Study Neutrosophic Quasi-Frobenius by Local and Artinian Rings

Omar A. Khashan[1,*], Majid M. Abed[2]
[1]Department of Mathematics, College of Computer Sciences and Mathematics, University of Tikrit, Tikrit, Iraq
[2]Department of Mathematics, College of Education for Pure Science, University Of Anbar, Ramadi, Anbar, Iraq
Emails: omar1990@st.tu.edu.iq; majid_math@uoanbar.edu.iq

Abstract

In this paper, we study the relationships between the Neutrosophic quasi-Frobenius rings and the Neutrosophic of local rings and Artinian rings. In addition, we present study the relationship between the Neutrosophic quasi-Frobenius ring and some concepts such as Neutrosophic semisimple ring, Neutrosophic module injective and Neutrosophic Noetherian ring. Finally, we introduce some mathematical formulas with an commutative, coherent and Neutrosophic perfect ring, through which we obtain the Neutrosophic quasi-Frobenius ring.

Keywords: Quasi-Frobenius rings; Artinian rings; Local rings; Noetherian ring

1. Introduction

A Neutrosophic Artinian ring is a Neutrosophic ring that satisfies the descending chain condition on (one-sided) ideals with conditions of Neutrosophic concept; that is, there is no infinite descending sequence of ideals. A Neutrosophic local (NL) ring is a Neutrosophic commutative ring has a unique Neutrosophic maximal ideal. A Neutrosophic ring $(R \cup I)$ is referred to, as quasi-Frobenius (QF) if it is (left or right) Neutrosophic self-injective and (left or right) artinian, equivalently, on which side it is self-injective and which side it is Neutrosophic Noetherian. Many authors generalize the concept of rings that are Frobenius and quasi-Frobenius. In (1941), Nakayama, Tadasi presented a study on Frobeniusian algebra [1]. In (1950), Ikeda, Masatoshi and Tadasi Nakayama submitted a study about supplementary remarks on Frobeniusian algebras [2]. In (1956), the authors studied both Hiroyuki Tachikawa, Kiiti Morita, and Morita Quasi Frobenius rings, character modules, and free module submodules [3]. In (1958), Dieudonne and Jean provided some remarks about quasi-Frobenius rings [4]. In (1964), E. A. Walker and Carl Faith proposed the concept about quasi-Frobenius rings on characterizations [5]. In addition, in (1996), Dinh van huynh demonstrated in her study A Note on quasi- Frobenius rings [6]. In addition, the term of neutrosophic set was defined with membership, non-membership and indeterminacy degrees by [7]. Sets of fuzzy and Sets of intuitionistic fuzzy have a generalization, which is the neutrosophic set. According to neutrosophic reasoning, There are three levels to a proposition: truth (T), indeterminacy (I), and falsehood (F). There is a membership function for each component level of involvement in the uncertainty issue, according to fuzzy set theory [8]. After that, in 1983, K. Atanassov expanded fuzzy sets to include intuitionistic fuzzy sets on universe X. In these sets, in addition to Membership degree $\mu_A(x_0) \in [0,1]$ for any element x_0 to set A, non-members may also get a degree to function $\nu_A(x_0) \in [0,1]$ that are present, where $x_0 \in X, \mu_A(x_0) + \nu_A(x_0) \leq 1$ [9].

2. Basic Concepts

In the present section, some fundamental terms that will be used later on are defined.

Definition 2.1 [7]: An object of the form $B = \{\langle x, T_B(x), I_B(x), F_B(x)\rangle\}; x \in X$ is a neutrosophic set with three membership functions:

1) $T_B(x)$: The membership function for truth.
2) $I_B(x)$: The membership function for indeterminacy.
3) $F_B(x)$: The membership function for falsity.

