



Logarithmic neutrosophic logical communicated to basic interaction aggregating operators using various finite weighted with extension

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Abstract

In this paper, we present novel techniques for the logarithmic neutrosophic interaction (LogNI) aggregating operator. The new averaging and geometric operations of LogNI numbers are studied using the universal aggregation function. The LogNI are satisfied some algebraic properties. Four novel aggregating operators are presented: LogNI weighted averaging, LogNI weighted geometric, generalized LogNI weighted averaging, and generalized LogNI weighted geometric.

Keywords: Aggregating operator; LogNIWA; LogNIWG; GLogNIWA; GLogNIWG

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1 Introduction

The uncertainties have led to the development of fuzzy set (FS),¹ intuitionistic FS (IFS),² Pythagorean FS (PFS),^{3,4} neutrosophic set (NSS)⁵ and Fermatean FS (FFS).⁶ For those who make decisions, Zadeh's FS¹ suggests a membership degree (MG). For $\gamma, \zeta \in [0, 1]$, each object has MG γ and non-membership degree (NMG) ζ , and fulfills $0 \leq \gamma + \zeta \leq 1$. For this reason, A tanassov² proposed an IFS notion. The condition that $\gamma^2 + \zeta^2 \leq 1$ states that PFSs are characterized by their MGs and NMGs, which were created by Yager.³ IFSs and PFSs have been widely utilized and investigated in a wide range of fields. Palanikumar et al.⁷ deals that the concept of aggregation operators via Pythagorean neutrosophic soft set and square root neutrosophic normal interval-valued sets. Cuong and associates⁸ by developing the concept of picture FSs (PicFSs). PicFSs has been found to accommodate certain more ambiguity because it is an enhanced version of IFSs. PicFSs states that the MG γ , neutral ζ , and NMG $\check{m}$ all have $0 \leq \gamma + \zeta + \check{m} \leq 1$ for $\gamma, \zeta, \check{m} \in [0, 1]$. Using the PicFS, it will ensure that expert opinion messages such as "yes," "abstain," "no," and "refusal" are transmitted. It will also ensure consistency between the assessment data and the actual decision environment and avoid evaluation information from being left out. Despite a large number of PicFSs implementations and studies, the concept has not been fully explored. Shahzaib et al.⁹ was established the spherical FS (SFS) for certain

AOs with MADM. $0 \leq \gamma^2 + \zeta^2 + \check{m}^2 \leq 1$ is required by the SFS, as opposed to $0 \leq \gamma + \zeta + \check{m} \leq 1$. In MADM problems, the linguistic SFS AOs were discussed by Jin et al.¹⁰ Rafiq and colleagues introduced SFSs and their applications in DM.¹¹ The concept of an FFS was first proposed by Senapati and associates⁶ in 2019. This condition applies to both the MG and NMG: $0 \leq \gamma^3 + \zeta^3 \leq 1$.

2 Basic concepts

Definition 2.1.³ Let $\mathbb{K}$ be the universe set. The PFS $\top = \{x, \langle \gamma(\iota), \delta(\iota) \rangle | x \in \mathbb{K}\}$, where $\gamma, \delta : \mathbb{K} \rightarrow [0, 1]$ refers the MG and NMG of $x \in \mathbb{K}$ to $\top$, respectively and $0 \leq (\gamma(\iota))^2 + (\delta(\iota))^2 \leq 1$. For, $\top = \langle \gamma, \delta \rangle$ is called the Pythagorean fuzzy number (PFN).

Definition 2.2.⁶ A Fermatean fuzzy set $\top = \{x, \langle \gamma(\iota), \delta(\iota) \rangle | x \in \mathbb{K}\}$, where $\gamma(\iota)$ and $\delta(\iota)$ denote MG and NMG of u respectively, where $\gamma, \delta : \mathbb{K} \rightarrow [0, 1]$ and $0 \leq (\gamma(\iota))^3 + (\delta(\iota))^3 \leq 1$. Here, $\top = \langle \gamma, \delta \rangle$ is represent a Fermatean fuzzy number (FFN).

Definition 2.3. For any PFNs, $\top = \langle \gamma, \delta \rangle$, $\top_1 = \langle \gamma_1, \delta_1 \rangle$ and $\top_2 = \langle \gamma_2, \delta_2 \rangle$, γ, δ denote MG and NMG of u respectively. Then

1. $\top_1 \nabla \top_2 = [\sqrt{(\gamma_1)^2 + (\gamma_2)^2 - (\gamma_1)^2 \cdot (\gamma_2)^2}, (\delta_1 \cdot \delta_2)]$
2. $\top_1 \triangle \top_2 = [(\gamma_1 \cdot \gamma_2), \sqrt{(\delta_1)^2 + (\delta_2)^2 - (\delta_1)^2 \cdot (\delta_2)^2}]$
3. $\top \cdot \top = [\sqrt{1 - (1 - (\gamma)^2)^{\check{m}}}, (\delta)^{\check{m}}]$
4. $\top^{\check{m}} = [(\gamma)^{\check{m}}, \sqrt{1 - (1 - (\delta)^2)^{\check{m}}}]$

Definition 2.4. If $\top_1 = \langle \gamma_1, \delta_1 \rangle$ and $\top_2 = \langle \gamma_2, \delta_2 \rangle$ are any two PFNs. Then the interaction AO is defined as

1. $\top_1 \nabla \top_2 = \left[\frac{\sqrt{(\gamma_1)^2 + (\gamma_2)^2 - (\gamma_1)^2 \cdot (\gamma_2)^2}}{\sqrt{(\delta_1)^2 + (\delta_2)^2 - (\delta_1)^2 \cdot (\delta_2)^2 - (\delta_1)^2 \cdot (\gamma_2)^2 - (\gamma_1)^2 \cdot (\delta_2)^2}}, \right]$
2. $\top_1 \triangle \top_2 = \left[\frac{\sqrt{(\gamma_1)^2 + (\gamma_2)^2 - (\gamma_1)^2 \cdot (\gamma_2)^2 - (\gamma_1)^2 \cdot (\delta_2)^2 - (\delta_1)^2 \cdot (\gamma_2)^2}}{\sqrt{(\delta_1)^2 + (\delta_2)^2 - (\delta_1)^2 \cdot (\delta_2)^2}}, \right]$
3. $\top \cdot \top_1 = \left[\sqrt{1 - (1 - (\gamma_1)^2)^{\check{m}}}, \sqrt{(1 - (\gamma_1)^2)^{\check{m}} - (1 - (\gamma_1 + \delta_1)^2)^{\check{m}}} \right]$
4. $\top_1^{\check{m}} = \left[\sqrt{(1 - (\delta_1)^2)^{\check{m}} - (1 - (\gamma_1 + \delta_1)^2)^{\check{m}}}, \sqrt{1 - (1 - (\delta_1)^2)^{\check{m}}} \right]$

where $\check{m}$ be a positive integers.

3 Different AOs for LogNN

Throughout this paper $\log_{\check{z}_i} = \Pi_{i \rightarrow 1}^j \{\gamma_i, \varpi_i, \delta_i\}$.

