)=5(3+2I)\otimes_{10}5(3+3I)=5(5+5I)=5+5I=\phi((3+2I)(3+3I))$. This means that ϕ is a neutrosophic ring homomorphism.
Example 4.3 Let R be real numbers ring. the mapping $\phi:\langle R\cup I\rangle\longrightarrow\langle M_{2\times2}(R)\cup I\rangle\ such\ that\ for\ all\ x\in\langle R\cup I\rangle:$

$\phi(x)=\begin{cases}\begin{bmatrix}0&0\\0&0\end{bmatrix},if\ \ x\neq I\\ \begin{bmatrix}I&0\\0&I\end{bmatrix},if\ \ x=I\end{cases}$ is neutrosophic ring homomorphism.

Proposition 4.3 Let $\phi:\langle R\cup I\rangle\longrightarrow\langle S\cup I\rangle$ be a neutrosophic ring homomorphism.
1. $if\ \langle R\cup I\rangle$ is a commutative ring, then $im(\phi)$ is a commutative subring of $\langle S\cup I\rangle$.
2. $let\ \langle R\cup I\rangle\ and\ \langle S\cup I\rangle$ are two neutrosophic rings with identities.
$if\ \phi\ is\ onto, then\ \phi(1)=1.$
Proof: 1. clear that $im(\phi)\ is\ subring\ of\ \langle S\cup I\rangle$,Now :
we want prove that $im(\phi)$ is commutative.

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$let\ \phi(x = (a + bI)), \phi(y\ = c + dI)\ \in im(\phi), then\ (a + bI), (c + dI)\ \in \langle R\ \cup\ I\rangle$,
$since\ \langle R\ \cup\ I\rangle\ is\ commutative, then\ (a + bI)(c + dI) = (c + dI)(a + bI)$,
$since\ \ \phi\big((a + bI)(c + bI)\big) = \phi(a + bI)\phi(c + dI)\ \Rightarrow\ \phi\big((a + bI)(c + dI)\big) = \phi\big((c + dI)(a + bI)\big), implies\ that\ \phi(a + bI)\phi(c + dI) = \ \phi(c + dI)\phi(a + bI), then\ im(\phi) is\ a\ commutative\ subring\ of\ \langle S\ \cup\ I\rangle$.
2. for all $x \in \langle R\ \cup\ I\rangle$ is $x\ 1 = 1\ x = x\ , and\ for\ all\ \phi(x) \in \langle S\ \cup\ I\rangle is$
$\phi(x)1 = 1\phi(x) = \phi(x), so{:}\ \phi(x1) = \phi(x)\phi(1)\ , \phi(x\ 1) = \phi(x)$, also $\phi(x)\ 1 = \ \phi(x)$,and
$\phi(1)\phi(x) = \phi(x)\phi(1)\ , 1\ \phi(x) = \phi(x)$, emplies that $\phi(1) = 1$.

Proposition 4.4 Let ϕ be a neutrosophic ring homomorphism of $\langle R\ \cup\ I\rangle\ into\ \langle S\ \cup\ I\rangle$, and let P be nonempty subset of $\langle R\ \cup\ I\rangle$. then:
1. if P is a subring of $\langle R\ \cup\ I\rangle$, then $\phi(P)$ is a subring of $\langle S\ \cup\ I\rangle$.
2. if P is a neutrosophic subring of $\langle R\ \cup\ I\rangle$, then $\phi(P)$ can not be a neutrosophic subring of $\langle S \cup I\rangle$.
3. If P is an ideal of $\langle R \cup I\rangle$, then $\phi(P)$ is an ideal of $\langle S\ \cup\ I\rangle$.
4. If P is a neutrosophic ideal of $\langle R\ \cup\ I\rangle$, then $\phi(P)$ can not be a neutrosophic ideal of $\langle S\ \cup\ I\rangle$.