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These functions map an element $x \in X$ to the unit interval $[0,1]$ independently, meaning:

$$T_B(x), I_B(x), F_B(x): X \longrightarrow [0,1]$$

For each element $x \in X$, the values of $T_B(x), I_B(x)$, and $F_B(x)$ represent the degree of truth, the degree of indeterminacy, and the degree of falsity, respectively, and they satisfy the following condition:

$$0 \leq T_B(x) + I_B(x) + F_B(x) \leq 3$$

Example 2.2: Suppose that the domain of discourse $U = \{x_1, x_2, x_3\}$, where x_1 describes the capacity, x_2 describes the reliability and x_3 represents the item prices Another such assumption is that that values of x_1, x_2 and x_3 located in $[0,1]$ These are derived from surveys completed by professionals. Professionals have the option to enforce their views in three areas: the degree to which things' qualities may be explained by their goodness, indeterminacy, or poverty. Assume that A is a set of Neutrosophic (NS) on U, where,

$A = \{< x_1, (0.3, 0.5, 0.6) >, < x_2, (0.3, 0.2, 0.3) >, < x_3, (0.3, 0.5, 0.6) >\}$, such that 0.3 is the degree for capacity goodness, degree for capability indeterminacy is 0.5 and 0.6 is the degree of capacity falsehood, etc.

Definition 2.3 [10]: A Fuzzy set (FS) on universal set X defined as mathematical structure $B = \{\langle x, \mu_B(x)\rangle : x \in X\}$ with mapping $\mu_B: X \longrightarrow [0,1]$ such that $X \neq \emptyset$ is any set.

Example 2.4: The universal set X is the group of people. The issue of whether person x is young is addressed to that extent by the definition of B fuzzy subset young? To each person in the universal set, We must allocate a degree for membership inside the fuzzy subset "young." The most straightforward method to do this is using the function of membership based on the individual's age.

$$\mu_B(x) = \begin{cases} 1, & age(x) \leq 20 \\ (30 - age(x))/10, & 20 \leq age(x) \leq 30 \\ 0, & age(x) > 30 \end{cases}$$

Definition 2.5 [11]: Consider a ring $(R, +, \cdot)$, where I represents the indeterminate element with specific characteristic $I^2 = I$. Then the set $R(I) = (R \cup I) = \{a + kI; a, k \in R\}$ is called the Neutrosophic ring(NR) formed by R and I, with respect to the same binary operations on R.

Example 2.6: Consider $(Z_2, +_2, \cdot_2)$ is the ring of integers modulo 2 such that $Z_2 = \{\bar{0}, \bar{1}\}$. Note that $Z_2(I) = (Z_2 \cup I) = \{\bar{0}, \bar{1}, I, \bar{1} + I\}$ with the same operations on Z_2 is the neutrosophic ring.

Definition 2.7 [12]: Remember that in order for a Neutrosophic ring $(R \cup I)$ to be considered Noetherian, it must meet the following three comparable requirements:

(1) Maximum elements (the maximum condition) exist in all nonempty sets from Neutrosophic ideals of $(R \cup I)$.

(2) Every ascending sequences of Neutrosophic ideals are stationary (the ascending chain condition (a.c.c.))

(3) Every Neutrosophic ideal of $(R \cup I)$ is Neutrosophic f-generated.

Defintion 2.8 [13]: A NR $(R \cup I)$ with identity is referred as Neutrosophic divisible if all Neutrosophic regular (reg.) element (not zero-divisor) represents the unit (have inverse). This means that all reg. member of the Neutrosophic type is a unit.

Defintion 2.9 [14]: A NR $(R \cup I)$ with 1 is Neutrosophic semisimple, or left semisimple to be precise, if the left $(R \cup I)$-module underlying $(R \cup I)$ is a sum of Neutrosophic simple $(R \cup I)$-module.

Defintion 2.10 [14]: A Neutrosophic ring $(R \cup I)$ is referred as Neutrosophic Kasch ring (NKR) if all Neutrosophic simple $(R \cup I)$-module embeds into $(R \cup I)$.

Defintion 2.11 [15]: A Neutrosophic ring is coherent if every Neutrosophic f-generated ideal of this Neutrosophic ring is Neutrosophic f-presented. Equivalently, a Neutrosophic ring is coherent if and only if it is a coherent Neutrosophic module over itself.

Every Neutrosophic noetherian ring is a coherent ring.

3. Auxiliary results

We begin with the following lemmas which needs its in the our main conclusions.

Lemma 3.1 [16]:

(a) Let R be a right injective, Subsequently, we possess:

1) for any A, B are Sub-object from R : $l(A \cap B) = l(A) + l(B)$.