Definition 3.1. Suppose that $\top_1 = \langle \gamma_1, \varpi_1, \delta_1 \rangle$ and $\top_2 = \langle \gamma_2, \varpi_2, \delta_2 \rangle$ be the any two LogNNs. Then

$$1. \top_1 \nabla \top_2 = \left[\frac{\sqrt{\langle \log_{\check{z}_i}^2 \gamma_1 \rangle + \langle \log_{\check{z}_i}^2 \gamma_2 \rangle - \langle \log_{\check{z}_i}^2 \gamma_1 \rangle \cdot \langle \log_{\check{z}_i}^2 \gamma_2 \rangle}, \right. \\ \left. \frac{\sqrt{\langle \log_{\check{z}_i}^2 \varpi_1 \rangle + \langle \log_{\check{z}_i}^2 \varpi_2 \rangle - \langle \log_{\check{z}_i}^2 \varpi_1 \rangle \cdot \langle \log_{\check{z}_i}^2 \varpi_2 \rangle}, \right. \\ \left. \frac{\sqrt{\langle \log_{\check{z}_i}^2 \delta_1 \rangle + \langle \log_{\check{z}_i}^2 \delta_2 \rangle - \langle \log_{\check{z}_i}^2 \delta_1 \rangle \cdot \langle \log_{\check{z}_i}^2 \delta_2 \rangle}}{\sqrt{-\langle \log_{\check{z}_i}^2 \delta_1 \rangle \cdot \langle \log_{\check{z}_i}^2 \gamma_2 \rangle - \langle \log_{\check{z}_i}^2 \gamma_1 \rangle \cdot \langle \log_{\check{z}_i}^2 \delta_2 \rangle}} \right]$$

$$\begin{aligned}
2. \top_1 \triangle \top_2 &= \left[\frac{\sqrt{\langle \log_{\top_1}^2 \gamma_1 \rangle + \langle \log_{\top_2}^2 \gamma_2 \rangle - \langle \log_{\top_1}^2 \gamma_1 \rangle \cdot \langle \log_{\top_2}^2 \gamma_2 \rangle}}{\sqrt{\langle \log_{\top_1}^2 \varpi_1 \rangle + \langle \log_{\top_2}^2 \varpi_2 \rangle - \langle \log_{\top_1}^2 \varpi_1 \rangle \cdot \langle \log_{\top_2}^2 \varpi_2 \rangle}}, \frac{\sqrt{-\langle \log_{\top_1}^2 \gamma_1 \rangle \cdot \langle \log_{\top_2}^2 \delta_2 \rangle - \langle \log_{\top_1}^2 \delta_1 \rangle \cdot \langle \log_{\top_2}^2 \gamma_2 \rangle}}{\sqrt{\langle \log_{\top_1}^2 \delta_1 \rangle + \langle \log_{\top_2}^2 \delta_2 \rangle - \langle \log_{\top_1}^2 \delta_1 \rangle \cdot \langle \log_{\top_2}^2 \delta_2 \rangle}} \right] \\
3. \top_1 \cdot \top_2 &= \left[\frac{\sqrt{1 - \langle 1 - \langle \log_{\top_1}^2 \gamma_1 \rangle \rangle^{\zeta_1}}, \sqrt{1 - \langle 1 - \langle \log_{\top_2}^2 \varpi_1 \rangle \rangle^{\zeta_2}}}{\sqrt{\langle 1 - \langle \log_{\top_1}^2 \gamma_1 \rangle \rangle^{\zeta_1} - \langle 1 - \langle \log_{\top_1}^2 \gamma_1 + \log_{\top_2}^2 \delta_1 \rangle \rangle^{\zeta_1}}} \right] \\
4. \top_1^{\zeta_1} &= \left[\frac{\sqrt{\langle 1 - \langle \log_{\top_1}^2 \delta_1 \rangle \rangle^{\zeta_1} - \langle 1 - \langle \log_{\top_1}^2 \gamma_1 + \log_{\top_2}^2 \delta_1 \rangle \rangle^{\zeta_1}}}{\sqrt{1 - \langle 1 - \langle \log_{\top_1}^2 \varpi_1 \rangle \rangle^{\zeta_2}}, \sqrt{1 - \langle 1 - \langle \log_{\top_1}^2 \delta_1 \rangle \rangle^{\zeta_1}}} \right]
\end{aligned}$$

3.1 LogNIWA operator

Definition 3.2. Let $\top_i = \langle \gamma_i, \varpi_i, \delta_i \rangle$ be the LogNNs, $i = 1, 2, \dots, j$, ζ_i be the weight of $\top_i$ and $\zeta_i \geq 0$, $\sum_{i=1}^j \zeta_i = 1$. Then the LogNIWA operator $\langle \top_1, \top_2, \dots, \top_j \rangle = \sum_{i=1}^j \zeta_i \top_i$.

Theorem 3.3. Let $\top_i = \langle \gamma_i, \varpi_i, \delta_i \rangle$ be the LogNNs, $i = 1, 2, \dots, j$.

$$\text{Then, LogNIWA}(\top_1, \top_2, \dots, \top_j) = \left[\frac{\sqrt{1 - \sum_{i=1}^j \zeta_i \langle 1 - \langle \log_{\top_i}^2 \gamma_i \rangle \rangle^{\zeta_i}}, \sqrt{1 - \sum_{i=1}^j \zeta_i \langle 1 - \langle \log_{\top_i}^2 \varpi_i \rangle \rangle^{\zeta_i}}}{\sqrt{\sum_{i=1}^j \zeta_i \langle 1 - \langle \log_{\top_i}^2 \gamma_i \rangle \rangle^{\zeta_i} - \sum_{i=1}^j \zeta_i \langle 1 - \langle \log_{\top_i}^2 \gamma_i + \log_{\top_i}^2 \delta_i \rangle \rangle^{\zeta_i}}} \right].$$

Proof. If $i = 2$, $\text{LogNIWA}(\top_1, \top_2) = \zeta_1 \top_1 \nabla \zeta_2 \top_2$,

where,

$$\zeta_1 \top_1 = \left[\frac{\sqrt{1 - \langle 1 - \langle \log_{\top_1}^2 \gamma_1 \rangle \rangle^{\zeta_1}}, \sqrt{1 - \langle 1 - \langle \log_{\top_1}^2 \varpi_1 \rangle \rangle^{\zeta_1}}}{\sqrt{\langle 1 - \langle \log_{\top_1}^2 \gamma_1 \rangle \rangle^{\zeta_1} - \langle 1 - \langle \log_{\top_1}^2 \gamma_1 + \log_{\top_1}^2 \delta_1 \rangle \rangle^{\zeta_1}}} \right]$$

and

$$\zeta_2 \top_2 = \left[\frac{\sqrt{1 - \langle 1 - \langle \log_{\top_2}^2 \gamma_2 \rangle \rangle^{\zeta_2}}, \sqrt{1 - \langle 1 - \langle \log_{\top_2}^2 \varpi_2 \rangle \rangle^{\zeta_2}}}{\sqrt{\langle 1 - \langle \log_{\top_2}^2 \gamma_2 \rangle \rangle^{\zeta_2} - \langle 1 - \langle \log_{\top_2}^2 \gamma_2 + \log_{\top_2}^2 \delta_2 \rangle \rangle^{\zeta_2}}} \right]$$