Proof: 2 and 4 are clear, we want just prove 1 and 3.
1. $\phi(P)\ is\ not\ empty, because\ \phi\ is\ homomorphism.$
$let\ \ \phi(x = a + bI), \phi(y = c + dI) \in \phi(P) \Longrightarrow (a + bI), (c + dI) \in P\ , so$
$((a + bI) - (c + dI))\ , (a + bI)(c + dI) \in P\ (because\ P\ is\ subring), then$
$$\phi\big((a + bI) - (c + dI)\big), \phi\big((a + bI)(c + dI)\big) \in \phi(P), but\ \phi\big((a + bI) - (c + dI)\big) = \phi(a + bI) - \phi(c + dI), and\ \phi((a + bI)(c + dI)) = \phi(a + bI)\phi(c + dI), implies\ that:$$
$\phi(a + bI) - \phi(c + dI) \in \phi(P), and\ \phi(a + bI)\phi(c + dI) \in \phi(P), then$
$\phi(P)\ is\ subring\ of\ \langle S\ \cup\ I\rangle.$
3. $We\ want\ prove\ that\ for\ all\ \phi(a + bI) \in \phi(P)\ and\ for\ all\ \phi(e + fI) \in \langle S\ \cup\ I\rangle\ is$
$\phi(a + bI)\phi(e + fI), \phi(e + fI)\phi(a + bI) \in \phi(P).$
$Let\ \phi(a + bI) \in \phi(P), \phi(e + fI) \in \langle S\ \cup\ I\rangle, then:$
$(a + bI) \in P, (e + fI) \in \langle R\ \cup\ I\rangle, and\ (a + bI)(e + fI), (e + fI)(a + bI) \in P.$
$(because\ P\ is\ ideal), So\ \phi((a + bI)(e + fI))\ , \phi((e + fI)(a + bI)) \in \phi(P), but$
$\phi\big((a + bI)(e + fI)\big) = \phi(a + bI)\phi(e + fI), and\ \phi\big((e + fI)(a + bI)\big) = \phi(e + fI)\phi(a + bI).$
$this\ means\ that\ \phi(a + bI)\phi(e + fI) \in \phi(P),\ \ \phi(e + fI)\phi(a + bI) \in \phi(P), then$
$\phi(P)\ is\ an\ ideal\ of\ \langle S\ \cup\ I\rangle.$

Theorem 4.1 [6,10,11,14] ($First\ Isomorphism\ Theorem\ for\ a\ neutrosophic$)
Let $\phi{:}\ \langle R\ \cup\ I\rangle \longrightarrow \langle S\ \cup\ I\rangle$ be a neutrosophic ring homomorphism, then the mapping ψ from quotient ring $\langle R\ \cup\ I\rangle/ker(\phi)$ to ring $im(\phi)$ is isomorphism, and is denote by
$\langle R\ \cup\ I\rangle/ker(\phi) \approx im(\phi).$