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2) For any finitely generated C Sub-object of left injective $R : lr(C) = C$.

(b) If (a) is true for both 1) and 2), then any homomorphism from R into itself that takes a finitely generated right ideal may be derived by left multiplying by an element of R.

Lemma 3.2 [16]: For a ring R the following statements are equivalent:

1) R is a right perfect.
2) All R-modules on the flat right are projective.
3) The descending chain condition for a cyclic left ideals is satisfied by R.
4) Every non-zero left R-module has a non-zero socle, and R does not contain an infinite set for orthogonal idempotent.
5) $R/Rad(R)$ is semi simple and $Rad(R)$ is a left t-nilpotent**.**

Lemma 3.3 [16]: Let R be a right perfect ring, subsequently, we possess:

(a) All left R-modules of noetherians are artinian.
(b) All right R-module of artinians are noetherian.
(c) Let R be a right Noetherian ring, then it is also a right Artinian ring.

Lemma 3.4 [16]:

(a) If M is right artinian resp. noetherian and a ring R is right artinian, then M is also right noetherian and artinian.
(b) If a ring R is right artinian, then R is right-noetherian.
(c) If a ring R is right-artinian and left-noetherian then R is left-artinian.

Lemma 3.5 [16]: Let R be a right Noetherian ring, Subsequently, we possess:

(a) The following statements are equivalent:
1) R is right-injective.
2) R is a right cogenerator.
3) R is left injective.
4) R is a left cogenerator.
5) $\forall$ A Sub-object of right injective R $[rl(A) = A]$ and $\forall$ B Sub-object of left injective R $[lr(B) = B]$.

(b) Assuming (a) holds, R is artinian from both sides.

Lemma 3.6 [16]: Let a ring R be right-Noetherian and if (1) and (2) in Lemma 3.1. are satisfied then R is right injective.

Lemma 3.7 [17]: The following conditions are equivalent to the ring R:

1) R is semis imple.
2) A left R-module is semi simple.
3) $R = \oplus_{i\in I} K_i$ s.t. K_i are minimal left-ideals.
4) R is a direct sum of finitely many minimal left ideals**.**

Lemma 3.8 [17]: Consider M to be an R-module and R to be a ring.

1) M has a composition series if and only if M is Artinian and Noetherian.
2) If M contains a sequence of compositions of length n, then all series from M possesses length of not more than n and may be elevated to the level of a series of compositions.

Remark 3.9 [17]: We characterized semi simple rings via left-modules, simple left-submodules, and minimum left-ideals, thereby allowing us to refer to it as left-semi simple. We may equivalently define right-semi simple using a right-modules, simple right-submodules, and minimum right-ideals over R. By revisiting our analysis of this concept of right-semi simple, we will arrive to the same Wedderburn-Artin theorem, which is independent of the notions of "left" or "right". The concepts for left and right-semi simple rings are same, thus we refer to them simply as semi simple**.**

Lemma 3.10 [18]: Assuming R is a right-KR with an injective-envelope $E(R)$ that embeds in the free-module for all cyclic submodule. Assume all direct summand from $E(R)$ has an essential projective-module P where $P/(P \cdot Z(R))$ is f-generated. Consequently, R has the essential socle f-generated.

Lemma 3.11 [19]: Let R be a commutative ring, then R is called semi-perfect iff it is a f-direct product of (commutative) L-rings.

Lemma 3.12 [19]: The following statements are equivalent for any ring R:

1) R is a right-perfect ring.

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2) R is satisfied DCC on the principal left-ideals.
3) Each left-module N over R provides DCC on cyclic submodules.
4) A set of not zero idempotents that is infinitely orthogonal to R does not exist, and any not zero left A basic submodule is present in R-module N.

Lemma 3.13 [20]: Let P be f.p., and $\beta : Q \longrightarrow P$ be an epimorphism. If Q is f.g., then so is $\ker(\beta)$.

4. Main Results

In this section of our work can be formulated as following:

Definition 4.1 [21]: Let $(R \cup I)$ be a commutative Neutrosophic ring. Then $(R \cup I)$ is said to be Neutrosophic local ring if $(R \cup I)$ has a unique Neutrosophic max. ideal.