We get

$$\begin{aligned}
\zeta_1 \top_1 \nabla \zeta_2 \top_2 &= \left[\frac{\sqrt{\langle 1 - \langle 1 - \langle \log_{\top_1}^2 \gamma_1 \rangle \rangle^{\zeta_1} \rangle + \langle 1 - \langle 1 - \langle \log_{\top_2}^2 \gamma_2 \rangle \rangle^{\zeta_2} \rangle}}{\sqrt{-\langle 1 - \langle 1 - \langle \log_{\top_1}^2 \gamma_1 \rangle \rangle^{\zeta_1} \rangle \cdot \langle 1 - \langle 1 - \langle \log_{\top_2}^2 \gamma_2 \rangle \rangle^{\zeta_2} \rangle}}, \frac{\sqrt{\langle 1 - \langle 1 - \langle \log_{\top_1}^2 \varpi_1 \rangle \rangle^{\zeta_1} \rangle + \langle 1 - \langle 1 - \langle \log_{\top_2}^2 \varpi_2 \rangle \rangle^{\zeta_2} \rangle}}{\sqrt{-\langle 1 - \langle 1 - \langle \log_{\top_1}^2 \varpi_1 \rangle \rangle^{\zeta_1} \rangle \cdot \langle 1 - \langle 1 - \langle \log_{\top_2}^2 \varpi_2 \rangle \rangle^{\zeta_2} \rangle}}, \\
&\quad \frac{\sqrt{\langle 1 - \langle 1 - \log_{\top_1}^2 \delta_1 \rangle \rangle^{\zeta_1} \rangle + \langle 1 - \langle 1 - \log_{\top_2}^2 \delta_2 \rangle \rangle^{\zeta_2} \rangle}}{\sqrt{-\langle 1 - \langle 1 - \log_{\top_1}^2 \delta_1 \rangle \rangle^{\zeta_1} \rangle \cdot \langle 1 - \langle 1 - \log_{\top_2}^2 \delta_2 \rangle \rangle^{\zeta_2} \rangle}} \\
&= \left[\frac{\sqrt{1 - \sum_{i=1}^2 \zeta_i \langle 1 - \langle \log_{\top_i}^2 \gamma_i \rangle \rangle^{\zeta_i}}, \sqrt{1 - \sum_{i=1}^2 \zeta_i \langle 1 - \langle \log_{\top_i}^2 \varpi_i \rangle \rangle^{\zeta_i}}}{\sqrt{\sum_{i=1}^2 \zeta_i \langle 1 - \langle \log_{\top_i}^2 \gamma_i \rangle \rangle^{\zeta_i} - \sum_{i=1}^2 \zeta_i \langle 1 - \langle \log_{\top_i}^2 \gamma_i + \log_{\top_i}^2 \delta_i \rangle \rangle^{\zeta_i}}} \right].
\end{aligned}$$

Ug induction $i \geq 3$, $\text{LogNIWA}(\top_1, \top_2, \dots, \top_j)$

$$= \left[\frac{\sqrt{1 - \ast_{i-1}^j \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i \rangle \right\rangle^{\zeta_i}}, \sqrt{1 - \ast_{i-1}^j \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_i \rangle \right\rangle^{\zeta_i}}}{\sqrt{\ast_{i-1}^j \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i \rangle \right\rangle^{\zeta_i} - \ast_{i-1}^j \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i + \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i}}}, \right].$$

If $i \rightarrow j+1$, then $\text{LogNIWA}(\top_1, \top_2, \dots, \top_j, \top_{j+1})$

$$= \left[\frac{\sqrt{\frac{\boxplus_{i-1}^j \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i \rangle \right\rangle^{\zeta_i} \right\rangle + \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_{j+1} \rangle \right\rangle^{\zeta_{j+1}} \right\rangle}{- \ast_{i-1}^j \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i \rangle \right\rangle^{\zeta_i} \right\rangle \cdot \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_{j+1} \rangle \right\rangle^{\zeta_{j+1}} \right\rangle}, \frac{\boxplus_{i-1}^j \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_i \rangle \right\rangle^{\zeta_i} \right\rangle + \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_{j+1} \rangle \right\rangle^{\zeta_{j+1}} \right\rangle}{- \ast_{i-1}^j \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_i \rangle \right\rangle^{\zeta_i} \right\rangle \cdot \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_{j+1} \rangle \right\rangle^{\zeta_{j+1}} \right\rangle}, \right. \\ \left. \sqrt{\frac{\boxplus_{i-1}^j \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i} \right\rangle + \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \delta_{j+1} \rangle \right\rangle^{\zeta_{j+1}} \right\rangle}{- \ast_{i-1}^j \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i} \right\rangle \cdot \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \delta_{j+1} \rangle \right\rangle^{\zeta_{j+1}} \right\rangle}} \right. \\ \left. \sqrt{- \ast_{i-1}^j \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i + \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i} \cdot \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_{j+1} + \log_{\perp_i}^2 \delta_{j+1} \rangle \right\rangle^{\zeta_{j+1}}} \right] \\ = \left[\frac{\sqrt{1 - \ast_{i-1}^j \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i \rangle \right\rangle^{\zeta_i}} \cdot \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_{j+1} \rangle \right\rangle^{\zeta_{j+1}}}{\sqrt{1 - \ast_{i-1}^j \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_i \rangle \right\rangle^{\zeta_i}} \cdot \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_{j+1} \rangle \right\rangle^{\zeta_{j+1}}}, \frac{\sqrt{1 - \ast_{i-1}^j \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i \rangle \right\rangle^{\zeta_i}} \cdot \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_{j+1} \rangle \right\rangle^{\zeta_{j+1}}}{\sqrt{1 - \ast_{i-1}^j \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_i \rangle \right\rangle^{\zeta_i}} \cdot \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_{j+1} \rangle \right\rangle^{\zeta_{j+1}}}}, \right. \\ \left. \sqrt{\frac{\left\langle \ast_{i-1}^j \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i \rangle \right\rangle^{\zeta_i} - \ast_{i-1}^j \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i + \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i} \right\rangle}{\left\langle \left\langle \log_{\perp_i}^2 \gamma_{j+1} \right\rangle \right\rangle^{\zeta_{j+1}} - \left\langle \log_{\perp_i}^2 \gamma_{j+1} + \log_{\perp_i}^2 \delta_{j+1} \right\rangle^{\zeta_{j+1}}}} \right] \\ = \left[\frac{\sqrt{1 - \ast_{i-1}^{j+1} \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i \rangle \right\rangle^{\zeta_i}}, \sqrt{1 - \ast_{i-1}^{j+1} \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_i \rangle \right\rangle^{\zeta_i}}}{\sqrt{\ast_{i-1}^{j+1} \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i \rangle \right\rangle^{\zeta_i} - \ast_{i-1}^{j+1} \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i + \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i}}}, \right]$$

3.2 Interaction weighted geometric(LogNIWG)operator

Definition 3.4. Let $\top_i = \langle \gamma_i, \varpi_i, \delta_i \rangle$ be the LogNNs, ζ_i be the weight of $\top_i$. Then the LogNIWG operator $\langle \top_1, \top_2, \dots, \top_j \rangle = \ast_{i=1}^j \top_i^{\zeta_i}$.