Proof: We define a map ψ as following:
$\psi : \langle R\ \cup\ I\rangle/ker(\phi) \longrightarrow im(\phi)\ via{:}\ \psi(u) = v$ where $u \in \langle R\ \cup\ I\rangle/ker(\phi), v \in im(\phi)$, such that
$u = r + ker(\phi), r = a + bI \in \langle R\ \cup\ I\rangle, and\ v = c + d \in im(\phi), then\ \psi\ is\ define\ as\ following$
$\psi((a + bI) + K) = \phi(a + bI), where\ K = ker(\phi), \phi(a + bI) = c + dI.$
Firstly: We $want\ prove\ that\ \psi\ is\ ring\ homomorphism.$
$Let\ \ (a_1 + b_1I) + K = (a_2 + b_2I) + K\ \Longrightarrow\ (a_1 + b_1I) - (a_2 + b_2I) \in K$
$\Longrightarrow \phi\big((a_1 + b_1I) - (a_2 + b_2I)\big) = 0\ \Longrightarrow\ \phi(a_1 + b_1I) = \phi\ (a_2 + b_2I)$
$\Longrightarrow \psi\big((a_1 + b_1) + K\ \big) = \psi\big((a_2 + b_2I) + K\big)$, emplies that ψ is mapping. Also
$$\psi\Big(\big((a_1 + b_1I) + K\big) + \big((a_2 + b_2I) + K\big)\Big)$$
$$= \psi\Big(\big((a_1 + b_1I) + (a_2 + b_2I)\big) + K\Big)$$
$= \phi\big((a_1 + b_1I) + (a_2 + b_2I)\big) = \phi(a_1 + b_1I) + \phi(a_2 + b_2I)$, And
$\psi\Big(\big((a_1 + b_1I) + K\big)\ \big((a_2 + b_2I) + K\big)\Big) = \psi\Big(\big((a_1 + b_1I)(a_2 + b_2I) + K\big)\Big)$
$= \phi\big((a_1 + b_1I)\ (a_2 + b_2I)\big) = \phi(a_1 + b_1I)\ \phi(a_2 + b_2I)$, Also
$\psi(I + K) = \phi(I) = I$. this means that ψ is ring homomorphism.
Now: we want prove that ψ is bijective.
Let $\psi\big((a_1 + b_1I) + K\big) = \psi\big((a_2 + b_2I) + K\big)$, then $\phi(a_1 + b_1I) = \phi(a_2 + b_2I)\ \Longrightarrow$
$$\phi(a_1 + b_1I) - \phi(a_2 + b_2I) = 0\ \Longrightarrow\ \phi\big((a_1 + b_1I) - (a_2 + b_2I)\big) = 0\ \Longrightarrow$$
$(a_1 + b_1I) - (a_2 + b_2I) \in K\ \Longrightarrow (a_1 + b_1I) + K = (a_2 + b_2I) + K$, implies that ψ is injective,
Also ψ is surjective, because $If\ (c + dI) \in im(\phi)$,
$then\ there\ is\ \ (a + bI) \in \langle R \cup I\rangle$ such that $\phi(a + bI) = c + dI$, and

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$(a+bI)+K \in \langle R \cup I\rangle/ker(\phi)$, this means that $c+dI=\psi((a+bI)+K)$, Hence $\langle R \cup I\rangle/ker(\phi) \approx im(\phi)$.

Theorem 4.2 ($Second\ Isomorphism\ Theorem\ for\ neutrosophic$)

Let $\langle R \cup I\rangle$ be a neutrosophic ring, and let P_1, P_2 are ideals of $\langle R \cup I\rangle$. then

A map ψ from a quotient ring $P_2/(P_1 \cap P_2)$ into a quotient ring $(P_1+P_2)/P_1$ is isomorphism, and is denote by $P_2/(P_1 \cap P_2) \approx (P_1+P_2)/P_1$.

Proof: Clear that P_1+P_2, $P_1 \cap P_2$ are ideal of $\langle R \cup I\rangle$.

Firstly, we define ψ as following: $\psi: P_2/(P_1 \cap P_2) \longrightarrow (P_1+P_2)/P_1$ via:

$\psi(u)=v$ where $u \in P_2/(P_1 \cap P_2)$, $v \in (P_1+P_2)/P_1$, such that:

$u=(a+bI)+(P_1 \cap P_2)$ such that $(a+bI) \in P_1$, And

$v=(c+dI)+P_1\ such\ that\ (c+dI) \in P_2$, implies that

$\psi\big((a+bI)+(P_1+P_2)\big)=(c+dI)+P_1$.

For prove that ψ is ring isomorphism, define a map ϕ as following: $\phi: P_2 \longrightarrow (P_1+P_2)/P_1$ via

$\phi(a+bI)=(a+bI)+P_1\ such\ that\ (a+bI) \in P_2$

Now: we want prove that ϕ is ring homomorphism and surjective.

Let $(a_1+b_1I), (a_2+b_2I) \in P_2$ such that $(a_1+b_1I)=(a_2+b_2I)$).