Example 4.2: Let $((\mathrm{Z}_8 \cup I), +_8, \cdot_8)$ be a commutativ Neutrosophic ring with unity

$\mathrm{I} = <2>, \mathrm{J} = <4>$ such that $<4> \subseteq <2> \subseteq \mathrm{Z}_8$, $\therefore <2>$ is only maximal ideal, hence $(\mathrm{Z}_8 \cup I)$ is Neutrosophic local ring.

Definition 4.3 [22]: A ring $(R \cup I)$ is a Neutrosophic Artinian if it satisfies the DCC (Descending Chain Condition): all descending chain from ideals of R, $\mathrm{I}_0 \supseteq \mathrm{I}_1 \supseteq \cdots \supseteq \mathrm{I}_n \supseteq \mathrm{I}_{n+1} \supseteq \cdots$ is stationary.

Example 4.4: Assume $(\mathrm{K(t)} \cup I)$ is the polynomial Neutrosophic ring in the variable t accompanied with coefficients in a Neutrosophic field K. Then for any positive integer n, the residue ring consisting of $(\mathrm{K(t)} \cup I)/(\mathrm{t}^n)$ is both Neutrosophic artinian and noetherian. Reason being, the vector space $(\mathrm{K(t)} \cup I)/(\mathrm{t}^n)$ is finite and has n dimensions.

Theorem 4.5 [20]: All local artinian ring is Q-F ring.

Theorem 4.6: Assume that $(R \cup I)$ is a N-divisible ring. Then $(R \cup I)$ is NQF if $(Z \cup I)$ is included within a proper Neutrosophic ideal. In this instance, $(Z \cup I)$ constitutes an ideal and $(R \cup I)$ is right injective and Neutrosophic Noetherian.

Proof: Because of the $(R \cup I)$ is Neutrosophic divisible, then every appropriate Neutrosophic right-ideal is involved inside $(Z \cup I)$. If $(Z \cup I)$ is contained inside a suitable N-ideal, then $(Z \cup I)$ is necessarily a N-ideal. Consequently, $(R \cup I)$ constitutes a NLR, with $(Z \cup I)$ serving as the only maximum Neutrosophic ideal. Now Since $(R \cup I)$ is Neutrosophic injective by Lemma 3.1, we have $lr(C \cup I) = (C \cup I)$ for every N f-generated left-ideals $(C \cup I)$ Sub-object of left injective $(R \cup I)$. Since $(R \cup I)$ is Neutrosophic Noetherian, Consequently, $(R \cup I)$ fulfills DCC for every f-generated ideals, specifically for all Neutrosophic cyclic ideals. Hence by Lemma 3.2, $(R \cup I)$ is right Neutrosophic perfect. Then Lemma 3.3, means that $(R \cup I)$ is a Neutrosophic artinian. Consequently, it ensues from $lr(C \cup I) = (C \cup I)$ that left injective $(R \cup I)$ fulfills ACC for Neutrosophic f-generated left ideals. We conclude that Neutrosophic injective $(R \cup I)$ is, in fact, Neutrosophic Noetherian. If this had not been the case, then an ideal B Sub-object of left injective $(R \cup I)$ must exist that is not f-generated. For any f-generated subideal of $(B \cup I)$, there exists a proper bigger f-generated subideal. Within $(B \cup I)$, one may inductively create an endless, correctly ascending chain for f-generated subideals, contradicting the preceding statement. Because $(R \cup I)$ is likewise left noetherian and right artinian, it follows from Lemma 3.4, that $(R \cup I)$ is also Neutrosophic artinian. Consequently from theorem 4.5. $(R \cup I)$ is Neutrosophic Q-F ring.

Theorem 4.7: Let $(R \cup I)$ be a N-uniform ring that is commutative and divisible, right Noetherian and 5) of Lemma 3.5 holds, then $(R \cup I)$ is Neutrosophic Q-F ring.