Theorem 3.5. If $\top_i = \langle \gamma_i, \varpi_i, \delta_i \rangle$ be the LogNNs. Then,

$$\text{LogNIWG}(\top_1, \top_2, \dots, \top_j) = \left[\frac{\sqrt{\ast_{i=1}^j \left\langle 1 - \langle \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i} - \ast_{i=1}^j \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i + \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i}}}{\sqrt{1 - \ast_{i=1}^j \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_i \rangle \right\rangle^{\zeta_i}}, \sqrt{1 - \ast_{i=1}^j \left\langle 1 - \langle \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i}}} \right]$$

Proof. If $i \rightarrow 2$, then $\text{LogNIWG}(\top_1, \top_2) = \top_1^{\zeta_1} \triangle \top_2^{\zeta_2}$, where,

$$\top_1^{\zeta_1} = \left[\frac{\sqrt{\left\langle 1 - \langle \log_{\perp_i}^2 \delta_1 \rangle \right\rangle^{\zeta_1} - \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_1 + \log_{\perp_i}^2 \delta_1 \rangle \right\rangle^{\zeta_1}}}{\sqrt{1 - \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_1 \rangle \right\rangle^{\zeta_1}}, \sqrt{1 - \left\langle 1 - \langle \log_{\perp_i}^2 \delta_1 \rangle \right\rangle^{\zeta_1}}} \right]$$

$$\mathbb{T}_2^{\zeta_2} = \left[\frac{\sqrt{\left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \delta_2 \rangle \right\rangle^{\zeta_2}} - \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \gamma_2 + \log_{\mathbb{J}_i}^2 \delta_2 \rangle \right\rangle^{\zeta_2}}{\sqrt{1 - \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \varpi_2 \rangle \right\rangle^{\zeta_2}} \sqrt{1 - \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \delta_2 \rangle \right\rangle^{\zeta_2}}} \right]$$

We get,

$$\mathbb{T}_1^{\zeta_1} \Delta \mathbb{T}_2^{\zeta_2} = \left[\begin{array}{l} \left\langle 1 - \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \delta_1 \rangle \right\rangle^{\zeta_1} \right\rangle + \left\langle 1 - \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \delta_2 \rangle \right\rangle^{\zeta_2} \right\rangle \\ - \left\langle 1 - \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \delta_1 \rangle \right\rangle^{\zeta_1} \right\rangle \cdot \left\langle 1 - \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \delta_2 \rangle \right\rangle^{\zeta_2} \right\rangle \\ - \left\langle \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \gamma_1 + \log_{\mathbb{J}_i}^2 \delta_1 \rangle \right\rangle^{\zeta_1} \cdot \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \gamma_2 + \log_{\mathbb{J}_i}^2 \delta_2 \rangle \right\rangle^{\zeta_2} \right\rangle \\ \sqrt{\frac{\left\langle 1 - \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \varpi_1 \rangle \right\rangle^{\zeta_1} \right\rangle + \left\langle 1 - \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \varpi_2 \rangle \right\rangle^{\zeta_2} \right\rangle}{- \left\langle 1 - \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \varpi_1 \rangle \right\rangle^{\zeta_1} \right\rangle \cdot \left\langle 1 - \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \varpi_2 \rangle \right\rangle^{\zeta_2} \right\rangle}} \\ \sqrt{\frac{\left\langle 1 - \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \delta_1 \rangle \right\rangle^{\zeta_1} \right\rangle + \left\langle 1 - \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \delta_2 \rangle \right\rangle^{\zeta_2} \right\rangle}{- \left\langle 1 - \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \delta_1 \rangle \right\rangle^{\zeta_1} \right\rangle \cdot \left\langle 1 - \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \delta_2 \rangle \right\rangle^{\zeta_2} \right\rangle}} \end{array} \right]$$

$$\text{Hence, } \text{LogNIWG}(\mathbb{T}_1, \mathbb{T}_2) = \left[\frac{\sqrt{\ast_{i \rightarrow 1}^2 \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \delta_i \rangle \right\rangle^{\zeta_i}} - \ast_{i \rightarrow 1}^2 \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \gamma_i + \log_{\mathbb{J}_i}^2 \delta_i \rangle \right\rangle^{\zeta_i}}{\sqrt{1 - \ast_{i \rightarrow 1}^2 \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \varpi_i \rangle \right\rangle^{\zeta_i}} \sqrt{1 - \ast_{i \rightarrow 1}^2 \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \delta_i \rangle \right\rangle^{\zeta_i}}} \right]$$

$$\text{LogNIWG}(\mathbb{T}_1, \mathbb{T}_2, \dots, \mathbb{T}_j) = \left[\frac{\sqrt{\ast_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \delta_i \rangle \right\rangle^{\zeta_i}} - \ast_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \gamma_i + \log_{\mathbb{J}_i}^2 \delta_i \rangle \right\rangle^{\zeta_i}}{\sqrt{1 - \ast_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \varpi_i \rangle \right\rangle^{\zeta_i}} \sqrt{1 - \ast_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\mathbb{J}_i}^2 \delta_i \rangle \right\rangle^{\zeta_i}}} \right]$$

If $i \rightarrow j+1$, then $\text{LogNIWG}(\top_1, \dots, \top_j, \top_{j+1})$

$$\begin{aligned}
 &= \sqrt[2]{\frac{\left(\bigoplus_{i=0}^j \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i} \right\rangle + \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \delta_{j+1} \rangle \right\rangle^{\zeta_{j+1}} \right\rangle}{- *_{i=0}^j \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i + \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i} \cdot \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_{j+1} + \log_{\perp_i}^2 \delta_{j+1} \rangle \right\rangle^{\zeta_{j+1}} \right\rangle}} \right. \\
 &= \sqrt[2]{\frac{\left(\bigoplus_{i=0}^j \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_i \rangle \right\rangle^{\zeta_i} \right\rangle + \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_{j+1} \rangle \right\rangle^{\zeta_{j+1}} \right\rangle}{- *_{i=0}^j \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_i \rangle \right\rangle^{\zeta_i} \cdot \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_{j+1} \rangle \right\rangle^{\zeta_{j+1}} \right\rangle}} \right. \\
 &= \sqrt[2]{\frac{\left(\bigoplus_{i=0}^j \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i} \right\rangle + \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \delta_{j+1} \rangle \right\rangle^{\zeta_{j+1}} \right\rangle}{- *_{i=0}^j \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i} \cdot \left\langle 1 - \left\langle 1 - \langle \log_{\perp_i}^2 \delta_{j+1} \rangle \right\rangle^{\zeta_{j+1}} \right\rangle}} \right. \\
 &= \sqrt[2]{\frac{\left\langle *_{i=0}^j \left\langle 1 - \langle \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i} - *_{i=0}^j \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i + \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i} \right\rangle}{\left\langle \left\langle \log_{\perp_i}^2 \delta_{j+1} \right\rangle^{\zeta_{j+1}} - \langle \log_{\perp_i}^2 \gamma_{j+1} + \log_{\perp_i}^2 \delta_{j+1} \rangle^{\zeta_{j+1}} \right\rangle}} \cdot \sqrt[2]{\frac{\left\langle 1 - *_{i=0}^j \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_i \rangle \right\rangle^{\zeta_i} \cdot \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_{j+1} \rangle \right\rangle^{\zeta_{j+1}} \right\rangle}{\left\langle 1 - *_{i=0}^j \left\langle 1 - \langle \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i} \cdot \left\langle 1 - \langle \log_{\perp_i}^2 \delta_{j+1} \rangle \right\rangle^{\zeta_{j+1}} \right\rangle}} \\
 &= \sqrt[2]{\frac{\left\langle *_{i=0}^{j+1} \left\langle 1 - \langle \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i} - *_{i=0}^{j+1} \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i + \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i} \right\rangle}{\left\langle 1 - *_{i=0}^{j+1} \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_i \rangle \right\rangle^{\zeta_i} \right\rangle} \sqrt[2]{\left\langle 1 - *_{i=0}^{j+1} \left\langle 1 - \langle \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i} \right\rangle}}
 \end{aligned}$$

3.3 Generalized LogNIWA (GLogNIWA) operator

Definition 3.6. Let $\top_i = \langle \gamma_i, \varpi_i, \delta_i \rangle$ be the LogNNs, ζ_i be a weight of $\top_i$. Then, the GLogNIWA operator $\langle \top_1, \top_2, \dots, \top_j \rangle = \left\langle \bigoplus_{i=0}^j \zeta_i \top_i^2 \right\rangle^{1/2}$.