Then $\phi(a_1+b_1I)=(a_1+b_1I)+P_1$, and $\phi(a_2+b_2I)=(a_2+b_2I)+P_1$,Since

$(a_1+b_1I)=(a_2+b_2I) \Longrightarrow (a_1+b_1I)+P_1=(a_2+b_2I)+P_1$, implies that

$\phi(a_1+b_1I)=\phi(a_2+b_2I)$, then ϕ is mapping. Also

$\phi((a_1+b_1I)+(a_2+b_2I))=\phi((a_1+a_2)+(b_1+b_2)I)=\phi(e+fI)=(e+fI)+P_1$, and

$\phi(a_1+b_1I)+\phi(a_2+b_2I)=\big((a_1+b_1I)+P_1\big)+\big((a_2+b_2I)+P_1\big)=$

$\big((a_1+a_2)+(b_1+b_2)I+P_1\big)=(e+fI)+P_1$, then $\phi\big((a_1+b_1I)+(a_2+b_2I)\big)=$

$\phi(a_1+b_1I)+\phi(a_2+b_2I)$, And $\phi\big((a_1+b_1I)(a_2+b_2I)\big)=$

$\phi(a_1a_1+(a_1b_2+b_1a_2+b_1b_2)I)=\phi(k+mI)=(k+mI)+P_1$, and

$\phi(a_1+b_1I)\,\phi(a_2+b_2I)=\big((a_1+b_1I)+P_1\big)\big((a_2+b_2I)+P_1\big)=$

$(a_1a_2+(a_1b_2+b_1a_2+b_1b_2)I)+P_1=(k+mI)+P_1$, then $\phi\big((a_1+b_1I)(a_2+b_2I)\big)=\phi(a_1+b_1I)\,\phi(a_2+b_2I)$,Also $\phi(I)=(I+P_1)$,since P_1 is the zero element of $(P_1+P_2)/P_1$.

Then $\phi(I)=I$, and for all $(a+bI)+P_1 \in (P_1+P_2)/P_1$ is $(a+bI) \in P_2$,implies that ϕ is surjective. Then ϕ is ring homomorphism and onto.

Now: we want prove that the kernel of ϕ is $P_1 \cap P_2$.

Let $a+bI \in ker(\phi) \Longrightarrow \phi(a+bI)=P_1 \Longrightarrow (a+bI) \in P_1 \Longrightarrow (a+bI) \in P_1 \cap P_2 \Longrightarrow ker(\phi) \subset P_1 \cap P_2$,Since $(P_1 \cap P_2) \subset P_1 \Longrightarrow ker(\phi)=P_1 \cap P_2$ .Hence and by ($First\ Isomorphism\ Theorem$) ,then ψ is Isomorphism.

Theorem 4.3 ($Third\ Isomorphism\ Theorem\ of\ neutrosophic$)

Let $\langle R \cup I\rangle$ be a neutrosophic ring, and let P_1, P_2 are ideals of $\langle R \cup I\rangle$ such that $P_1 \subseteq P_2$. then

A map ψ of quotient ring $(\langle R \cup I\rangle/P_1)/(P_2/P_1)$ into quotient ring $\langle R \cup I\rangle/P_2$ is isomorphism, and denote by $(\langle R \cup I\rangle/P_1)/(P_2/P_1) \approx \langle R \cup I\rangle/P_2$.

Proof: Firstly, we define ψ as following: $\psi: (\langle R \cup I\rangle/P_1)/(P_2/P_1) \longrightarrow \langle R \cup I\rangle/P_2$ via:

$\psi(u)=v$, where $u \in (\langle R \cup I\rangle/P_1)/(P_2/P_1)$, $v \in \langle R \cup I\rangle/P_2$,suth that

$u=(r+P_1)+(p+P_1)=(r+p)+P_1$; $r \in \langle R \cup I\rangle, p \in P_2$,and $v=r+P_2$; $r \in \langle R \cup I\rangle$,

Now; define a map ϕ as following: $\phi: \langle R \cup I\rangle/P_1 \longrightarrow \langle R \cup I\rangle/P_2$ via:

$\phi(r+P_1)=r+P_2$; $r \in \langle R \cup I\rangle$.

For prove that ψ isomorphism. Must prove that ϕ is surjective ring homomorphism and the kernel of ϕ is P_2/P_1.