Proof: Assume aI and bI represent any pair of non-units neutrosophic elements in $(R \cup I)$. Because $(R \cup I)$ is a neutrosophic divisible, $aI, bI \in (Z \cup I)$, i.e., $(aI)(xI) = 0 = (bI)(yI)$, $(xI), (yI) \neq 0$. Since $(R \cup I)$ is neutrosophic uniform, $(xI)(nI) = (yI)(mI) \neq 0$ for some $(nI), (mI) \in (R \cup I)$. Then $(aI + bI)(xI)(nI) = (bI)(xI)(nI) = (bI)(yI)(mI) = 0$. Hence $aI + bI \in (Z \cup I)$. That means $aI + bI$ represent a not unit and therefore $(R \cup I)$ is NLR. Now we wish to apply Lemma 3.6. For this purpose, we must show the right ideals $(A \cup I)$ and $(B \cup I)$ from $(R \cup I)$ that $l((A \cup I) \cap (B \cup I)) = l(A \cup I) + l(B \cup I)$. By 5) we have $rl((A \cup I) \cap (B \cup I)) = (A \cup I) \cap (B \cup I) = rl(A \cup I) \cap rl(B \cup I) = r(l(A \cup I) + l(B \cup I))$, in which the final equality is readily verified. Consequently, from the application of l, it follows that $l((A \cup I) \cap (B \cup I)) = lr(l(A \cup I) + l(B \cup I)) = l(A \cup I) + l(B \cup I)$. From Lemma 3.6, we deduce then that $(R \cup I)$ is right injective. Therefore, by the second part of theorem 4.6. $(R \cup I)$ is Neutrosophic artinian on both sides. Consequently from theorem 4.5. $(R \cup I)$ is Neutrosophic Q-F ring.

Theorem 4.8: Assume $(R \cup I)$ is a Neutrosophic right Noetherian; two-sided, Neutrosophic uniform ring that is divisible and semisimple. Then $(R \cup I)$ is Neutrosophic Q-F ring.

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Proof: Assume aI and bI represent any pair of non-units in $(R \cup I)$. Then $aI, bI \in (Z \cup I)$. Let $(aI)(xI) = 0 = (bI)(yI)$. Because $(xI)(nI) = (yI)(mI) \neq 0$ by right Neutrosophic uniform property, $(aI + bI)(xI)(nI) = (bI)(xI)(nI) = (bI)(yI)(mI) = 0$. Hence $aI + bI \in (Z \cup I)$. Similarly if $(xI)(aI) = (yI)(bI) = 0$, we possess $aI + b\mathrm{I} \in (Z \cup I)$, by Neutrosophic uniform attribute. Now let $(aI)(xI) = 0 = (yI)(bI)$, because $(R \cup I)$ is Neutrosophic Noetherian, so $(aI)^n = 0$, $(aI)^{n-1} \neq 0$ by [23, Th. 6.1]. Followed by Neutrosophic uniform, $(nI)(aI)^{n-1} = (mI)(yI)$. Hence $(nI)(aI)^{n-1}(aI + b\mathrm{I}) = (nI)(aI)^{n-1}(bI) = (mI)(yI)(bI) = 0$, i.e., $aI + bI \in (Z \cup I)$. Thus $aI + bI$ is non-unit, so a NR $(R \cup I)$ is local with a N-ideal $(Z \cup I)$ as the unique maximal. Now by Lemma 3.7, $(R \cup I) = (K \cup I)_1 \oplus \cdots \oplus (K \cup I)_n$, where $n \in N$ and $(K \cup I)_1, \ldots, (K \cup I)_n$ represent a minimal left ideas from $(R \cup I)$, i.e., minimal neutrosophic submodules from left $(R \cup I)$-module $(R \cup I)$. From this, it is evident that

$$\{0\} \subsetneq (K \cup I)_1 \subsetneq (K \cup I)_1 \oplus (K \cup I)_2 \subsetneq \cdots \subsetneq (K \cup I)_1 \oplus \cdots \oplus (K \cup I)_{n-1} \oplus (R \cup I)$$

constitutes a series of composition to $(R \cup I)$. Now, we can apply Lemma 3.8, to $(R \cup I)$-module to derive an inference about Neutrosophic Artinian and Neutrosophic Noetherian. The comments about Neutrosophic Artinian and Neutrosophic Noetherian follow of Remark 3.9. continuing the previous debate with Neutrosophic semi simple will obtain that a Neutrosophic semi simple ring is Neutrosophic Artinian and Neutrosophic Noetherian. Consequently from theorem 4.5. $(R \cup I)$ is Neutrosophic Q-F ring.