Theorem 3.7. Let $\top_i = \langle \gamma_i, \varpi_i, \delta_i \rangle$ be the LogNNs. Then $G\text{LogNIWA}(\top_1, \top_2, \dots, \top_j) =$

$$\left[\left\langle \sqrt[2]{1 - *_{i=0}^j \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i \rangle \right\rangle^{\zeta_i}} \right\rangle, \left\langle \sqrt[2]{1 - *_{i=0}^j \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_i \rangle \right\rangle^{\zeta_i}} \right\rangle, \left\langle \sqrt[2]{*_{i=0}^j \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i \rangle \right\rangle^{\zeta_i} - *_{i=0}^j \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i + \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i}} \right\rangle \right].$$

Proof. First, we have find that

$$\bigoplus_{i=0}^j \zeta_i \top_i^2 = \left[\sqrt[2]{1 - *_{i=0}^j \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i \rangle \right\rangle^{\zeta_i}}, \sqrt[2]{1 - *_{i=0}^j \left\langle 1 - \langle \log_{\perp_i}^2 \varpi_i \rangle \right\rangle^{\zeta_i}}, \sqrt[2]{*_{i=0}^j \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i \rangle \right\rangle^{\zeta_i} - *_{i=0}^j \left\langle 1 - \langle \log_{\perp_i}^2 \gamma_i + \log_{\perp_i}^2 \delta_i \rangle \right\rangle^{\zeta_i}} \right].$$

$$\text{If } i \rightarrow 2, \text{ then } \zeta_1 \top_1^2 = \left[\sqrt[2]{1 - \left\langle 1 - \langle \log_{\perp_1}^2 \gamma_1 \rangle \right\rangle^{\zeta_1}}, \sqrt[2]{1 - \left\langle 1 - \langle \log_{\perp_1}^2 \varpi_1 \rangle \right\rangle^{\zeta_1}}, \sqrt[2]{\left\langle 1 - \langle \log_{\perp_1}^2 \gamma_1 \rangle \right\rangle^{\zeta_1} - \left\langle 1 - \langle \log_{\perp_1}^2 \gamma_1 + \log_{\perp_1}^2 \delta_1 \rangle \right\rangle^{\zeta_1}} \right]$$

and

$$\zeta_2 \top_2^2 = \left[\sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_2 \right\rangle^2 \right\rangle^{\zeta_1}}, \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \varpi_2 \right\rangle^2 \right\rangle^{\zeta_1}}, \right. \\ \left. \sqrt{\left\langle \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_2 \right\rangle^2 \right\rangle^{\zeta_1} - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_2 + \log_{\mathbb{J}_i}^2 \delta_2 \right\rangle^2 \right\rangle^{\zeta_1}} \right].$$

We get, $\zeta_1 \top_1 \nabla \zeta_2 \top_2 =$

$$\left[\sqrt{\frac{\left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_1 \right\rangle^2 \right\rangle^{\zeta_1}} \right\rangle^2 + \left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_2 \right\rangle^2 \right\rangle^{\zeta_1}} \right\rangle^2}{\left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_1 \right\rangle^2 \right\rangle^{\zeta_1}} \right\rangle^2 \cdot \left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_2 \right\rangle^2 \right\rangle^{\zeta_1}} \right\rangle^2}, \right. \\ \sqrt{\frac{\left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \varpi_1 \right\rangle^2 \right\rangle^{\zeta_1}} \right\rangle^2 + \left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \varpi_2 \right\rangle^2 \right\rangle^{\zeta_1}} \right\rangle^2}{\left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \varpi_1 \right\rangle^2 \right\rangle^{\zeta_1}} \right\rangle^2 \cdot \left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \varpi_2 \right\rangle^2 \right\rangle^{\zeta_1}} \right\rangle^2}, \\ \sqrt{\frac{\left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \delta_1 \right\rangle^2 \right\rangle^{\zeta_1}} \right\rangle^2 + \left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \delta_2 \right\rangle^2 \right\rangle^{\zeta_1}} \right\rangle^2}{\left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \delta_1 \right\rangle^2 \right\rangle^{\zeta_1}} \right\rangle^2 \cdot \left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \delta_2 \right\rangle^2 \right\rangle^{\zeta_1}} \right\rangle^2}, \\ - \left[\frac{\left\langle \sqrt{\left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_1 \right\rangle^2 \right\rangle^{\zeta_1} - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_1 + \log_{\mathbb{J}_i}^2 \delta_1 \right\rangle^2 \right\rangle^{\zeta_1}} \right\rangle^2}{\left\langle \sqrt{\left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_2 \right\rangle^2 \right\rangle^{\zeta_1} - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_2 + \log_{\mathbb{J}_i}^2 \delta_2 \right\rangle^2 \right\rangle^{\zeta_1}} \right\rangle^2} \right] \\ \left. \sqrt{\left[\frac{\left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_1 \right\rangle^2 \right\rangle^{\zeta_1}} \right\rangle^2 + \left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \varpi_1 \right\rangle^2 \right\rangle^{\zeta_1}} \right\rangle^2}{\left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_1 \right\rangle^2 \right\rangle^{\zeta_1}} \right\rangle^2 \cdot \left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \varpi_1 \right\rangle^2 \right\rangle^{\zeta_1}} \right\rangle^2}, \right.} \right. \\ \left. \left. \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_1 \right\rangle^2 \right\rangle^{\zeta_1}}, \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \varpi_1 \right\rangle^2 \right\rangle^{\zeta_1}}, \right. \\ \left. \sqrt{\left\langle \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_1 \right\rangle^2 \right\rangle^{\zeta_1} - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_1 + \log_{\mathbb{J}_i}^2 \delta_1 \right\rangle^2 \right\rangle^{\zeta_1}} \right] \right] \\ = \left[\sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_1 \right\rangle^2 \right\rangle^{\zeta_i}}, \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \varpi_1 \right\rangle^2 \right\rangle^{\zeta_i}}, \right. \\ \left. \sqrt{\left\langle \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_1 \right\rangle^2 \right\rangle^{\zeta_i} - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_1 + \log_{\mathbb{J}_i}^2 \delta_1 \right\rangle^2 \right\rangle^{\zeta_i}} \right]$$

In general,

$$= \left[\sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_1 \right\rangle^2 \right\rangle^{\zeta_i}}, \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \varpi_1 \right\rangle^2 \right\rangle^{\zeta_i}}, \right. \\ \left. \sqrt{\left\langle \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_1 \right\rangle^2 \right\rangle^{\zeta_i} - \left\langle 1 - \left\langle \log_{\mathbb{J}_i}^2 \gamma_1 + \log_{\mathbb{J}_i}^2 \delta_1 \right\rangle^2 \right\rangle^{\zeta_i}} \right].$$

If $i \rightarrow j+1$, then $\boxplus_{i \rightarrow j+1}^j \zeta_i \top_i^2 + \zeta_{j+1} \top_{j+1}^2 = \boxplus_{i \rightarrow j+1}^{j+1} \zeta_i \top_i^2$.