Let $r_1+P_1, r_2+P_1 \in \langle R \cup I\rangle/P_1$ such that $r_1+P_1=r_2+P_1$,So

$\phi(r_1+P_1)=r_1+P_2$, $\phi(r_2+P_1)=r_2+P_2$,since $r_1+P_1=r_2+P_1 \Rightarrow r_1=r_2$,so

$r_1+P_2=r_2+P_2$, implies that ϕ is well-define. Also

Let $(a+bI)+P_1, (c+dI)+P_1 \in \langle R \cup I\rangle/P_1$,then

$\phi\Big(\big((a+bI)+P_1)\big)+\big((c+dI)+P_1\big)\Big)=\phi\Big(\big((a+c)+(b+d)I\big)+P_1\Big)=$

$((a+c)+(b+d)I)+P_2$, and $\phi\big((a+bI)+P_1\big)+\phi\big((c+dI)+P_1\big)=$

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$((a+bI)+P_2)+\big((c+dI)+P_2\big)=((a+c)+(b+d)I)+P_2$, then

$\phi\big(((a+bI)+P_1)+((c+dI)+P_1)\big)=\phi\big((a+bI)+P_1\big)+\phi\big((c+dI)+P_1\big)$. Also

$\phi\left(((a+bI)+P_1)\left((c+dI)+P_1\right)\right)=\phi\big((a+bI)(c+dI)+P_1\big)=$

$\phi\big((ac+(ad+bc+bd)I)+P_1\big)=(ac+(ad+bc+bd)I)+P_2$, and

$\phi\big((a+bI)+P_1\big)\ \phi\big((c+dI)+P_1\big)=\big((a+bI)+P_2\big)\big((c+dI)+P_2\big)=$

$(ac+(ad+bc+bd)I)+P_2$, then

$\phi\left(((a+bI)+P_1)((c+dI)+P_1)\right)=\phi\big((a+bI)+P_1\big)\ \phi\big((c+dI)+P_1\big)$, since

$\phi(I+P_1)=I+P_2$,e.t $\phi(I)=I$,and

For all $((a+bI)+P_2\in\langle R\cup I\rangle/P_2$ there is $((a+bI)+P_1\in\langle R\cup I\rangle/P_1$,emplies that

ϕ is surjective. Hence

ϕ is a neutrosophic ring homomorphism and onto. Now we want prove that the kernel of ϕ is P_2/P_1.

Let $(a+bI)+P_1\in P_2/P_1$;$(a+bI)\in P_2$, $(a+bI)\notin P_1$,its clear that $(a+bI)+P_1\in\langle R\cup I\rangle/P_1$,then $\phi\big((a+bI)+P_1\big)=(a+bI)+P_2=P_2$ (because $(a+bI)\in P_2$) ,since

the zero element of $\langle R\cup I\rangle/P_2$ is P_2 . then $(a+bI)+P_1\in ker(\phi)$, emplies that

$P_2/P_1\subset ker(\phi)$, also, for all $(a+bI)+P_1\in ker(\phi)$ is $\phi\big((a+bI)+P_1\big)=P_2$,but $P_1\subseteq P_2$,and

the zero element of $\langle R\cup I\rangle/P_1$ is P_1 and for all $(a+bI)+P_1\in\langle R\cup I\rangle/P_1$ such that $(a+bI)\in P_1$ is $(a+bI)+P_1=P_1$, emplies that $ker(\phi)\subset P_2/P_1$,So $ker(\phi)=P_2/P_1$.

Hence; by $First\ neutrosophic\ ring\ Isomorphism\ Theorem$ .then

A map ψ of quotient ring $(\langle R\cup I\rangle/P_1)/(P_2/P_1)$ into quotient ring $\langle R\cup I\rangle/P_2$ is isomorphism.

5. Conclusion

In this paper we have studied the neutrosophic ring homomorphism, Isomorphism and Isomorphism theories for a neutrosophic Rings by using classical methods in Ring Theory.

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