Theorem 4.9: Assume $(R \cup I)$ is a NKR and $(Z \cup I)$ is N-ideal s.t. $(Z \cup I) \subseteq (J \cup I)$. Then $(R \cup I)$ is Neutrosophic Q-F ring if $(R \cup I)/(Z \cup I)$ is a Neutrosophic local ring and $(R \cup I)$ is right extending.

Proof: Assume $(R \cup I)/(Z \cup I)$ is a NLR. We refer the elements from $(R \cup I)/(Z \cup I)$ by $\bar{a}I, aI \in (R \cup I)$. If $(\bar{a}I)(\bar{x}I) = 1$, then $(aI)(xI) = 1 + n,\ n \in Z$. Since $(Z \cup I) \subseteq (J \cup I)$, $(aI)(xI)$ is a unit, i.e., $(aI)(xI)(yI) = 1$, This implies $(xI)(yI)(aI) = 1$ since $(Z \cup I) \subseteq (J \cup I)$. Hence aI is unit. It may be readily confirmed that aI is not unit if $\bar{a}I$ is not unit. Next, if aI and bI are not units, clearly, based on the aforementioned descriptions, $\bar{a}I$ and $\bar{b}I$ are not units. Therefore $\bar{a}I + \bar{b}I$ is not unit because $(R \cup I)/(Z \cup I)$ is a NLR. This means that $aI + bI$ is not unit. Hence $(R \cup I)$ is a NLR. Now if $(R \cup I)$ extending, clearly each direct summand from $E(R \cup I)$ comprises an essential direct summand from $(R \cup I)$. Therefore, by Lemma 3.10, $(R \cup I)$ has essential socle f-generated. Consequently, if $(R \cup I)$ is an expanding ring and every cyclic $(R \cup I)$-module embeds in a free-module, so every right-cyclic $(R \cup I)$-module has an essential socle that is f-generated, and the ring is an artinian. Consequently from Theorem 4.5; $(R \cup I)$ is Neutrosophic Q-F ring.

Theorem 4.10: Let $(R \cup I)$ be a Neutrosophic ting which through satisfying the following condition:

1) Right ideals containing $(J \cup I)$ are principal.
2) $(J \cup I)$ is Neutrosophic entirely prime and constitutes an important left ideal.
3) $(Z \cup I) \subseteq (J \cup I)$.
4) $(J \cup I) \neq 0$.

Then $(R \cup I)$ is Neutrosophic Q-F ring.

Proof: It is sufficient to demonstrate that $(J \cup I)$ comprises all not units. Assume $(J \cup I) = (R \cup I)(xI)$ and $aI \notin (J \cup I)$. Then by (1) $(J \cup I) + (aI)(R \cup I) = (bI)(R \cup I)$ and $bI \notin (J \cup I)$. Hence $xI = (bI)(yI)$. Since $(J \cup I)$ is completely prime, $yI \in (J \cup I)$. Therefore $yI = (cI)(Ix)$, i.e., $xI = (bI)(yI) = (bI)(cI)(xI)$. Thus $1 - (bI)(cI) \in (Z \cup I)$. This implies by (3) $1 - (bI)(cI) \in (J \cup I)$ and $(bI)(cI)$ is unit and so, bI is unit. Hence $(J \cup I) + (aI)(R \cup I) = (R \cup I)$. Consequently, it is evident that $1 = jI + (aI)(cI),\ jI \in (J \cup I)$ and $1 - (aI)(cI) \in (J \cup I)$ and $(aI)(cI)$ is unit. Accordingly, we deduce that aI is unit. Therefore, $(R \cup I)$ is a NLR. Now assume that $(R \cup I)$ is coherent and perfect. By Lemma 3.11, $(R \cup I) \cong (R \cup I)_1 \times \cdots \times (R \cup I)_n$ where the $(R \cup I)_i$ are Neutrosophic local rings and commutative. Any $(R \cup I)_i$ is perfect and coherent also. Thus, it is reasonable to presume that $(R \cup I)$ is local, for example, with the maximum ideal $(M \cup I)$. Through lemma 3.12, $(R \cup I)$ fulfills DCC for principal ideals. Specifically, $(R \cup I)$ has a minimum ideal $(M \cup I)$. Because $(R \cup I)$ is coherent, $(R \cup I) / (M \cup I) \cong (M \cup I)$ is f.p. from the precise sequence $0 \longrightarrow (M \cup I) \longrightarrow (R \cup I) \longrightarrow (R \cup I) / (M \cup I) \longrightarrow 0$, it follows (see Lemma 3.13) that $(M \cup I)$ is f-generated Because $(M \cup I)$ is nil (being $(R \cup I)$ is perfect), it is necessary to be nilpotent. Now any $(M \cup I)^i / (M \cup I)^{i+1}$ is semi simple and f-generated, thus has finite-length. Therefore, $(R \cup I)$ has finite-length, thus classifying $(R \cup I)$ as an Artinian ring. Consequently from Theorem 4.5. $(R \cup I)$ is Neutrosophic Q-F ring.