Now, $\boxplus_{i \rightarrow j+1}^j \zeta_i \top_i^2 + \zeta_{j+1} \top_{j+1}^2 = \zeta_1 \top_1^2 \nabla \zeta_2 \top_2^2 \nabla \dots \nabla \zeta_j \top_j^2 \nabla \zeta_{j+1} \top_{j+1}^2$

$$\begin{aligned}
&= \left[\sqrt{\frac{\left\langle \sqrt{1 - \ast_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\beta_i}^2 \gamma_i \rangle} \right\rangle^{\zeta_i}} \right\rangle^2 + \left\langle \sqrt{1 - \left\langle 1 - \langle \log_{\beta_i}^2 \gamma_{j+1} \rangle} \right\rangle^{\zeta_1}} \right\rangle^2}{\left\langle \sqrt{1 - \ast_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\beta_i}^2 \gamma_i \rangle} \right\rangle^{\zeta_i}} \right\rangle^2 \cdot \left\langle \sqrt{1 - \left\langle 1 - \langle \log_{\beta_i}^2 \gamma_{j+1} \rangle} \right\rangle^{\zeta_1}} \right\rangle^2} \right. \\
&\quad \left[\sqrt{\frac{\left\langle \sqrt{1 - \ast_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\beta_i}^2 \varpi_i \rangle} \right\rangle^{\zeta_i}} \right\rangle^2 + \left\langle \sqrt{1 - \left\langle 1 - \langle \log_{\beta_i}^2 \varpi_{j+1} \rangle} \right\rangle^{\zeta_1}} \right\rangle^2}{\left\langle \sqrt{1 - \ast_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\beta_i}^2 \varpi_i \rangle} \right\rangle^{\zeta_i}} \right\rangle^2 \cdot \left\langle \sqrt{1 - \left\langle 1 - \langle \log_{\beta_i}^2 \varpi_{j+1} \rangle} \right\rangle^{\zeta_1}} \right\rangle^2} \right. \\
&\quad \left[\sqrt{\frac{\left\langle \sqrt{1 - \ast_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\beta_i}^2 \gamma_i \rangle} \right\rangle^{\zeta_i}} \right\rangle^2 + \left\langle \sqrt{1 - \left\langle 1 - \langle \log_{\beta_i}^2 \gamma_{j+1} \rangle} \right\rangle^{\zeta_1}} \right\rangle^2}{\left\langle \sqrt{1 - \ast_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\beta_i}^2 \gamma_i \rangle} \right\rangle^{\zeta_i}} \right\rangle^2 \cdot \left\langle \sqrt{1 - \left\langle 1 - \langle \log_{\beta_i}^2 \gamma_{j+1} \rangle} \right\rangle^{\zeta_1}} \right\rangle^2} \right. \\
&\quad \left. - \left[\frac{\left\langle \sqrt{\ast_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\beta_i}^2 \gamma_i \rangle} \right\rangle^{\zeta_i}} - \ast_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\beta_i}^2 \gamma_i + \log_{\beta_i}^2 \delta_i \rangle} \right\rangle^{\zeta_i}}{\left\langle \sqrt{1 - \left\langle 1 - \langle \log_{\beta_i}^2 \gamma_{j+1} \rangle} \right\rangle^{\zeta_1}} - \left\langle 1 - \langle \log_{\beta_i}^2 \gamma_{j+1} + \log_{\beta_i}^2 \delta_{j+1} \rangle} \right\rangle^{\zeta_1}} \right]^2 \right] \\
&= \left[\sqrt{1 - \ast_{i \rightarrow 1}^{j+1} \left\langle 1 - \langle \log_{\beta_i}^2 \gamma_1 \rangle} \right\rangle^{\zeta_i}}, \sqrt{1 - \ast_{i \rightarrow 1}^{j+1} \left\langle 1 - \langle \log_{\beta_i}^2 \varpi_1 \rangle} \right\rangle^{\zeta_i}}, \right. \\
&\quad \left. \sqrt{\ast_{i \rightarrow 1}^{j+1} \left\langle 1 - \langle \log_{\beta_i}^2 \gamma_1 \rangle} \right\rangle^{\zeta_i}} - \ast_{i \rightarrow 1}^{j+1} \left\langle 1 - \langle \log_{\beta_i}^2 \gamma_1 + \log_{\beta_i}^2 \delta_1 \rangle} \right\rangle^{\zeta_i}} \right] \\
\text{and } \boxplus_{i \rightarrow 1}^{j+1} \langle \zeta_i \top_i^2 \rangle &= \left[\begin{array}{c} \left\langle \sqrt{1 - \ast_{i \rightarrow 1}^{j+1} \left\langle 1 - \langle \log_{\beta_i}^2 \gamma_i \rangle} \right\rangle^{\zeta_i}} \right\rangle, \\ \left\langle \sqrt{1 - \ast_{i \rightarrow 1}^{j+1} \left\langle 1 - \langle \log_{\beta_i}^2 \varpi_i \rangle} \right\rangle^{\zeta_i}} \right\rangle, \\ \left\langle \sqrt{\ast_{i \rightarrow 1}^{j+1} \left\langle 1 - \langle \log_{\beta_i}^2 \gamma_i \rangle} \right\rangle^{\zeta_i}} - \ast_{i \rightarrow 1}^{j+1} \left\langle 1 - \langle \log_{\beta_i}^2 \gamma_i + \log_{\beta_i}^2 \delta_i \rangle} \right\rangle^{\zeta_i}} \right\rangle \end{array} \right].
\end{aligned}$$

3.4 Generalized LogNIWG(GLogNIWG) operator

Definition 3.8. Let $\top_i = \langle \gamma_i, \varpi_i, \delta_i \rangle$ be the LogNNs, ζ_i be the weight of $\top_i$, where $i \rightarrow 1, 2, \dots, j$. Then, the GLogNIWG $\langle \top_1, \top_2, \dots, \top_j \rangle = \frac{1}{\#} \left\langle \ast_{i \rightarrow 1}^j \langle \cap \top_i \rangle^{\zeta_i} \right\rangle$.

Theorem 3.9. Let $\top_i = \langle \gamma_i, \varpi_i, \delta_i \rangle$ be the collection of LogNNs. Then the GLogNIWG operator $\langle \top_1, \top_2, \dots, \top_j \rangle =$

$$\left[\left\langle \sqrt{\ast_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\beta_i}^2 \delta_i \rangle} \right\rangle^{\zeta_i}} - \ast_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\beta_i}^2 \gamma_i + \log_{\beta_i}^2 \delta_i \rangle} \right\rangle^{\zeta_i}} \right\rangle, \right. \\
\left. \left\langle \sqrt{1 - \ast_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\beta_i}^2 \varpi_i \rangle} \right\rangle^{\zeta_i}} \right\rangle \left\langle \sqrt{1 - \ast_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\beta_i}^2 \delta_i \rangle} \right\rangle^{\zeta_i}} \right\rangle \right]$$

Proof. Ug the induction method,

$$\ast_{i \rightarrow 1}^j \langle \mathfrak{h} \top_i \rangle^{\zeta_i} = \left[\frac{\sqrt{\ast_{i \rightarrow 1}^j \left\langle 1 - \left\langle \log_{\mathfrak{J}_i}^2 \delta_i \right\rangle \right\rangle^{\zeta_i}} - \ast_{i \rightarrow 1}^j \left\langle 1 - \left\langle \log_{\mathfrak{J}_i}^2 \gamma_i + \log_{\mathfrak{J}_i}^2 \delta_i \right\rangle \right\rangle^{\zeta_i}}{\sqrt{1 - \ast_{i \rightarrow 1}^j \left\langle 1 - \left\langle \log_{\mathfrak{J}_i}^2 \varpi_i \right\rangle \right\rangle^{\zeta_i}} \sqrt{1 - \ast_{i \rightarrow 1}^j \left\langle 1 - \left\langle \log_{\mathfrak{J}_i}^2 \delta_i \right\rangle \right\rangle^{\zeta_i}}} \right]$$

If $i \rightarrow 2$, then

$$\langle \mathfrak{h} \top_1 \rangle^{\zeta_1} = \left[\frac{\sqrt{\left\langle 1 - \left\langle \log_{\mathfrak{J}_1}^2 \delta_1 \right\rangle \right\rangle^{\zeta_1}} - \left\langle 1 - \left\langle \log_{\mathfrak{J}_1}^2 \gamma_1 + \log_{\mathfrak{J}_1}^2 \delta_1 \right\rangle \right\rangle^{\zeta_1}}{\sqrt{1 - \left\langle 1 - \left\langle \log_{\mathfrak{J}_1}^2 \varpi_1 \right\rangle \right\rangle^{\zeta_1}} \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathfrak{J}_1}^2 \delta_1 \right\rangle \right\rangle^{\zeta_1}}} \right]$$

and

$$\langle \mathfrak{h} \top_2 \rangle^{\zeta_2} = \left[\frac{\sqrt{\left\langle 1 - \left\langle \log_{\mathfrak{J}_2}^2 \delta_2 \right\rangle \right\rangle^{\zeta_2}} - \left\langle 1 - \left\langle \log_{\mathfrak{J}_2}^2 \gamma_2 + \log_{\mathfrak{J}_2}^2 \delta_2 \right\rangle \right\rangle^{\zeta_2}}{\sqrt{1 - \left\langle 1 - \left\langle \log_{\mathfrak{J}_2}^2 \varpi_2 \right\rangle \right\rangle^{\zeta_2}} \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathfrak{J}_2}^2 \delta_2 \right\rangle \right\rangle^{\zeta_2}}} \right]$$

We get, $\langle \mathfrak{h} \top_1 \rangle^{\zeta_1} \triangle \langle \mathfrak{h} \top_2 \rangle^{\zeta_2}$

$$\begin{aligned} & \left[\frac{\left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathfrak{J}_1}^2 \delta_1 \right\rangle \right\rangle^{\zeta_1}} \right\rangle^2 + \left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathfrak{J}_2}^2 \delta_2 \right\rangle \right\rangle^{\zeta_2}} \right\rangle^2}{- \left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathfrak{J}_1}^2 \delta_1 \right\rangle \right\rangle^{\zeta_1}} \right\rangle^2 \cdot \left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathfrak{J}_2}^2 \delta_2 \right\rangle \right\rangle^{\zeta_2}} \right\rangle^2} \right. \\ & \quad \left. - \left[\frac{\left\langle \sqrt{\left\langle 1 - \left\langle \log_{\mathfrak{J}_1}^2 \delta_1 \right\rangle \right\rangle^{\zeta_1}} - \left\langle 1 - \left\langle \log_{\mathfrak{J}_1}^2 \gamma_1 + \gamma_1 \right\rangle \right\rangle^{\zeta_1}} \right\rangle^2}{\left\langle \sqrt{\left\langle 1 - \left\langle \log_{\mathfrak{J}_2}^2 \delta_2 \right\rangle \right\rangle^{\zeta_2}} - \left\langle 1 - \left\langle \log_{\mathfrak{J}_2}^2 \gamma_2 + \log_{\mathfrak{J}_2}^2 \delta_2 \right\rangle \right\rangle^{\zeta_2}} \right\rangle^2} \right] \\ & = \left[\frac{\left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathfrak{J}_1}^2 \varpi_1 \right\rangle \right\rangle^{\zeta_1}} \right\rangle^2 + \left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathfrak{J}_2}^2 \varpi_2 \right\rangle \right\rangle^{\zeta_2}} \right\rangle^2}{- \left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathfrak{J}_1}^2 \varpi_1 \right\rangle \right\rangle^{\zeta_1}} \right\rangle^2 \cdot \left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathfrak{J}_2}^2 \varpi_2 \right\rangle \right\rangle^{\zeta_2}} \right\rangle^2} \right. \\ & \quad \left. - \left[\frac{\left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathfrak{J}_1}^2 \delta_1 \right\rangle \right\rangle^{\zeta_1}} \right\rangle^2 + \left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathfrak{J}_2}^2 \delta_2 \right\rangle \right\rangle^{\zeta_2}} \right\rangle^2}{- \left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathfrak{J}_1}^2 \delta_1 \right\rangle \right\rangle^{\zeta_1}} \right\rangle^2 \cdot \left\langle \sqrt{1 - \left\langle 1 - \left\langle \log_{\mathfrak{J}_2}^2 \delta_2 \right\rangle \right\rangle^{\zeta_2}} \right\rangle^2} \right] \\ & = \left[\frac{\sqrt{\ast_{i \rightarrow 1}^2 \left\langle 1 - \left\langle \log_{\mathfrak{J}_i}^2 \delta_i \right\rangle \right\rangle^{\zeta_i}} - \ast_{i \rightarrow 1}^2 \left\langle 1 - \left\langle \log_{\mathfrak{J}_i}^2 \gamma_i + \log_{\mathfrak{J}_i}^2 \delta_i \right\rangle \right\rangle^{\zeta_i}}{\sqrt{1 - \ast_{i \rightarrow 1}^2 \left\langle 1 - \left\langle \log_{\mathfrak{J}_i}^2 \varpi_i \right\rangle \right\rangle^{\zeta_i}} \sqrt{1 - \ast_{i \rightarrow 1}^2 \left\langle 1 - \left\langle \log_{\mathfrak{J}_i}^2 \delta_i \right\rangle \right\rangle^{\zeta_i}}} \right] \end{aligned}$$

If $i \rightarrow j$, then

$$= \left[\frac{\sqrt{\ast_{i \rightarrow 1}^j \left\langle 1 - \left\langle \log_{\mathfrak{J}_i}^2 \delta_i \right\rangle \right\rangle^{\zeta_i}} - \ast_{i \rightarrow 1}^j \left\langle 1 - \left\langle \log_{\mathfrak{J}_i}^2 \gamma_i + \log_{\mathfrak{J}_i}^2 \delta_i \right\rangle \right\rangle^{\zeta_i}}{\sqrt{1 - \ast_{i \rightarrow 1}^j \left\langle 1 - \left\langle \log_{\mathfrak{J}_i}^2 \varpi_i \right\rangle \right\rangle^{\zeta_i}} \sqrt{1 - \ast_{i \rightarrow 1}^j \left\langle 1 - \left\langle \log_{\mathfrak{J}_i}^2 \delta_i \right\rangle \right\rangle^{\zeta_i}}} \right]$$

If $i \rightarrow j+1$, then $*_{i \rightarrow 1}^j \langle \cap T_i \rangle^{\zeta_i} \cdot \langle \cap T_{j+1} \rangle^{\zeta_{j+1}} = *_{i \rightarrow 1}^{j+1} \langle \cap T_i \rangle^{\zeta_i}$.