Theorem 4.11: Let (R ∪ I) be a Neutrosophic right injective ring with identity, right uniform and every Neutrosophic injective (R ∪ I)-module is direct sum from Neutrosophic injective hulls from simple (R ∪ I)-modules. Then (R ∪ I) is Neutrosophic Q-F ring.

Proof: Because a Neutrosophic ring (R ∪ I) possesses identity and is right-injective, $\mathrm{Hom}_{(R \cup I)}((R \cup I), (R \cup I))$ is isomorphic for the ring from every left-multiplications and therefore to (R ∪ I). However $\mathrm{Hom}_{(R \cup I)}((R \cup$

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I), (R ∪ I)) is a NLR because (R ∪ I) is uniform of right [24, Prop. 2.2]. Therefore (R ∪ I) is a Neutrosophic local ring. Now, let all Neutrosophic injective (R ∪ I)-module is direct sum from Neutrosophic injective hulls from simple (R ∪ I)-modules. If (R ∪ I) is a not artinian, assuming a descending $(J \cup I)_1 \supsetneq (J \cup I)_2 \supsetneq \cdots$ of Neutrosophic ideals $(J \cup I)_i$ and let $(J \cup I) = \cap (J \cup I)_i$. By assumption $E((R \cup I)/(J \cup I)) = \oplus_{i=1}^{n} E((V \cup I)_i)$ where any $(V \cup I)_i$ is simple (R ∪ I)-module. (The direct sum can only be a finite in this case because $(R \cup I)/(J \cup I)$ is a cyclic module). Assume $(V \cup I) = (V \cup I)_1 \oplus \ldots \oplus (V \cup I)_n$. The length of this module is finite, therefore the descending chain $(V \cup I) \cap ((J \cup I)_1/(J \cup I)) \supseteq (V \cup I) \cap ((J \cup I)_2/(J \cup I))$ ... must stabilize, say $(V \cup I) \cap ((J \cup I)_2/(J \cup I)) = (V \cup I) \cap ((J \cup I)_{k+1}/(J \cup I)) = \cdots$ for certain k. Now, through [20, Theorem 3.38], $(V \cup I) \subseteq_e E((R \cup I)/(J \cup I))$, so $(V \cup I) \cap ((J \cup I)_k/(J \cup I)) \neq 0$. After that, the final equation displays $\cap_{i=1}^{\infty}((J \cup I)_i/(J \cup I)) \neq 0$, a contradiction. Therefore (R ∪ I) is artinian. Consequently from theorem 4.5. (R ∪ I) is Neutrosophic Q-F ring.

5. Conclusion

In this research article, we studied the relation between the Neutrosophic quasi-Frobenius rings on the one hand and the Neutrosophic local rings and the Artinian rings on the other hand. In addition, we studied some concepts such as semisimple ring, right injective and Noetherian ring, which through we obtained the quasi-Frobenius ring. Finally, we studied some mathematical formulas with a commutative, coherent and perfect ring, through which we obtained the Neutrosophic quasi-Frobenius ring.

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