Now, $*_{i \rightarrow 1}^j \langle \cap T_i \rangle^{\zeta_i} \cdot \langle \cap T_{j+1} \rangle^{\zeta_{j+1}} = \langle \cap T_1 \rangle^{\zeta_1} \triangle \langle \cap T_2 \rangle^{\zeta_2} \triangle \dots \triangle \langle \cap T_j \rangle^{\zeta_j} \triangle \langle \cap T_{j+1} \rangle^{\zeta_{j+1}}$

$$\begin{aligned}
 &= \sqrt{\left[\frac{\left\langle \sqrt{1 - *_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \delta_i \rangle} \right\rangle^{\zeta_i}} \right\rangle^2 + \left\langle \sqrt{1 - \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \delta_{j+1} \rangle} \right\rangle^{\zeta_1}} \right\rangle^2}{\left\langle \sqrt{1 - *_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \delta_i \rangle} \right\rangle^{\zeta_i}} \right\rangle^2 \cdot \left\langle \sqrt{1 - \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \delta_{j+1} \rangle} \right\rangle^{\zeta_1}} \right\rangle^2} \right. \\
 &\quad \left. - \left[\frac{\left\langle \sqrt{*_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \delta_i \rangle} \right\rangle^{\zeta_i}} - *_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \gamma_i + \log_{\mathcal{D}_i}^2 \delta_i \rangle} \right\rangle^{\zeta_i}} \right\rangle^2}{\left\langle \sqrt{\left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \delta_{j+1} \rangle} \right\rangle^{\zeta_1}} - \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \gamma_{j+1} + \log_{\mathcal{D}_i}^2 \delta_{j+1} \rangle} \right\rangle^{\zeta_1}} \right\rangle^2} \right] \\
 &= \sqrt{\frac{\left\langle \sqrt{1 - *_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \varpi_i \rangle} \right\rangle^{\zeta_i}} \right\rangle^2 + \left\langle \sqrt{1 - \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \varpi_{j+1} \rangle} \right\rangle^{\zeta_1}} \right\rangle^2}{\left\langle \sqrt{1 - *_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \varpi_i \rangle} \right\rangle^{\zeta_i}} \right\rangle^2 \cdot \left\langle \sqrt{1 - \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \varpi_{j+1} \rangle} \right\rangle^{\zeta_1}} \right\rangle^2} \\
 &\quad \sqrt{\frac{\left\langle \sqrt{1 - *_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \delta_i \rangle} \right\rangle^{\zeta_i}} \right\rangle^2 + \left\langle \sqrt{1 - \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \delta_{j+1} \rangle} \right\rangle^{\zeta_1}} \right\rangle^2}{\left\langle \sqrt{1 - *_{i \rightarrow 1}^j \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \delta_i \rangle} \right\rangle^{\zeta_i}} \right\rangle^2 \cdot \left\langle \sqrt{1 - \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \delta_{j+1} \rangle} \right\rangle^{\zeta_1}} \right\rangle^2} \\
 &= \left[\frac{\left\langle \sqrt{*_{i \rightarrow 1}^{j+1} \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \delta_1 \rangle} \right\rangle^{\zeta_i}} - *_{i \rightarrow 1}^{j+1} \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \gamma_1 + \log_{\mathcal{D}_i}^2 \delta_1 \rangle} \right\rangle^{\zeta_i}} \right\rangle^{\zeta_i}}{\sqrt{1 - *_{i \rightarrow 1}^{j+1} \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \varpi_1 \rangle} \right\rangle^{\zeta_i}}, \sqrt{1 - *_{i \rightarrow 1}^{j+1} \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \delta_1 \rangle} \right\rangle^{\zeta_i}} \right]
 \end{aligned}$$

Hence

$$\begin{aligned}
 &\frac{1}{\cap} \left\langle *_{i \rightarrow 1}^{j+1} \langle \cap T_i \rangle^{\zeta_i} \right\rangle = \\
 &\quad \left[\frac{\left\langle \sqrt{*_{i \rightarrow 1}^{j+1} \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \delta_i \rangle} \right\rangle^{\zeta_i}} - *_{i \rightarrow 1}^{j+1} \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \gamma_i + \log_{\mathcal{D}_i}^2 \delta_i \rangle} \right\rangle^{\zeta_i}} \right\rangle^{\zeta_i}}{\left\langle \sqrt{1 - *_{i \rightarrow 1}^{j+1} \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \varpi_i \rangle} \right\rangle^{\zeta_i}} \right\rangle^{\zeta_i}} \left\langle \sqrt{1 - *_{i \rightarrow 1}^{j+1} \left\langle 1 - \langle \log_{\mathcal{D}_i}^2 \delta_i \rangle} \right\rangle^{\zeta_i}} \right\rangle^{\zeta_i} \right]
 \end{aligned}$$

References

- [1] L. A. Zadeh, Fuzzy sets, Information and control, 8(3), (1965), 338-353.
- [2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets and Systems, 20(1), (1986), 87-96.
- [3] R.R. Yager, Pythagorean membership degrees in multi criteria decision making, IEEE Trans. Fuzzy Systems, 22, (2014), 958-965.
- [4] D. Liang and Z. Xu, The new extension of TOPSIS method for multiple criteria decision making with hesitant Pythagorean fuzzy sets, Applied Soft Computing, 60, (2017), 167-179.
- [5] F. Smarandache, A unifying field in logics, Neutrosophy neutrosophic probability, set and logic, American Research Press, Rehoboth, (1999).
- [6] T. Senapati, R.R. Yager, Fermatean, fuzzy sets. J. Ambient Intell. Humaniz. Comput. 11, (2020), 663-674.

- [7] M. Palanikumar, K. Arulmozhi, A Iampan, Multi criteria group decision making based on VIKOR and TOPSIS methods for Fermatean fuzzy soft with aggregation operators, *ICIC Express Letters* 16 (10), (2022), 1129–1138.
- [8] Cuong B.C, Kreinovich V, Picture fuzzy sets a new concept for computational intelligence problems, third world congress on information and communication technologies, WICT 2013. IEEE, 1-6.
- [9] Ashraf S, Abdullah S and Mahmood T, Spherical fuzzy Dombi aggregation operators and their application in group decision making problems, *Journal of Ambient Intelligence and Humanized Computing*, (2019), 1-19.
- [10] Jin, H., Ashraf, S., Abdullah, S., Qiyas, M., Bano, M. and Zeng, S, Linguistic spherical fuzzy aggregation operators and their applications in multi-attribute decision making problems, *Mathematics*, 7(5), 2019, 413.
- [11] Rafiq, M., Ashraf, S., Abdullah, S., Mahmood, T. and Muhammad, S, The coe similarity measures of spherical fuzzy sets and their applications in decision making. *Journal of Intelligent & Fuzzy Systems*, 36(6), 2019, 6059-6073.